Initial boundary value problems for hyperbolic systems, and dispersive perturbations

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Abstract The goal of these notes is to point out similarities and differences between two kinds of initial boundary value problems in dimension one. The first one concerns hyperbolic systems (such as the nonlinear shallow water equations) while the second one concerns dispersive perturbations of such systems (such as Boussinesq systems). In the absence of a boundary, that is, for the initial value problem, the link between both classes is quite obvious but in the presence of a boundary, the situation is more complex and dispersive boundary layers must be understood if one wants to understand the links between both classes of problems. After reviewing several types of initial boundary value problems (some of which being free boundary problems) arising in the study of waves in shallow water, we sketch the general theory for hyperbolic initial boundary value problems developed in [18] and that encompasses all of the above examples that involve hyperbolic systems. Such a general theory does not exist for dispersive perturbations of hyperbolic systems, but we treat two important examples involving Boussinesq systems. In the first one, we show that the nature of the initial boundary value problem shares little in common with the hyperbolic configuration. For instance, the problem has the structure of an ODE and no higher order compatibility conditions on the data are required to have solutions of high regularity. These differences naturally raise the questions of the control of the time of existence and of the dispersionless limit; they are addressed in a second example motivated by a wave-structure interaction problem. We explain the approach developed in [13] to treat this problem, pointing out the role played by dispersive boundary layers.

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1 Introduction

These extended lecture notes correspond to a course given at the Winter School on Fluid Dynamics, Dispersive Equations and Quantum Fluids in Bressanone, in december 2018. The goal of this course was to provide theoretical tools to deal with several initial boundary value problems (IBVP) arising in the study of the propagation of waves in shallow water. In these examples, the boundary can be artificial, for instance when one wants to prescribe the water height at the entrance of a domain, or impose a transparent boundary condition at the end of the domain of interest; and of course, the boundary can be physical, for instance in the presence of a wall or when the waves interact with a partially immersed structure. There are also various kinds of models that can be used to describe the propagation of the waves. We have chosen here to focus our attention on two particular wave models that are used for many applications, namely, the nonlinear shallow water equations and the Boussinesq equations. We also consider here the case of a horizontal dimension $d = 1$ because the theory in higher dimensions is far less developed.

From a mathematical point of view, the nonlinear shallow water equations form a quasilinear hyperbolic system, while the Boussinesq systems keep the same structure, but with an additional dispersive term. The above hydrodynamical motivations therefore lead us to address several mathematical issues (that are also relevant in many other physical contexts), and among them,

- Provide a general theory to address various classes of hyperbolic IBVPs: standard boundary conditions, transmission problems, free boundary problems of kinematic type, and a new class of free boundary problems motivated by the analysis of the interaction of waves with floating structures.
- Point out some of the main issues that arise when a dispersive perturbation is added to an hyperbolic IBVP. As we shall see, the situation is drastically different. For instance, the IBVP for the classical Boussinesq-Abbott system has an ODE structure while the nonlinear shallow water equations are of course a system of PDEs. Also, m compatibility conditions are needed for solutions of m-th order regularity for hyperbolic IBVPs while higher order compatibility conditions are not required for high regularity solutions to the IBVP associated with the Boussinesq-Abbott system.
- Give some elements to understand the dispersionless limit for IBVP. For instance, it is possible to control the time of existence for the solutions of an IBVP involving the Boussinesq equations when the dispersion parameter tends to zero, and do these solutions converge to the solutions of the corresponding (dispersionless) hyperbolic IBVP ? As we shall see, the notion of *dispersive boundary layer* is crucial to answer these questions.

The material presented here essentially comes from [18] where an extensive theory for the analysis of IBVPs for hyperbolic systems in dimension $d = 1$ was proposed, from [25] where a standard IBVP was considered for a Boussinesq system (both from the PDE and numerical perspectives), and from [13], devoted to a wave-structure interaction problem involving a Boussinesq system and in which an analysis of the dispersive boundary layer was proposed to understand the dispersionless limit (and more generally to control the dependence of the existence time on the dispersion parameter). The goal of the present lecture notes is to propose a somehow general picture of the situation as far as IBVPs are concerned and to try to relate the well established theory for hyperbolic system to the behavior exhibited in the few articles that have dealt with IBVPs for dispersive perturbations of hyperbolic systems. We have therefore tried to follow a similar structure in the presentation of the results for both classes of systems, and we purposely skipped many technical details (for which we refer to the aforementioned works), hoping that it will help the reader to understand better the role of dispersion in IBVPs.

1.1 Organization of the paper

We review in Section 2 several IBVPs arising in the study of the propagation of waves in shallow water. Various problems involving the nonlinear shallow water equations are presented in §2.1 (the standard IBVP is presented in §2.1.2, transmission problems are discussed in §2.1.3, a free boundary problem related to the hydraulic piston is presented in §2.1.4 and another type of free boundary problem is introduced in §2.1.5 to describe the interaction of waves with a partially immersed object); in §2.2 we present two problems related to the Boussinesq equations (the standard IBVP in §2.2.2, and a wave-structure interaction problem in §2.2.3).

In Section 3 we present a general analysis of hyperbolic IBVPs starting with the linear case in §3.1, where we introduce the important notions of compatibility conditions, the Kreiss-Lopatinskiĭ condition and Kreiss symmetrizers. Quasilinear systems are then treated in §3.2, IBVPs on a moving domain in §3.3 when the boundary has a prescribed motion; we introduce in particular the so-called Alinhac good unknown. We then address in §3.4 a first class of free boundary value problems of "kinematic" type (such as the hydraulic piston), and finally in §3.5, we consider a more singular class of free boundary problems (to which the floating body problem of §2.1.5 belongs).

Finally, in Section 4 we present two IBVPs for dispersive perturbations of hyperbolic systems. We consider in §4.1 a standard initial boundary value problem for a classical Boussinesq system and observe several major differences with the hyperbolic case, such as the ODE nature of the problem or the absence of high order compatibility conditions for regular solutions. This change of behavior suggests that the dispersionless limit involves non trivial phenomena. This issue is addressed in §4.2 where a wave-structure interaction problem is considered for another type of Boussinesq system. We will insist in particular on the role played by *dispersive boundary layers* in the dispersionless limit and in the control of the interval of existence for the solution.

