Well-posedness of 1D free-congested Navier-Stokes equations

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Fluid equations under maximal density constraint

- free-congested Euler (/Navier-Stokes) equations

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) + \nabla p - 2\operatorname{div}(\mu \mathrm{D}(u)) - \nabla(\lambda \operatorname{div} u) = \rho f \\ 0 \le \rho \le 1, \ (1-\rho)p = 0, \ p \ge 0 \end{cases}$$

- free phase $\rho < 1$ and $p = 0$
- congested phase $\rho = 1, \ p \ge 0, \ \operatorname{div} u \ge 0$
- free boundary problem the interface between the two phases is unknown

Applications in physics - modeling of two-phase flows

- 1D biphasic equations for liquid-gas mixtures, α volume fraction of the liquid

$$\begin{cases} \partial_t(\alpha\rho_l) + \partial_x(\alpha\rho_l u_l) = 0 \\ \partial_t((1-\alpha)\rho_g) + \partial_x((1-\alpha)\rho_g u_g) = 0 \\ \partial_t(\alpha\rho_l u_l) + \partial_x(\alpha\rho_l u_l^2) + \alpha\partial_x p + \tau_l = M \\ \partial_t((1-\alpha)\rho_g u_g) + \partial_x((1-\alpha)\rho_g u_g^2) + (1-\alpha)\partial_x p + \tau_g = -M \end{cases}$$

$$M = \mu\alpha(1-\alpha)\rho_l(u_g - u_l),$$

$$\rho_l = \text{cst}, \quad p = a\rho_g^\gamma = a\overline{\rho}_g^\gamma \, (\widetilde{\rho}_g)^\gamma \quad \text{where } \widetilde{\rho}_g(t,x) = \frac{\rho_g(t,x)}{\overline{\rho}_g}$$

- formal limit system as $\varepsilon = \frac{\overline{\rho}_g}{\rho_l} \to 0$, $u_g = u_l + \varepsilon w$

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u_l) = 0 \\ \partial_t(\alpha u_l) + \partial_x(\alpha u_l^2) = 0 \end{cases}$$

pressureless equations: α can exceed 1 !!

ref: Bouchut, Brenier, Cortes, Ripoll (2001)

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$$M = \mu\alpha(1-\alpha)\rho_l(u_g - u_l),$$

$$\rho_l = \text{cst}, \quad p = a\rho_g^\gamma = a\overline{\rho}_g^\gamma\ (\widetilde{\rho}_g)^\gamma \quad \text{where } \widetilde{\rho}_g(t,x) = \frac{\rho_g(t,x)}{\overline{\rho}_g}$$

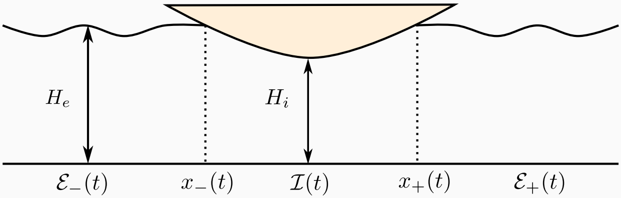
- formally: as $\varepsilon = \frac{\overline{\rho}_g}{\rho_l} \to 0$, $u_g = u_l + \varepsilon w$, $\varepsilon\overline{\rho}_g^{\gamma-1} p \to \pi$ with $(1-\alpha)\pi = 0$

$$\begin{cases} \partial_t\alpha + \partial_x(\alpha u_l) = 0 \\ \partial_t(\alpha u_l) + \partial_x(\alpha u_l^2) + \partial_x\pi = 0 \end{cases}$$

π ensures the constraint $\alpha \leq 1$

ref: Bouchut, Brenier, Cortes, Ripoll (2001)

Applications - wave-structure interaction



in the shallow water regime

$$\begin{cases} \partial_t h + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h} + \frac{g h^2}{2} \right) = -\frac{1}{\rho} h \partial_x p \\ 0 \leq h \leq h_{\text{sol}}, \ (h_{\text{sol}} - h)(p - p_{\text{atm}}) = 0 \end{cases}$$

ref: Lannes '17, Bocchi '18, Iguchi & Lannes '18

Free-congested equations

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla P(\rho) + \nabla p - \mathrm{div}(\mathbb{S}) = 0 \\ 0 \le \rho \le 1, \ (1-\rho)p = 0, \ p \ge 0 \end{cases}$$