1.2 Notations

The following notations will be used throughout the article.
- We write $\Omega_T = [0,T] \times \mathbb{R}_+$.
- We denote by $H^m(\mathbb{R}_+)$ and $H^m(\Omega_T)$ the standard Sobolev spaces

$$H^m(\mathbb{R}_+) = \{f \in L^2(\mathbb{R}_+), \partial_x^k f \in L^2(\mathbb{R}_+), k = 0, \dots, m\},$$
$$H^m(\Omega_T) = \{f \in L^2(\Omega_T), \partial_t^j \partial_x^k f \in L^2(\Omega_T), 0 \leq j + k \leq m\}.$$

- The functional space $\mathbb{W}^m(T)$ is defined in Definition 2.
- The quantities $|||u(t)|||_m$ and $|u_{|x=0}|_{m,t}$ are defined in Definition 2.
- We sometimes use the notation ∂ which is a stakeholder for both ∂_x and ∂_t.
- We write $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$, with $\mathcal{E}_- = (-\infty; -R)$ and $\mathcal{E}_+ = (R, \infty)$, for some $R > 0$.

2 Several examples arising in wave-structure interactions

We present in these lecture notes several initial boundary value problems that arise in the study of wave-structure interactions. We shall more specifically consider two classes of systems, depending on the model that is used to describe the propagation of waves,

- Models based on the shallow water equations. The corresponding wave-structure models fall into the category of hyperbolic initial boundary problems and we shall address several variants such as transmission problems and free boundary problems of several types.
- Models based on the Boussinesq equations. We shall focus on the mathematical difficulties raised by the presence of a dispersive term in the equations that changes deeply the behavior of the solutions.

2.1 Models based on the shallow water equations: the hyperbolic framework

After recalling in §2.1.1 the standard result for the initial value problem associated with the shallow water equations on the full line $\mathbb{R}$, we present several types of initial boundary value problems that arise in the presence of a boundary. These problems are presented by increasing complexity and we try to point the specific mathematical issues that they raise. After recalling some basing facts on the nonlinear shallow water equations in §2.1.1 we address in §2.1.2 the standard initial boundary value problem for these equations. We then briefly indicate in §2.1.3 how to extend these results to the case of transmission problems. In §2.1.4 we analyze a first example of free boundary problem, namely, the hydraulic piston. Finally, we study in §2.1.5 a floating body problem, which provides an example of another kind of free boundary value problem, more singular than the previous one.

2.1.1 The nonlinear shallow water equations

The nonlinear shallow water equations form a very common model to describe the propagation of waves; they couple the evolution of the elevation of the free surface ζ to the horizontal discharge q (the vertical integral of the horizontal velocity in the fluid domain),

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \big(\frac{1}{h} q^2 + \frac{1}{2} \mathtt{g} h^2\big) = 0, \end{cases} \qquad \big(h(t,x) = h_0 + \zeta(t,x)\big) \tag{1}$$

where h_0 is the (constant) depth at rest and $\mathtt{g}$ the acceleration of gravity. For regular solutions, and provided that the total water height h does not vanish (see [24] for a study devoted to the case where h vanishes), these equations can equivalently be written in terms of ζ and $\overline{v}$, where $\overline{v}$ is the vertical integral of the horizontal velocity in the fluid (so that $q = h\overline{v}$),

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{v}) = 0, \\ \partial_t \overline{v} + \overline{v}\partial_x \overline{v} + \mathtt{g}\partial_x \zeta = 0. \end{cases} \tag{2}$$

The initial value problem for (1) consists in studying the existence and uniqueness of solutions to (1) on $[0,T] \times \mathbb{R}$ for some $T > 0$ and satisfying initial conditions at $t = 0$ and for all $x \in \mathbb{R}$,

$$(\zeta, q)_{|_{t=0}} = (\zeta^{\rm in}, q^{\rm in}) \quad \text{on} \quad \mathbb{R}. \tag{3}$$

If $h > 0$, then (1) falls into the category of Friedrichs symmetrizable systems for which the local well-posedness is a classical result in Sobolev spaces of regularity $s > d/2 + 1$ (and therefore $s > 3/2$ in dimension $d = 1$) [5, 33]. The theorem below is however stated for Sobolev spaces of integer regularity index in order to ease the comparison with the theorems derived later for initial boundary value problems.

Theorem 1 *Let $n \geq 2$ and $(\zeta^{\rm in}, q^{\rm in}) \in H^n(\mathbb{R})^2$ be such that for some $c_0 > 0$, one has $h_0 + \zeta^{\rm in} \geq c_0$ on $\mathbb{R}$. Then there is $T > 0$ and a unique solution $(\zeta, q) \in C([0,T]; H^2(\mathbb{R})^2) \cap C^1([0,T]; H^1(\mathbb{R})^2)$ to* (1) *satisfying the initial condition* (3).

2.1.2 The initial boundary value problem for the nonlinear shallow water equations

In some situations, it is not relevant to solve the nonlinear shallow water equations (1) on the full line $\mathbb{R}$, but on a smaller domain, say, $\mathbb{R}_+$; in addition to the initial data, one also has to satisfy some conditions at the boundary of the domain, which is the point $\{x = 0\}$ if the domain is the half line $\mathbb{R}_+$. For instance, one can have a *wall boundary condition*, where the discharge vanishes at $x = 0$, or a *generating boundary condition*, where the surface elevation has to be equal to some prescribed function of time g (typically provided by measurements from a floating buoy located at $\{x = 0\}$),

$$q(t, x = 0) = 0 \qquad \text{(wall boundary condition)},$$
$$\zeta(t, x = 0) = g(t) \qquad \text{(generating boundary condition)}$$

(other kinds of boundary conditions, in particular nonlinear ones, will be considered later –*transparent* boundary conditions are for instance treated in Example 2). One must therefore solve an *initial boundary value problem* for (1). This consists in studying the existence and uniqueness of solution for (1) on $\Omega_T := [0, T] \times \mathbb{R}_+$ for some $T > 0$ and satisfying

- A boundary condition at $x = 0$ and for all $t \geq 0$. *Linear* boundary conditions such as the wall and generating boundary conditions considered above can be put under the form
$$\big(\nu_1 \zeta + \nu_2 q\big)_{|x=0} = g, \tag{4}$$
for some $\nu \in \mathbb{N}^2$ ($\nu \neq 0$) and some function g.
- Initial conditions at $t = 0$ and for all $x \geq 0$,
$$(\zeta, q)_{|t=0} = (\zeta^{\rm in}, q^{\rm in}) \quad \text{on} \quad \mathbb{R}_+, \tag{5}$$
where $\zeta^{\rm in}$ and $q^{\rm in}$ are some prescribed functions.

Compared to initial boundary value problems on the full line, initial boundary value problems (IBVP) raise new questions, such as

1. Number of boundary conditions. Is a boundary condition really needed ? Is *one* scalar boundary condition like (4) enough ? (we address this issue in §3.1.1)
2. Structure of the boundary condition. Does the choice of $\nu \in \mathbb{N}^2$ ($\nu \neq 0$) in the boundary condition (4) has an incidence on the well-posedness of the IBVP ? (this issue is also addressed in §3.1.1)
3. Compatibility conditions. Quite obviously, one cannot expect a solution to the IBVP to be continuous in time and space if $(\nu_1 \zeta^{\rm in} + \nu_2 q^{\rm in})(x = 0) \neq g(t = 0)$. Therefore, what *compatibility conditions* must the data satisfy to allow the existence of regular solutions ? (this issue is treated in §3.1.2)

We shall establish the local well-posedness of *linear* initial boundary value problems in §3.1, and extend this result to *quasilinear* systems (such as the nonlinear shallow water equations) in §3.2.

2.1.3 Example of transmission problems

A typical example of transmission problem consists in considering the propagation of a wave in shallow water above a topography step, as shown in Figure 1. On both

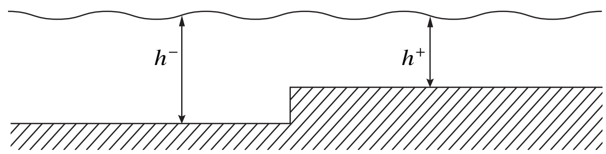

Fig. 1 Shallow water over a step (Figure reproduced from [18])

sides of the step (located at $x = 0$), the propagation of the waves is described by the nonlinear shallow water equations, but the height at rest is not the same: it is given by h_0^- if $x < 0$ and $h_0^+ > 0$ if $x > 0$. We also assume that the surface elevation ζ and the horizontal discharge q are continuous at $x = 0$, so that we need to solve the equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \big(\frac{1}{h^\pm} q^2\big) + \mathtt{g} h^\pm \partial_x \zeta = 0, \end{cases} \qquad \text{for } \pm x > 0 \qquad \text{and with} \qquad h^\pm = h_0^\pm + \zeta, \quad (6)$$

and transmission conditions

$$\zeta(t, 0^+) = \zeta(t, 0^-) \qquad \text{and} \qquad q(t, 0^+) = q(t, 0^-). \qquad (7)$$

Together with initial conditions on (ζ, q), (6)-(7) forms a 2×2 transmission problem. The equations on the left part of the step can be transformed into an equation on $\{x > 0\}$ by setting $(\widetilde{\zeta}, \widetilde{q})(t, x) = (\zeta, q)(t, -x)$ and this transmission problem can therefore be transformed into a 4×2 initial boundary value problem for $(\zeta, q, \tilde{\zeta}, \tilde{q})$ on $\{x > 0\}$. As shown in [18], the analysis presented in §3 for 2×2 hyperbolic systems can be extended to systems of larger size, so that the well-posedness of the above transmission problem is a direct consequence of the results presented in [18], where this problem is specifically addressed, together with other transmission problems such as the stability of shocks.

Another transmission problem consists in studying the interaction of waves (described by the nonlinear shallow water equations) with a floating object allowed to move vertically only and with vertical side-walls; we do not provide details on this example here (see however Remark 1 for the mathematical formulation of this problem in the case of a fixed object) because we treat two more complicated cases in these notes,

- in §2.1.5 we consider the motion of a floating objects with non-vertical side-walls; the added difficulty is that finding the position of the contact points between the surface of the fluid and the solid is a free boundary problem;

- in §2.2.3 we consider vertical side-walls but the wave model under consideration is provided by the Boussinesq equations that include dispersive terms.

This floating body problem, introduced in [22], can be solved easily using the results of [18]. We also mention [26] for a generalization including viscous effects, and [8] for the extension to the $2D$ radial case. Also, an interesting application to a wave energy convertor called *oscillating water column* can be found in [9] and a numerical approach based on a compressible approximation was proposed in [17].

2.1.4 The hydraulic piston

In the previous section, we considered two typical boundary conditions (wall and generating) for the nonlinear shallow water equations in the half-line. More generally, we considered a general linear boundary condition (4). A crucial point was that the boundary of the fluid domain was located at $\{x = 0\}$ and was therefore *fixed*. There are situations where this is no longer the case, and where the equations are cast on a domain which is moving, and whose position is itself determined by the solution in the domain. Such situations are called *free boundary value problems* and are more complex to handle; the *hydraulic piston* falls into this category.

As in the case of the NSW equations with a wall boundary condition considered above, the fluid occupies a semi-infinite canal over a flat bottom which is delimited by a lateral wall. The difference is that the wall is now moving and located at time t at some point $x = \underline{x}(t)$, and that it moves under the action of the hydrodynamic force created by the waves and of a spring force that tends to bring it back to its equilibrium position (see Figure 2).

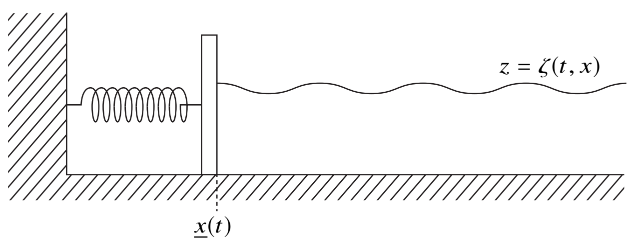

Fig. 2 Waves interacting with a lateral piston (figure reproduced from [18]).

Using the formulation (2) of the NSW equations, we must solve the equations on the *moving* half-line $(\underline{x}(t), \infty)$,

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{v}) = 0 & \text{in} \quad (\underline{x}(t), \infty), \\ \partial_t \overline{v} + \overline{v}\partial_x \overline{v} + \mathtt{g}\partial_x h = 0 & \text{in} \quad (\underline{x}(t), \infty); \end{cases} \tag{8}$$

solving the free boundary problem consists in finding the position of the boundary $\underline{x}$ on a time interval $[0, T]$ for some $T > 0$ and $(\zeta, \overline{v})$ that solve (8) and satisfy

- A boundary condition. Here, it is naturally given by the fact that at the boundary, the velocity of the fluid must match the velocity of the wall,

$$\overline{v}(t, \underline{x}(t)) = \dot{\underline{x}}(t). \tag{9}$$

- An equation for the position of the boundary. Here, this is given by Newton's equation since the wall moves under the action of the hydrodynamic force exerted by the fluid and of the spring force,

$$\mathtt{m}\ddot{\underline{x}} = -\mathtt{k}(\underline{x} - \underline{x}_{\rm eq}) + \frac{1}{2}\rho\mathtt{g}\big((h_0 + \zeta_{|x=\underline{x}})^2 - h_0^2\big), \tag{10}$$

  where $\mathtt{m}$ and $\mathtt{k}$ are respectively the mass of the piston and the stiffness of the spring, while $\underline{x}_{\rm eq}$ is the equilibrium position of the piston (see [18] for more details on the derivation of this equation).
- Initial conditions. They must be specified on ζ and $\overline{v}$, but also on the initial position and velocity of the piston; assuming without loss of generality that the piston is located at $x = 0$ at $t = 0$, the initial data take the form

$$(\zeta, \overline{v})_{|t=0} = (\zeta^{\rm in}, \overline{v}^{\rm in}) \quad \text{on} \quad \mathbb{R}_+, \qquad (\underline{x}, \dot{\underline{x}})_{|t=0} = (0, \underline{x}_1). \tag{11}$$