- existing results on the 2-phase compressible-incompressible pb for immiscible fluids
  ref: Colombo et al '16
- for the congestion problem the interface is not closed !
  → regularity of the pressure p, of the interface ?
  → what transmission conditions at the interface ?

  local well-posedness of the 1d floating body pb (Lannes, Iguchi '18) when
  $P(\rho) = \kappa \rho^2$, $\mathbb{S} = 0$ + continuity of the variables at the interface
- existence of global weak sol. via a singular compressible approx pb, $p_\varepsilon(\rho) \underset{\rho \to 1^-}{\to} +\infty$
  joint works with E. Zatorska '15, R. Bianchini '21, K. Saleh '21
  study of traveling waves $(\rho_\varepsilon, u_\varepsilon)(x - s_\varepsilon t)$ with A.-L. Dalibard '20

1D free-congested Navier-Stokes equations, $P(\rho) = 0$

in Lagrangian mass coordinates, $v = \frac{1}{\rho}$ being the specific volume

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p - \mu \partial_x \left(\dfrac{1}{v} \partial_x u \right) = 0 \\ v \geq 1, \quad (v-1)p = 0, \quad p \geq 0 \end{cases}$$

free boundary problem set on $\Omega = \mathbb{R}$:

- free zone $\{v > 1\}$: compressible pressureless dynamics;
- congested zone $\{v = 1\}$: incompressible dynamics

the result on the existence of a weak sol. does not give any info. on the congested zone

Explicit propagation front $(\overline{v},\overline{u},\overline{p})(t,x) = (\overline{v},\overline{u},\overline{p})(x-st)$

$$\begin{cases} -s\overline{v}' - \overline{u}' = 0 \\ -s\overline{u}' + \overline{p}' - \mu\left(\dfrac{\overline{u}'}{\overline{v}}\right)' = 0 \end{cases} \qquad + \text{ far field cond. } \ (\overline{v},\overline{u}) \underset{\pm\infty}{\to} (v_\pm, u_\pm)$$

we set $v_- = 1 < v_+,\ u_+ < u_-$, we look for a profile which is such that

- congested on $]-\infty,\xi^*[$, free on $]\xi^*,+\infty[$

  in the congested zone: $\overline{v}(\xi) = 1,\ \overline{u}(\xi) = u_-$

  in the free zone:

$$\begin{cases} s\overline{v}(\xi) + \overline{u}(\xi) = sv_+ + u_+ \\ s\overline{u}(\xi) + \mu\dfrac{\overline{u}'(\xi)}{\overline{v}(\xi)} = su_+ \end{cases} \implies \overline{v}' = \frac{s}{\mu}v(v_+ - \overline{v})$$

- $\overline{u}$, $\overline{v}$ and $\overline{p} - \mu\dfrac{\overline{u}'}{\overline{v}}$ continuous at ξ^*

$$\implies s = \frac{u_- - u_+}{v_+ - 1} > 0, \qquad \xi < \xi^* \quad p(\xi) \equiv p_- = -\mu \lim_{\xi\to(\xi^*)^+} \overline{u}'(\xi) = s^2(v_+ - 1)$$

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we set $v_- = 1 < v_+$, $u_+ < u_-$, we look for a profile which is such that

- congested on $]-\infty,\xi^*[$, free on $]\xi^*,+\infty[$;
- $\overline{u}$, $\overline{v}$ and $\overline{p} - \mu\dfrac{\overline{u}'}{\overline{v}}$ continuous at ξ^*