Compared to standard IBVP, free boundary problems raise specific issues, in particular

1. Fixing the domain. It can be tricky to find a good functional setting for the solutions, since for instance, ζ and $\overline{v}$ take their values on a time dependent domain. The best way to deal with this difficulty consists in going back to the case of a fixed boundary by using a "good" diffeomorphism that maps $\mathbb{R}_+$ to $(\underline{x}(t), \infty)$ for all times.
2. Singular coefficients. The equations on the fixed domain Ω_T deduced from (8) by using a diffeomophism have coefficients that depends on this diffeomorphism, which is itself related to ζ and $\overline{v}$ through the evolution equation (10) for the boundary position $\underline{x}(t)$. This dependence induces a loss of derivative that makes the general theory for IBVP fail. This singularity can however be removed by introducing Alinhac's good unknown (see (42) below).

The problem of the hydraulic piston is a free boundary problem that can essentially be reduced (see [18]) to a general class of hyperbolic free boundary problems where the evolution of the free boundary is of "kinematic" type. This general class is studied in §3.4.

2.1.5 Floating objects in shallow water

A general approach to describe interactions between waves and floating structures was proposed in [22]; according to this approach, the pressure exerted by the fluid on the object is seen as the Lagrange multiplier associated with the constraint that, under the surface of the object, the surface of the fluid has to coincide with the bottom of the object. This method can be implemented with various models for the description of the propagation of the waves; we describe here the equations obtained if we use the nonlinear shallow water equations. Note also that similar mathematical structures arise in other configurations where congestion phenomena are present (see for instance [15, 12, 31]).

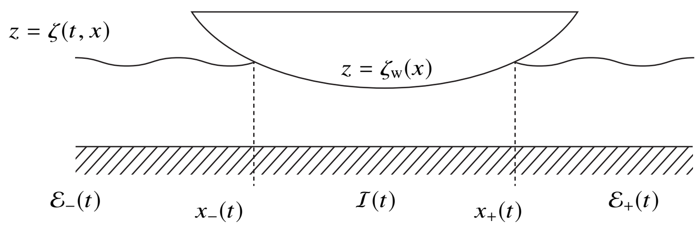

Fig. 3 Waves interacting with a floating body

We consider an object which is partially immersed in the water and assume that the projection of the immersed part of the object in an interval $\mathcal{I}(t) = (x_-(t), x_+(t))$ with $x_-(t) < x_+(t)$. This interval $\mathcal{I}(t)$ is called the *interior region*, while the *exterior region* is given by

$$\mathcal{E}(t) = \mathcal{E}_-(t) \cup \mathcal{E}_+(t), \qquad \mathcal{E}_-(t) = (-\infty, x_-(t)), \qquad \mathcal{E}_-(t) = (x_+(t), \infty).$$

For the sake of simplicity, and because this does not remove the mathematical structure we want to comment on here, we assume that the immersed solid is fixed and that its bottom is parametrized by a function $\zeta_w(x)$ (the general case of a floating object moving under the action of the waves is considered and solved in [18]). As shown in [22, 18], this problem can be reduced to the standard nonlinear shallow water equations on the two half-lines $\mathcal{E}_\pm$,

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 & \text{in} \quad \mathcal{E}(t), \\ \partial_t q + \partial_x \big(\frac{q^2}{h} + \frac{1}{2} g h^2\big) = 0 & \text{in} \quad \mathcal{E}(t). \end{cases} \tag{12}$$

Once again, the problem under consideration is a free boundary problem because the evolution of the boundaries $x_\pm(t)$ of $\mathcal{E}_\pm$ is part of the problem; we therefore need to find $x_\pm(t)$ on a time interval $[0, T]$ for some $T > 0$ and (ζ, q) that solve (12) and satisfy

- A boundary condition at $x_-(t)$ and $x_+(t)$. This condition is provided by the fact that the discharge is continuous across the contact points, and since it is constant under the object if this one is fixed, this yields

$$q(t, x_\pm(t)) = \langle q_{\rm i}\rangle(t), \tag{13}$$

where $\langle q_{\rm i}\rangle$ is the value of the discharge under the object; it is found by solving an ODE of the form

$$\frac{d}{dt}\langle q_{\rm i}\rangle = F(\langle q_{\rm i}\rangle, x_-(t), x_+(t)), \tag{14}$$

where F is a smooth function whose exact expression is of no importance here (see [18] for details).
- An equation for the position of the boundary. It is here given implicitly by using the fact that at the contact points, the surface elevation matches the parametrization $\zeta_{\rm w}$ of the bottom of the object,

$$\zeta(t, x_\pm(t)) = \zeta_{\rm w}(x_\pm(t)). \tag{15}$$

- Initial conditions. We must give initial data for ζ, q, $\langle q_{\rm i}\rangle$ and $x_\pm$,

$$(\zeta, q)_{|_{t=0}} = (\zeta^{\rm in}, q^{\rm in}) \quad \text{on} \quad \mathcal{E}(0), \qquad \langle q_{\rm i}\rangle_{|_{t=0}} = \langle q_{\rm i}\rangle^{\rm in}, \qquad x_\pm(0) = x_\pm^{\rm in}. \tag{16}$$

What we stated as an evolution equation for $x_\pm$, namely, equation (15), can also be seen as a boundary condition on ζ. Therefore, one must solves two IBVP (on $\mathcal{E}_-$ and on $\mathcal{E}_+$), and on each of them, we impose *two* boundary conditions (one on q and one on ζ). This is, as we shall see, overdetermined for the IBVP under consideration, and it is this overdetermination that yields the necessary equations for the evolution of the boundaries. We shall see that this kind of free boundary problem is more singular than the previous one, and requires the understanding of IBVP with fully nonlinear boundary conditions and the introduction of a "second order" Alinhac unknown (see §3.5).

Remark 1 If the solid were fixed and with vertical sidewalls located at $x = \pm R$, then the problem would be a simple transmission problem on $\mathcal{E} = \mathcal{E}_+ \cup \mathcal{E}_-$ with $\mathcal{E}_- = (-\infty, -R)$ and $\mathcal{E}_+ = (R, \infty)$, namely

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x\big(\frac{q^2}{h} + \mathtt{g}\frac{1}{2}h^2\big) = 0 \end{cases} \qquad \text{on} \quad \mathcal{E}$$

with transmission conditions

$$[\![q]\!] = 0 \quad \text{and} \quad \langle q\rangle = q_{\rm i},$$

where we recall that $[\![q]\!] = q(R) - q(-R)$ and $2\langle q\rangle = q(R) + q(-R)$, and where $\langle q_{\rm i}\rangle$ solves

$$-\alpha\frac{d}{dt}\langle q_{\rm i}\rangle = [\![\mathtt{g}\zeta + \frac{1}{2}\frac{q^2}{h^2}]\!], \qquad (\alpha = \int_{-R}^{R}\frac{1}{h_{\rm w}}),$$

where h_w denotes the height of the water under the object. This transmission problem can be treated along the lines described in §2.1.3. It can be also be extended to non-fixed objects, and, in the so called return to equilibrium problem (where the object is dropped from an out of equilibrium position in a fluid at rest), the motion of the object can be reduced to a simple nonlinear ODE on the position of the center of mass [22]. Generalizations to the $2D$ radial case can be found in [8], to the $1D$ viscous case in [26] and to the Boussinesq equations in [4].

2.2 Models based on the Boussinesq equations: a dispersive perturbation of the hyperbolic framework

The nonlinear shallow water equations form a robust approximation of the water waves (or free surface Euler) equations in shallow water. They miss however some important physical effects related to dispersion; for instance, they are unable to account for the propagation of solitary waves. In the same physical regime, the Serre-Green-Naghdi equations are known to provide a better approximation; they include in particular the dispersive terms neglected in the shallow water equations. They are quite complicated though, and this is the reason why the so-called Boussinesq equations are quite popular. For this latter model, it is assumed that the waves are of small amplitude and many terms of the Serre-Green-Naghdi equations are neglected (see for instance [21, 23]). Boussinesq models still contain dispersive terms though, and they are able to explain the possibility of solitary waves. From a mathematical viewpoint, they are a dispersive perturbation of an hyperbolic system. We will see in §4 that this dispersive perturbation drastically changes the mathematical properties of the related initial boundary value problems.

In §2.2.1 we recall some basic facts on the Boussinesq equations and then turn to present two initial-boundary-value problems involving these equations. The first one, in §2.2.2, consists in studying a standard initial boundary value problem for the Boussinesq equation with boundary condition on ζ or q (in the case of the nonlinear shallow water equations, this problem has been considered in §2.1.2). The second one, presented in §2.2.3 deals with the interactions of waves with a partially immersed structure.

2.2.1 The Boussinesq equations

We recall here some classical facts about the derivation of the Boussinesq equations and about their precision as approximations to the full water waves equations; we refer to [21, 23] for more precisions on these points. In order to introduce the Boussinesq equations, it is convenient to work with dimensionless quantities. Denoting by a the typical amplitude of the waves, by L the typical horizontal length, by h_0 the depth at rest, we write

$$\tilde{x} = \frac{x}{\lambda}, \qquad \tilde{t} = \frac{t}{L\sqrt{gh_0}}, \qquad \tilde{\zeta} = \frac{\zeta}{a}, \qquad \tilde{q} = \frac{q}{a\sqrt{gh_0}},$$

as well as the dimensionless parameters

$$\varepsilon = \frac{a}{h_0}, \qquad \mu = \frac{h_0^2}{L^2} \quad \text{and} \quad \kappa = \sqrt{\frac{\mu}{3}}$$

the first one is called the nonlinearity parameter and the second one the shallowness parameter; the last one is just introduce for notational convenience since the quantity $\sqrt{\mu/3}$ will play an important role in the next sections. In dimensionless variables, the nonlinear shallow water equations (1) become (omitting the tildes for the sake of clarity)

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x \big(\varepsilon \frac{1}{h} q^2 + \frac{1}{2\varepsilon} h^2 \big) = 0, \end{cases} \qquad \big(h(t,x) = 1 + \varepsilon \zeta(t,x) \big). \tag{17}$$

They are known to approximate the water waves equation with a precision $O(\mu)$, without any smallness assumption on ε. The Boussinesq equations offer a better approximation of precision $O(\mu^2)$, but under the weak nonlinearity assumption $\varepsilon = O(\mu)$. There are actually many formally equivalent Boussinesq systems, one of which being the so-called Boussinesq-Abbott system

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t q + \partial_x \big(\varepsilon \frac{1}{h} q^2 + \frac{1}{2\varepsilon} h^2 \big) = 0; \end{cases} \tag{18}$$

this system is obviously a perturbation of order $O(\mu)$ of the nonlinear shallow water equations (17). This perturbation is provided by the term $-\frac{\mu}{3}\partial_x^2 \partial_t q$ which is of dispersive nature. Dealing with Boussinesq systems therefore leads us to investigate *dispersive perturbations of hyperbolic systems*.

Remark 2 As said above, there are actually many formally equivalent Boussinesq systems that differ one from each other by terms of order $O(\mu^2)$, which is the precision of the approximation. For instance, since $h = 1 + \varepsilon\zeta$ and under the assumption that $\varepsilon = O(\mu)$, the nonlinear term $\varepsilon \frac{q^2}{h}$ in (18) can be written $\varepsilon \frac{q^2}{h} = \varepsilon q^2 + O(\mu^2)$, and a somewhat simpler Boussinesq system is given by

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t q + \partial_x \big(\varepsilon q^2 + \frac{1}{2\varepsilon} h^2 \big) = 0. \end{cases} \tag{19}$$

The generic result for Boussinesq systems cast on the full line $\mathbb{R}$ is that, if they are linearly well posed, then they are (nonlinearly) well-posed over a time scale $O(1/\varepsilon)$ [14, 32]. These results hold for instance for (19) (the case of the Boussinesq-Abbott system (18) is not covered by [14, 32] because the nonlinearity is not quadratic, but the proof can be adapted easily, as sketched in [4] for instance). When the domain of consideration has a boundary, the situation is more complicated as there is no general result for IBVPs for dispersive perturbations of hyperbolic systems. As we shall see, the presence of the dispersive perturbation drastically changes the theory.

2.2.2 The initial boundary value problem for the Boussinesq-Abbott equations

For the same reasons as those explained in §2.1.2 for the nonlinear shallow water equations, one is naturally led to consider initial boundary value problems for Boussinesq systems when they are cast on a domain with boundary, for instance $\mathbb{R}_+$. Note that the comments we shall make here are valid for the Boussinesq systems (18) (and also for the simplified system (19)) but not necessarily for other variants.