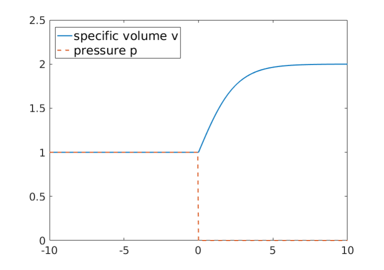
Lemma

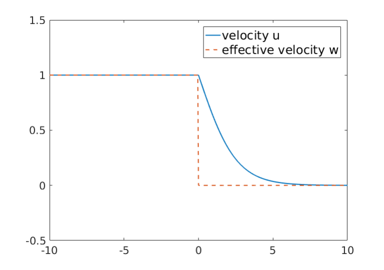
Let $s = \dfrac{u_- - u_+}{v_+ - 1} > 0$, *there exists a unique (up to a shift) propagation front:*

$$\overline{v}(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \dfrac{v_+}{1+(v_+-1)\exp(-\frac{sv_+}{\mu}x)} & \text{if } x > 0 \end{cases} \qquad \overline{p}(x) = \begin{cases} s^2(v_+-1) & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

$$\overline{u}(x) = u_+ + sv_+ - s\overline{v}(x)$$

Perturbations of the limit profile $(\overline{v},\overline{u})$





→ what happens if we perturb this profile ?

- we perturb initially only the free zone $\{x > 0\}$
- we look for a solution (v,u) (not a traveling wave a priori) which is
  - congested on $]-\infty,\widetilde{x}(t)[$
  - free on $]\widetilde{x}(t),+\infty[$
- as for the profile $(\overline{v},\overline{u},\overline{p})$, we impose the continuity of $v,u,p-\mu\partial_x\left(\frac{\partial_x u}{v}\right)$ at $\widetilde{x}$

Dynamics of the interface $\widetilde{x}(t)$

$$\begin{cases} \partial_t v - \partial_x u = 0 \\ \partial_t u + \partial_x p - \mu \partial_x \left(\frac{1}{v} \partial_x u \right) = 0 \\ v \geq 1, \quad (v-1)p = 0, \quad p \geq 0 \end{cases}$$

- we look for a solution (v,u) which is congested on $]-\infty, \widetilde{x}(t)[$, free on $]\widetilde{x}(t), +\infty[$
- recall that the interface is not closed $\rightarrow$ $\widetilde{x}$ is not given by a kinematic condition
- the dynamics of $\widetilde{x}$ is actually encoded in the continuity conditions:

$$v(t, \widetilde{x}(t)) = 1 \quad \forall\ t \quad \Longrightarrow \quad \boxed{\widetilde{x}'(t) = -\frac{\partial_t v(t, \widetilde{x}(t))}{\partial_x v(t, \widetilde{x}(t))} = -\frac{\partial_x u(t, \widetilde{x}(t))}{\partial_x v(t, \widetilde{x}(t))}}$$

see also: Iguchi, Lannes '19

Well-posedness results

Local in time result

Assuming some compatibility and non-degeneracy conditions on the initial data, there exists a unique solution $(v,u,p)(t,x) = (v_s,u_s,p_s)(t,x-\widetilde{x}(t))$ on short time intervals $[0,T]$ with $\widetilde{x} \in H^2(0,T)$,

$$v_s - \overline{v},\ u_s - \overline{u} \in L^\infty([0,T];H^3(\mathbb{R}_+)), \quad p_s(t,x) = -\mu(\partial_x u_s)_{x=0^+} \in H^1(0,T).$$

Global in time result

Assuming moreover that the “initial energy is small”, then the solution is global and

$$|\widetilde{x}'(t) - s| + \sup_{x\in\mathbb{R}} |(u_s,v_s,p_s)(t,x) - (\overline{u},\overline{v},\overline{p})(x)| \to 0 \quad \text{as } t \to +\infty.$$

Local in time result

Local in time result - Strategy

- (v,u) should be constant on $x<\widetilde{x}(t)$ → we only study (v,u) on $x>\widetilde{x}(t)$

  let $(v_s,u_s)=(v,u)(t,x-\widetilde{x}(t))$

$$\begin{cases} \partial_t v_s - \widetilde{x}'(t)\partial_x v_s - \partial_x u_s = 0 \\ \partial_t u_s - \widetilde{x}'(t)\partial_x u_s - \mu\partial_x\left(\dfrac{1}{v_s}\partial_x u_s\right) = 0 \end{cases} \quad \text{for} \quad t>0,\ x>0$$

$$\text{boundary cond.} \quad [v_s]_{x=0}=1, \quad [u_s]_{x=0}=u_-, \quad \lim_{x\to+\infty}(v,u)=(v_+,u_+)$$

- dynamics of the interface coupled with the dynamics of (v_s,u_s)

$$\widetilde{x}'(t) = -\left[\frac{\partial_x u_s}{\partial_x v_s}\right]_{|x=0^+}$$