The goal of this section being to point out the main differences with respect to the hyperbolic case, we shall consider only boundary conditions on q or on ζ (this latter case is often referred to as a *generating* boundary condition and has been studied mathematically and numerically implemented in [25]), namely,

$$q(t, x = 0) = g(t) \quad \text{or} \quad \zeta(t, x = 0) = g(t).$$

In the hyperbolic case, the initial-boundary value problem for the nonlinear shallow water equations with such boundary conditions are locally well posed, as shown in §3.2, provided that some compatibility conditions are satisfied. For a generalization of this result to the Boussinesq-Abbott equations (18), we need to address several issues, such as,

1. Number of boundary conditions. Does the presence of the dispersive perturbation require additional boundary conditions ?
2. Compatibility conditions. Are the compatibility conditions that were derived in the hyperbolic case affected by the presence of the dispersion ?

The analysis performed in §4.1 will reveal important differences between the initial boundary value problem for the Boussinesq-Abbott equations (18) and the corresponding one for the nonlinear shallow water equations (17). These latter equations are formally deduced from the Boussinesq-Abbott equations (18) by setting $\mu = 0$. These different behaviors raise several questions on the dispersionless limit $\mu \to 0$ and suggest the presence of a *dispersive boundary layer*.

2.2.3 A wave-structure interaction problem for a Boussinesq system of equations

In this section, we consider a model describing the interaction between waves at the surface of a fluid with a partially immersed fixed structure with vertical side walls; contrary to the situation considered in §2.1.5, the horizontal coordinates of the contact points between the free surface and the object are time independent, say, $x = \pm R$ (see Figure 4). Since the object is fixed, its bottom is parametrized (in dimensionless variables) by a time independent function $\varepsilon\zeta_{\rm w}(x)$. We consider here a weakly nonlinear dispersive model, namely the Boussinesq system (19) (which is

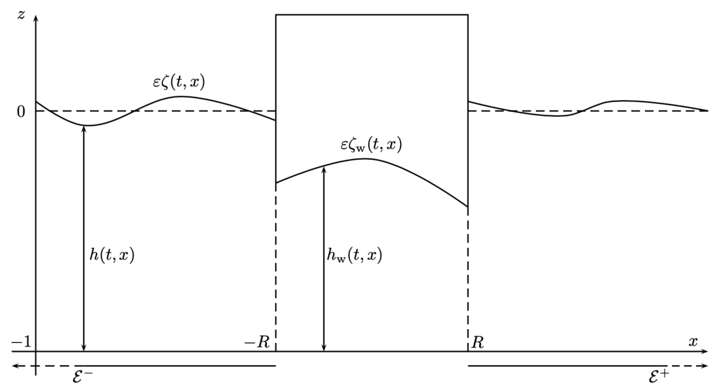

Fig. 4 Waves interacting with a partially immersed body

deduced from the Boussinesq-Abbott system by keeping only linear and quadratic terms; handling the Boussinesq-Abbott system is more delicate, see [4]). The presence of the floating object is taken into account by a pressure term in the so called interior region $\mathcal{I} = (-R, R)$. The equations can therefore be written as

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{1}{3}\mu\partial_x^2)\partial_t q + \varepsilon\partial_x(q^2) + h\partial_x \zeta = -h\partial_x \underline{P}_{\rm i}, \end{cases}$$

where $\underline{P}_{\rm i} = 0$ in the exterior region $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+ = (-\infty, -R) \cup (R, \infty)$, so that the equations exactly coincide in this region with the standard equations (19). In the interior region, the pressure term $\underline{P}_{\rm i}$ is non zero and must be viewed as a Lagrange multiplier associated with the fact that the surface of the fluid must satisfy the constraint $\zeta = \zeta_{\rm w}$. The equations are complemented with a condition of continuity of the horizontal discharge at the contact points $x = \pm R$ and another one ensuring the total conservation of the energy. The problem can then be reduced to the following transmission problem (we refer to [13] for the details of the derivation).

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{1}{3}\mu\partial_x^2)\partial_t q + \varepsilon\partial_x(q^2) + h\partial_x \zeta = 0 \end{cases} \quad \text{on} \quad (-\infty, -R) \cup (R, +\infty); \tag{20}$$

with transmission conditions

$$[\![q]\!] = 0, \qquad \langle q \rangle = q_{\rm i}, \tag{21}$$

where

$$[\![q]\!] := q(R) - q(-R), \qquad \langle q \rangle := \frac{1}{2}(q(R) + q(-R)),$$

and $q_{\rm i}$ is a function of time only provided by

$$-\alpha \dot{q}_{\rm i} = -\frac{\mu}{3}\partial_t [\![\partial_x q]\!] + [\![\zeta + \varepsilon\frac{1}{2}\zeta^2]\!], \qquad (\alpha = \int_{-R}^{R} \frac{1}{h_{\rm w}}). \tag{22}$$

The system is complemented by the initial condition

$$(\zeta, q)_{|_{t=0}} = (\zeta^{\rm in}, q^{\rm in}). \tag{23}$$

Remark 3 By setting $\mu = 0$, the problem is reduced to an hyperbolic transmission problem. The reason why it does not coincide with the system written in Remark 1 is because the Boussinesq system (19) does not degenerate into the shallow water equations if we set $\mu = 0$. If we had worked with the Boussinesq-Abbott system (18) instead, we would have found the transmission problem of Remark 1 by setting $\mu = 0$ (see [4], where the Boussinesq-Abbott system is used to investigate the more general case of an object allowed to move vertically).

This system of equations is studied in [13] where, in addition to the local well-posedness of the equations, the dispersionless limit is considered and the singularities caused by dispersive boundary layer studied with great detail. A sketch of this analysis is presented in §4.2. We also mention [4] for the adaptation of this approach to the Boussinesq-Abbott system (69) and its generalization to freely floating (rather than fixed) objects and to [19, 11] for a numerical approach based on a direct numerical coupling.

3 Mathematical analysis of hyperbolic free boundary problems

We present here a general theory that allows to handle the various initial boundary value problems presented in §2.1. For the sake of simplicity, we consider only here 2×2 systems of equations (which is enough to handle the nonlinear shallow water equations); it is possible to extend the analysis to systems of larger size. For this and other generalizations, as well as for the full details of the proofs, we refer to [18] from which most of the material presented in this section is extracted.

We first address in §3.1 the case of linear initial boundary value problems and discuss the number and structure of the boundary conditions that yield a well-posed problem (§3.1.1); we also discuss the important notion of compatibility conditions (§3.1.2). The main result is stated in §3.1.3 and the proof sketched in §3.1.4; the key ingredient is the construction of a so-called Kreiss symmetrizer under the Kreiss-Lopatinskĭ condition.

It is then shown in §3.2 how to extend these results to cover the case of quasilinear systems such as the nonlinear shallow water equations.