- effective velocity $w=u-\mu\partial_x \ln v$

$$\rightsquigarrow \ \partial_t w_s - \widetilde{x}'(t)\partial_x w_s = 0 \implies w_s(t,x)=w^0(x+\widetilde{x}(t)) \ \text{ provided } \widetilde{x}'>0$$

$$\rightsquigarrow \quad \partial_t v_s - \widetilde{x}'(t)\partial_x v_s - \mu\partial_x^2 \ln v_s = \partial_x w^0(x+\widetilde{x}(t))$$

Strategy - Fixed point argument

given $\widetilde{y} \in H^2_{loc}(\mathbb{R}_+)$ such that $\widetilde{y}(0) = 0$, $\widetilde{y}'(0) = -\left[\frac{\partial_x u^0}{\partial_x v^0}\right]_{|x=0^+}$

we look for a fixed point of

$$\begin{aligned} \mathscr{T} \ : \ H^2_{loc}(\mathbb{R}_+) &\longrightarrow H^2_{loc}(\mathbb{R}_+) \\ \widetilde{y} &\mapsto \widetilde{z} \end{aligned}$$

where $\widetilde{z}$ is defined through the following steps:

1. define and find high order energy estimates on v_s solution to:

$$\begin{cases} \partial_t v_s - \widetilde{y}'(t)\partial_x v_s - \mu\partial_x^2 \ln v_s = \partial_x w^0(x+\widetilde{y}(t)) & \text{for} \quad t>0,\ x>0 \\ [v_s]_{x=0} = 1, \quad \lim_{x\to+\infty} v = v_+, \quad v_s(t=0) = v^0 \end{cases}$$

   in particular show that $v_s(t,x) > 1$ for $x > 0$

2. define and find high order energy estimates on u_s solution to:

$$\begin{cases} \partial_t u_s - \widetilde{y}'(t)\partial_x u_s - \mu\partial_x\left(\dfrac{1}{v_s}\partial_x u_s\right) = 0 & \text{for} \quad t>0,\ x>0 \\ [u_s]_{x=0} = u_-, \quad \lim_{x\to+\infty} u = u_+, \quad u_s(t=0) = u^0 \end{cases}$$

3. set

$$\widetilde{z}(t) = -\int_0^t \frac{\partial_x u_s(\tau,0)}{\partial_x v_s(\tau,0)}\,d\tau = -\mu\int_0^t \frac{\partial_x u_s(\tau,0)}{u_- - w^0(\widetilde{y}(\tau))}\,d\tau$$

Remarks

- H^2_t is the minimal regularity required on $\widetilde{x}$ to formally close the fixed point argument

- regularity of the solution

$$v_s - \overline{v},\ u_s - \overline{u} \in L^\infty([0,T[; H^3(\mathbb{R}_+))$$

$$\partial_t(v_s - \overline{v}),\ \partial_t(u_s - \overline{u}) \in L^\infty(0,T; H^1(\mathbb{R}_+)) \cap L^2(0,T; H^2(\mathbb{R}_+))$$

- extension of the solution (v_s, u_s) by setting $(v_s, u_s) = (1, u_-)$ on $\mathbb{R}_-$
  $p_s(t,x) := p_-(t) = -\mu[\partial_x u_s]_{|x=0^+}$ for $x < 0$, and $p_- \in H^1(0,T)$

Local in time result - Sketch of proof - Eq. on v

$$\begin{cases} \partial_t v_s - \tilde{y}'(t)\partial_x v_s - \mu \partial_x^2 \ln v_s = \partial_x w^0(x+\tilde{y}(t)) & \text{for} \quad t>0,\ x>0 \\ [v_s]_{x=0} = 1, \quad \lim_{x\to+\infty} v = v_+, \quad v_s(t=0) = v^0 \end{cases}$$

Lemma (Maximum principle - Local-in-time L^∞ estimate)

Assume that $\forall\ t \in [0,T]$, $\tilde{y}'(t) \in [M^{-1}, M]$, $\inf_{x\in[0,1]} \partial_x v^0(x) \geq M^{-1}$ *for some* $M>1$.

Then there exists T_0*, dep. on* $\|\partial_x w^0\|_{L^\infty(\mathbb{R}_+)}$, $\sup v^0$, M *and the parameters of the system such that, for all* $t \in [0,T_0]$: $1 < v_s(t,x) \leq 2\sup v^0$.

Lemma (Existence and uniqueness of v)

Let $\tilde{y} \in H^2(0,T)$ *such that* $\tilde{y}(0)=0$, $\tilde{y}'(0) = -\left[\frac{\partial_x u^0}{\partial_x v^0}\right]_{|x=0^+}$ + *cond. previous lemma.*
There exists $T_0 \leq T$ *dep. on* M *and the parameters of the system, such that there exists a unique* v_s *satisfying*

$$v_s \in L_t^\infty H^1_{loc,x} \cap L_t^2 H_x^2, \quad \partial_t v \in L^2_{t,x}.$$

$$\|v_s - \overline{v}\|^2_{L^\infty H^1} + \|\partial_x(v_s - \overline{v})\|^2_{L^2L^2} + \|\partial_t v_s\|^2_{L^2L^2}$$

$$\leq C\left(\|v^0 - \overline{v}\|^2_{H^1} + \|\sqrt{x}\partial_x w^0\|^2_{L^2} + \|\tilde{y}' - s\|^2_{L^2}\right)e^{CT_0}$$

Sketch of proof - More estimates on v

$$\begin{cases} \partial_t v_s - \widetilde{y}'(t)\partial_x v_s - \mu \partial_x^2 \ln v_s = \partial_x w^0(x+\widetilde{y}(t)) \qquad \text{for} \quad t>0,\ x>0 \\ [v_s]_{x=0} = 1, \quad \lim_{x\to+\infty} v = v_+, \quad v_s(t=0) = v^0 \end{cases}$$

- remove the exponential dep. wrt time by using $V(t,x) = -\int_x^{+\infty}(v_s - \overline{v})(t,z)dz$

$$\partial_t V - \widetilde{y}'\partial_x V - \mu\partial_x \ln\left(1+\frac{\partial_x V}{\overline{v}}\right) = \partial_x W^0 - \frac{\mu}{s}(\widetilde{y}' - s)\partial_x \ln\frac{\overline{v}}{v_+}$$

"nice" structure of the diffusion and source terms

$$\mu\int_{R_+} \ln\left(1+\frac{\partial_x V}{\overline{v}}\right)\partial_x V \geq c_0 \int_{R_+} |\partial_x V|^2$$

source terms controlled via integration by parts and Cauchy-Schwarz ineq.

thanks to the control of $\|\partial_x V\|_{L^2_{t,x}}$ we avoid Gronwall ineq. when estimating $v_s - \overline{v}$

Sketch of proof - Estimates on (v_s, u_s)

$$\mathcal{E}_0 := \|v^0 - \overline{v}\|^2_{H^3(\mathbb{R}_+)} + \|u^0 - \overline{u}\|^2_{H^3(\mathbb{R}_+)} + \|V^0\|^2_{L^2(\mathbb{R}_+)} + \|(1+\sqrt{x})\partial_x^k W^0\|^2_{L^2(\mathbb{R}_+)}$$

$$\mathcal{E}_T := \mathcal{E}_0 + \|\widetilde{y}' - s\|^2_{H^1(0,T)}$$

Proposition

- *Assume the cond. of previous lemmas, then for small times and for some $q > 1$:*

$$\|\partial_t \partial_x v_s\|^2_{L^\infty L^2 \cap L^2 H^1} + \|\partial_x^3 (v_s - \overline{v})\|^2_{L^\infty L^2} + \|\partial_t^2 v_s\|_{L^2 L^2} \leq C\mathcal{E}_T (1 + \mathcal{E}_T)^q.$$

- *Assume furthermore the compatibility condition*

$$\partial_x u^0_{|0} \; (\widetilde{y}'(0) - \mu(\partial_x v^0)_{|0}) + \mu(\partial_x^2 u^0)_{|0} = 0$$

then

$$\|u_s - \overline{u}\|^2_{H^1 H^2} + \|u_s - \overline{u}\|^2_{W^{1,\infty} H^1} + \|u_s - \overline{u}\|^2_{H^2 L^2} \leq C\mathcal{E}_T (1 + \mathcal{E}_T)^q.$$

this eventually allows to prove that the map $\mathcal{T} : \widetilde{y} \mapsto \widetilde{z}$, $\quad \widetilde{z}(t) = -\int_0^t \frac{\partial_x u_s(\tau,0)}{\partial_x v_s(\tau,0)} d\tau$
has an invariant set $\mathcal{I}_M$ and is a contraction on $\mathcal{I}_M$ $\Longrightarrow$ local-in-time result