We then address the case of initial boundary value problems cast on moving domains. We first consider in §3.3 the case where the boundary has a prescribed motion; the key ingredient is that one must use a diffeomorphism to bring the problem back to an IBVP cast on a fixed domain. The dependence of this new IBVP on the diffeomorphism raises regularity issues that can be lifted by introducing Alinhac's good unknown.
We then address the case of *free boundary* problems, for which the motion of the boundary is not prescribed but found by solving an equation coupled to the solution of the IBVP. We first consider in §3.4 a first class of boundary value problems of "kinematic" type, for which the equation for the evolution of the boundary involves a nonlinear function of the traces at the boundary of the solutions to the IBVP (such a situation occurs for instance for the hydraulic piston problem presented in §2.1.4). Finally, in §3.5 we consider more singular free boundary value problems for which the boundary equation involve also traces of the derivatives of the solution to the IBVP (this is typically the situation one has to face in the floating body problem of §2.1.5).

3.1 Linear initial boundary value problems

We first consider linear initial boundary value problems, more precisely, we consider systems of the form

$$\begin{cases} \partial_t u + A(t,x)\partial_x u + B(t,x)u = f(t,x) & \text{in} \quad \Omega_T, \\ u_{|_{t=0}} = u^{\text{in}}(x) & \text{on} \quad \mathbb{R}_+, \\ \nu(t)\cdot u_{|_{x=0}} = g(t) & \text{on} \quad (0,T), \end{cases} \tag{24}$$

where u, u^{in}, f, and ν are $\mathbb{R}^2$-valued functions and g is a real-valued function, while A and B take their values in the space of 2×2 real-valued matrices. As seen in §2.1.2, the analysis of initial boundary value problems raises new issues compared to initial value (or Cauchy) problems: the number of boundary conditions needed (there is *one* boundary condition in (24), but why is it the correct number?), the structure of these conditions, and the necessity of compatibility conditions the data must satisfy. We briefly comment on these issue before stating and proving the main theorem for (24).

3.1.1 The correct number and the structure of the boundary conditions

We want to investigate here the issue of the number and nature of the boundary conditions that can be imposed to an hyperbolic IBVP. For the sake of simplicity, we take $B = 0$ and $f = 0$ here, and assume that A is a 2×2 matrix with constant coefficients, and possessing two eigenvalues λ_+ and $-\lambda_-$, with $\lambda_\pm > 0$, with corresponding unit eigenvectors $\mathbf{e}_\pm$. We therefore consider an equation

$$\partial_t u + A\partial_x u = 0 \quad \text{in} \quad [0,T]\times\mathbb{R}_+ \tag{25}$$

with a general boundary condition of the form

$$Nu_{|x=0} = \mathbf{g}(t), \tag{26}$$

where N is a $p\times 2$ matrix of maximal rank (the number of boundary conditions is therefore p) and $\mathbf{g}$ takes its values in $\mathbb{R}^p$; we also impose the initial condition

$$u_{|t=0} = 0. \tag{27}$$

It is possible to solve explicitly these equations by using the Laplace transform

$$\mathcal{L}f(s) = \widehat{f}(s) := \int_0^\infty e^{-st} f(t)\mathrm{d}t \quad \text{with } \Re s > 0.$$

Applying the Laplace transform to the above equations yields

$$\begin{aligned} s\widehat{u} + A\frac{d}{dx}\widehat{u} = 0 \quad &\text{in} \quad \mathbb{R}_+,\\ N\widehat{u} = \widehat{g} \quad &\text{at } x = 0. \end{aligned}$$

The general solution of this ODE is given by

$$\widehat{u}(s,x) = \widehat{c}_+(s)\exp\big(-\frac{s}{\lambda_+}x\big)\mathbf{e}_+ + \widehat{c}_-(s)\exp\big(\frac{s}{\lambda_-}x\big)\mathbf{e}_-,$$

where the scalar functions $\widehat{c}_\pm$ are integration constants. Imposing that $\widehat{u}$ vanishes at infinity implies that $\widehat{c}_- \equiv 0$. The boundary condition reads therefore

$$(N\mathbf{e}_+)\widehat{c}_+ = \widehat{\mathbf{g}}.$$

Clearly, it is possible to solve this equation for all $\mathbf{g}$ if an only if $p = 1$, and therefore $N = \nu^{\mathrm{T}}$ with $\nu \in \mathbb{R}^2$, and if $\nu\cdot\mathbf{e}_+ \neq 0$, in which case one can take

$$\widehat{c}_+ = \frac{1}{\nu\cdot\mathbf{e}_+}\widehat{g},$$

where we wrote simply $g = \mathbf{g}$ because it is a scalar valued function. The conclusion is that in order for a boundary value problem of the form (25)-(26), with initial condition (27) to be well posed, one needs *one* boundary condition of the form

$$\nu\cdot u_{|x=0} = g$$

and that ν must in addition satisfy the condition

$$\nu\cdot\mathbf{e}_+ \neq 0; \tag{28}$$

this condition is called the *Kreiss-Lopatinskiĭ condition* and is not restricted to 2×2 problems nor to one-dimensional problems. We refer for instance to [18] for a generalization of this analysis to initial boundary value problem of larger size and to [27, 28, 29, 5] for problems in dimension $d > 1$; note also that these considerations are also relevant for variables coefficients as well as nonlinear IBVP as we shall see below.

3.1.2 Compatibility conditions

In most of the statements of the theorems presented below, we refer to *compatibility conditions* without making them explicit because the formulas can be quite heavy. We refer to [18] for precisions, and prefer here to insist instead on the general idea that is behind these conditions and that we present here in the simplest framework, when A has constant coefficients, $B = 0$ and $\nu(t) = \underline{\nu}$ is time independent. The general case involves commutator terms that make the expressions more complicated, but does not raise any additional difficulty. Denoting $u_k = \partial_t^k u$, we have

$$u_1 = -A\partial_x u + f$$

and more generally, if we time differentiate the equation k-times we obtain the induction relation

$$u_{k+1} = -\partial_x u_k + \partial_t^k f.$$

For a smooth solution u, $u_k^{\rm in} = u_k{}_{|_{t=0}}$ is therefore given inductively by $u_0^{\rm in} = u^{\rm in}$ and

$$u_{k+1}^{\rm in} = -\partial_x u_k^{\rm in} + (\partial_t^k f)_{|_{t=0}}. \tag{29}$$

On the other hand, the boundary condition $\underline{\nu} \cdot u_{|_{x=0}} = g$ also implies that

$$\underline{\nu} \cdot u_k{}_{|_{x=0}} = \partial_t^k g.$$

On the edge $\{t = 0, x = 0\}$, smooth enough solutions must therefore satisfy

$$\underline{\nu} \cdot u_k^{\rm in}{}_{|_{x=0}} = \partial_t^k g_{|_{t=0}}. \tag{30}$$

If we consider solutions of finite smoothness, then the relations (30) will be necessary as long as they make sense.