Global in time result

Towards a global result

Assume that $\mathcal{E}_0 \leq c_0\delta^2$ (small) and let T^* be the maximal time of existence of (v_s, u_s) . Set

$$\overline{T} := \sup \left\{T \in \,]0, T^*[, \quad \|\widetilde{x}' - s\|_{H^1(0,T)} \leq \delta\right\}.$$

The global existence of the solution relies on the following bootstrap argument

Proposition

There exist constants $c_0, \delta_0 > 0$ depending only on the parameters of the problem s, μ, v_+, such that:
for all $\delta \in (0, \delta_0)$, if $\mathcal{E}_0 \leq c_0\delta^2$, and $\|(1+\sqrt{x})\partial_x^k w^0\|_{L^2(\mathbb{R}_+)} \leq c_0\delta^{3/2}$, $k = 1,2,3$, then

$$\forall \ T \in [0, \overline{T}], \quad \|\widetilde{x}' - s\|_{H^1(0,T)} \leq \frac{\delta}{2}.$$

the above proposition does not derive directly from the previous set of estimates...

Coercivity of the linearized operator - New variable

To simplify let us set $w^0 1_{x>0} \equiv u_+$ then, on $[0, T^*[$, $g := v_s - \overline{v}$, $\beta = \widetilde{x}' - s$, is solution to

$$\begin{cases} \partial_t g + \partial_x \mathscr{A} g = \beta \partial_x \overline{v} + \beta \partial_x g + \mu \partial_x^2 \left(\ln\left(1 + \frac{g}{\overline{v}}\right) - \frac{g}{\overline{v}} \right) \\ g_{|x=0} = 0, \quad \lim_{x\to+\infty} g(t,x) = 0, \quad g_{t=0} = v^0 - \overline{v} \end{cases}$$

- $\partial_x \mathscr{A}$: linearized operator around $\overline{v}$ where $\mathscr{A} := -s\mathrm{Id} - \mu \partial_x \left(\frac{\cdot}{\overline{v}} \right)$

  good coercivity properties on $\partial_x \mathscr{A}$ and $\mathscr{A} \partial_x$
- 2nd and 3rd terms of the RHS are quadratic → treated perturbatively
- observe that $\partial_x \overline{v} \in \mathrm{Ker}\, \mathscr{A}$, hence $g_1 = \mathscr{A} g$ satisfies

$$\begin{cases} \partial_t g_1 + \mathscr{A} \partial_x g_1 = \beta \mathscr{A} \partial_x g + \mu \mathscr{A} \partial_x^2 \left(\ln\left(1 + \frac{g}{\overline{v}}\right) - \frac{g}{\overline{v}} \right) \\ g_{1|x=0} = 0, \quad \lim_{x\to+\infty} g_1(t,x) = 0, \quad g_{t=0} = \mathscr{A}(v^0 - \overline{v}) \end{cases}$$

only quadratic terms in the RHS ! recall that $\|g\| \lesssim \mathscr{E}_0^{1/2} + \|\beta\|$

Estimates on $g_1 = \mathscr{A}(v_s - \overline{v})$ and $\beta(t) = \widetilde{x}'(t) - s$

- $L^\infty(L^2) \cap L^2(H^1)$: we use $\int_{\mathbb{R}_+} (\mathscr{A}\partial_x \varphi)\varphi \geq \frac{\mu}{v_+}\int_{\mathbb{R}_+} (\partial_x \varphi)^2 + \frac{s}{2}(\varphi(0))^2 + \mu\varphi'(0)\varphi(0)$ with $\varphi = g_1$

$$\|g_1\|^2_{L^\infty L^2} + \frac{\mu}{v_+}\|\partial_x g_1\|^2_{L^2 L^2} \leq \mathscr{E}_0 + C\big(\mathscr{E}_0 + \|\beta\|^2_{H^1}\big)^2$$

- $L^\infty(H^1) \cap L^2(H^2)$: we multiply the eq. on $\partial_x g_1$ by $\partial_x g_1 \frac{\rho}{\overline{v}}$ for some well-chosen ρ

and use
$$\int_{\mathbb{R}_+} (\partial_x \mathscr{A}\varphi)\frac{\rho\varphi}{\overline{v}} \geq \mu \int_{\mathbb{R}_+} (\partial_x \frac{\varphi}{\overline{v}})^2 \rho - C\|\rho\|_{W^{2,\infty}} \int_{\mathbb{R}_+} \varphi^2$$
$$+ \big(\frac{s}{2}\rho(0) - \frac{\mu\rho'(0)}{2}\big)(\varphi(0))^2 + \mu\partial_x(\frac{\varphi}{\overline{v}})(0)\varphi(0)\rho(0)$$

replace the BC: $\partial_x g_{1|x=0} = \beta \frac{s(v_+ - 1)}{\mu}, \quad \mu\partial_x\left(\frac{\partial_x g_1}{\overline{v}}\right)_{|x=0} = -(\beta + s)\partial_x g_{1|x=0}$

$$\implies \|\partial_x g_1\|^2_{L^\infty L^2} + \|\partial_x \frac{\partial_x g_1}{\overline{v}}\|^2_{L^2 L^2} + \|\beta\|^2_{L^2} \leq \mathscr{E}_0 + C\big(\mathscr{E}_0 + \|\beta\|^2_{H^1}\big)^2 + C\big(\mathscr{E}_0^{1/2} + \|\beta\|_{H^1}\big)\|\beta\|^2_{L^2}$$

idea: we perform the same type of estimates on $\partial_t g_1 \rightsquigarrow$ control of $\|\beta\|_{H^1}$

Control of $\beta(t) = \widetilde{x}'(t) - s$

- general $w^0 \to$ additional source term in the eq. + modified BC at $x = 0$
  we get from the estimate of $\partial_t \partial_x g_1$ in $L^\infty(L^2)$:

$$\|\beta\|^2_{H^1(0,\overline{T})} \le C\left(\mathscr{E}_0 + \left(\mathscr{E}_0 + \|\beta\|^2_{H^1(0,\overline{T})}\right)^2 + \sum_{k=1}^{3} \|(1+\sqrt{x})\partial_x^k w^0\|^{4/3}_{L^2(\mathbb{R}_+)}\right)$$

- assume that $\mathscr{E}_0 \le c_0 \delta^2$, $\|(1+\sqrt{x})\partial_x^k w^0\|_{L^2(\mathbb{R}_+)} \le c_0 \delta^{3/2}$, then for small c_0

$$\|\beta\|^2_{H^1(0,\overline{T})} \le C c_0 \delta^2 + C\delta^4 \le \frac{\delta^2}{4}$$

Conclusion

Final results - Hypotheses

(H1) **Partially congested initial data:** $(u^0, v^0) \in (\overline{u}, \overline{v}) + L^1(\mathbb{R})$, and such that $u^0(x) = \overline{u}(x) = u_-$, $v^0(x) = \overline{v}(x) = 1$ if $x < 0$;

(H2) **Regularity:** $1_{x>0}(u^0 - \overline{u}, v^0 - \overline{u}) \in H^3(\mathbb{R}_+)$ and $1_{x>0}\sqrt{x}\partial_x^k w^0 \in L^2(\mathbb{R}_+)$ for $k = 1, 2$, where $w^0 = u^0 - \mu\partial_x \ln v^0$;

(H3) **Compatibility:** $u^0(0^+) = u_-$, $v^0(0^+) = 1$, and

$$\left[-\frac{(\partial_x u^0)^2}{\partial_x v^0} - \mu\partial_x v^0 \partial_x u^0 + \mu\partial_x^2 u^0 \right]_{|x=0^+} = 0;$$

(H4) **Non-degeneracy:** $\partial_x v^0(0^+) > 0$, $\partial_x u^0(0^+) < 0$ and $v^0(x) > 1$ for $x > 0$;

(H5) **Decay:** $1_{x>0} V^0 \in L^2(\mathbb{R}_+)$, where $V^0(x) := -\int_x^\infty (v^0 - \overline{v})$, and $1_{x>0}(1 + \sqrt{x}) W^0 \in L^2(\mathbb{R}_+)$, where $W^0 := -\int_x^\infty (w^0 - u_+)$.

Theorem (Local in time result)

Let (u^0, v^0) satisfying the assumptions (H1)-(H5).

Then there exists $T > 0$ and $\widetilde{x} \in H^2_{loc}([0,T[)$, with $\widetilde{x}(0) = 0$, $\widetilde{x}'(0) = -[\frac{\partial_x u^0}{\partial_x v^0}]_{x=0}$, such that the FC Navier-Stokes eq. have a unique maximal solution (u,v) of the form

$$(u,v)(t,x) = (u_s, v_s)(t, x - \widetilde{x}(t)) \quad \textit{on the interval } [0,T[,$$

where $u_s(t,x) = u_-$, $v_s(t,x) = 1$ and $p_s(t,x) = -\mu(\partial_x u_s)_{|x=0^+}$ for $x < 0$.

Furthermore,

$$v_s(t,x) > 1 \quad \textit{for all } t \in [0,T[,\ x > 0,$$

and the solution (u_s, v_s) has the following regularity in the free domain:

$$v_s - \overline{v},\ u_s - \overline{u} \in L^\infty([0,T[; H^3(\mathbb{R}_+)),$$
$$\partial_t(v_s - \overline{v}),\ \partial_t(u_s - \overline{u}) \in L^\infty([0,T[; H^1(\mathbb{R}_+)) \cap L^2(]0,T[; H^2(\mathbb{R}_+)).$$

Eventually, the pressure in the congested domain satisfies

$$p_s \in H^1(0,T).$$

Theorem (Global in time result)

Let (u^0, v^0) satisfying the assumptions (H1)-(H5), and let

$$\mathscr{E}_0 := \|v^0 - \overline{v}\|^2_{H^3(\mathbb{R}_+)} + \|u^0 - \overline{u}\|^2_{H^3(\mathbb{R}_+)} + \|w^0 - u_+\|^2_{L^2(\mathbb{R}_+)} + \|V^0\|^2_{L^2(\mathbb{R}_+)} + \|(1+\sqrt{x})W^0\|^2_{L^2(\mathbb{R}_+)} + \|(1+\sqrt{x})\partial_x w^0\|^2_{L^2(\mathbb{R}_+)} + \|(1+\sqrt{x})\partial_x^2 w^0\|^2_{L^2(\mathbb{R}_+)}.$$

Assume moreover that $1_{\{x>0\}}\sqrt{x}\partial_x^3 w^0 \in L^2(\mathbb{R})$. Then, there exist constants $c_0, \delta_0 > 0$ depending only on the parameters of the pb $s, \mu, v_+, u_\pm$, such that $\forall\ \delta \in (0, \delta_0)$, if

$$\mathscr{E}_0 \le c_0 \delta^2, \quad \|(1+\sqrt{x})\partial_x^k w^0\|_{L^2(\mathbb{R}_+)} \le c_0 \delta^{3/2}, \quad \forall\ \delta \in (0, \delta_0),\ k = 1, 2, 3,$$

then the solution $(\widetilde{x}, u_s, v_s)$ is global, and

$$\|v_s - \overline{v}\|_{L^\infty(\mathbb{R}_+, H^3(\mathbb{R}_+))} + \|u_s - \overline{u}\|_{L^\infty(\mathbb{R}_+, H^3(\mathbb{R}_+))} + \|\widetilde{x}' - s\|_{H^1(\mathbb{R}_+)} \le C\delta.$$

Furthermore

$$|\widetilde{x}'(t) - s| + \sup_{x \in \mathbb{R}} |(u_s, v_s, p_s)(t, x) - (\overline{u}, \overline{v}, \overline{p})(x)| \longrightarrow 0 \quad \text{as } t \to +\infty,$$

where $\overline{p}(x) = p_- 1_{x<0} = s^2(v_+ - 1) 1_{x<0}$.

Conclusion - Perspectives and open issues

- more general perturbations
- multi-D: explicit partially congested solutions + stability
- beyond Navier-Stokes equations, congestion phenomena in other fluid eq.