Definition 1 Let $m \geq 1$ be an integer. We say that the data $u^{\rm in} \in H^m(\mathbb{R}_+)$, $f \in H^m(\Omega_T)$, and $g \in H^m(0,T)$ for the initial boundary value problem (24) satisfy the compatibility condition at order $m-1$ if the $\{u_j^{\rm in}\}_{j=0}^m$ defined in (29) satisfy (30) for $k = 0, 1, \dots, m-1$.

3.1.3 Statement of the main result

Based on the analysis performed in §3.1.1, it is natural to make the following assumption.

Assumption 1 There exists $c_0 > 0$ such that the following assertions hold.

i. $A \in W^{1,\infty}(\Omega_T),\ B \in L^\infty(\Omega_T),\ \nu \in C([0,T])$.
ii. For any $(t,x) \in \Omega_T$, the matrix $A(t,x)$ has eigenvalues $\lambda_+(t,x)$ and $-\lambda_-(t,x)$ satisfying

$$\lambda_\pm(t,x) \geq c_0.$$

iii. (The uniform Kreiss–Lopatinskiĭ condition.) Denoting by $\mathbf{e}_+(t,x)$ a unit eigenvector associated with the eigenvalue $\lambda_+(t,x)$ of $A(t,x)$, for any $t \in [0,T]$ we have

$$|\nu(t,0)\cdot \mathbf{e}_+(t,0)| \geq c_0.$$

Example 1 A typical example of application is to consider the linearized shallow water equations with a boundary condition on the horizontal water flux q. This system has the form

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + 2\frac{\underline{q}}{\underline{h}}\partial_x q + \Big(\mathtt{g}\underline{h} - \frac{\underline{q}^2}{\underline{h}^2}\Big)\partial_x \zeta = 0 \end{cases}$$

with initial and boundary conditions

$$(\zeta,q)_{|_{t=0}} = (\zeta^{\rm in}, q^{\rm in}) \quad \text{ and } \quad q_{|_{x=0}} = g,$$

where $\mathtt{g}$ is the gravitational constant. This problem is of the form (24) with $u = (\zeta,q)^{\rm T}$, $B = 0$, $f = 0$, $\nu = (0,1)^{\rm T}$, and

$$A(t,x) = A(\underline{u}) = \begin{pmatrix} 0 & 1 \\ \mathtt{g}\underline{h} - \frac{\underline{q}^2}{\underline{h}^2} & 2\frac{\underline{q}}{\underline{h}} \end{pmatrix}. \tag{31}$$

The eigenvalues $\pm\lambda_\pm$ and the corresponding unit eigenvectors $\mathbf{e}_\pm$ of A are given by $\lambda_\pm = \sqrt{\mathtt{g}\underline{h}} \pm \frac{\underline{q}}{\underline{h}}$ and $\mathbf{e}_\pm = \frac{1}{\sqrt{1+\lambda_\pm^2}}(1,\pm\lambda_\pm)^{\rm T}$, so that Assumption 1 is satisfied provided that $\underline{h},\underline{q} \in W^{1,\infty}(\Omega_T)$, and

$$\underline{h}(t,x) \geq c_0, \qquad \sqrt{\mathtt{g}\underline{h}(t,x)} \pm \frac{\underline{q}(t,x)}{\underline{h}(t,x)} \geq c_0$$

with some positive constant c_0 independent of $(t,x) \in \Omega_T$.

Before stating the main result of this section we need to define the spaces $\mathbb{W}^m(T)$ that are used to measure the regularity of functions in $\Omega_T = [0,T]\times\mathbb{R}_+$. We have in particular $H^{m+1}(\Omega_T) \subset \mathbb{W}^m(T) \subset H^m(\Omega_T)$.

Definition 2 i. We introduce the space $\mathbb{W}^m(T)$ as

$$\mathbb{W}^m(T) = \bigcap_{j=0}^{m} C^j([0,T]; H^{m-j}(\mathbb{R}_+)),$$

with associated norm

$$\|u\|_{\mathbb{W}^m(T)} = \sup_{t\in[0,T]} |||u(t)|||_m \quad \text{with} \quad |||u(t)|||_m = \sum_{j=0}^{m} \|\partial_t^j u(t)\|_{H^{m-j}(\mathbb{R}_+)}.$$

ii. In order to control the boundary regularity of the solution, we use the norm

$$|u_{|x=0}|_{m,t} = \Big(\sum_{j=0}^{m} |(\partial_x^j u)_{|x=0}|^2_{H^{m-j}(0,t)}\Big)^{\frac{1}{2}} = \Big(\sum_{|\alpha|\le m} |(\partial^\alpha u)_{|x=0}|^2_{L^2(0,t)}\Big)^{\frac{1}{2}}.$$

The main result of this section shows the well-posedness of the IBVP (24) under appropriate boundary conditions and provides a control of the time and space derivatives (up to order m) of the solution, but also a control on the trace of the solution and of its derivatives at the boundary $x = 0$. This control of the trace is crucial for many applications because it avoids the loss of half a derivative one would obtain by deriving it from classical trace estimates and the bound of the solution in $\mathbb{W}^m(T)$. The general idea, due to Kreiss [20] in a general setting, is that under the uniform Kreiss-Lopatinskiĭ condition made in Assumption 1, it is possible to construct a symmetrizer that transforms the IBVP (24) into another one with a boundary condition enjoying appropriate dissipativity properties.

Theorem 2 *Let $m \geq 1$ be an integer, $T > 0$, and assume that Assumption 1 is satisfied for some $c_0 > 0$. Assume moreover that there are constants $0 < K_0 \leq K$ such that*

$$\begin{cases} \frac{1}{c_0}, \|A\|_{L^\infty(\Omega_T)}, |\nu|_{L^\infty(0,T)} \leq K_0, \\ \|A\|_{W^{1,\infty}(\Omega_T)}, \|B\|_{L^\infty(\Omega_T)}, \|(\partial A, \partial B)\|_{\mathbb{W}^{m-1}(T)}, |\nu|_{W^{m,\infty}(0,T)} \leq K. \end{cases}$$

Then, for any data $u^{\rm in} \in H^m(\mathbb{R}_+)$, $g \in H^m(0,T)$, and $f \in H^m(\Omega_T)$ satisfying the compatibility conditions up to order $m-1$ in the sense of Definition 1, there exists a unique solution $u \in \mathbb{W}^m(T)$ to the initial boundary value problem (24). *Moreover, the following estimate holds*

$$\begin{aligned} &|||u(t)|||_m + |u_{|x=0}|_{m,t} \\ &\leq C(K_0)e^{C(K)t}\Big(|||u(0)|||_m + |g|_{H^m(0,t)} + |f_{|x=0}|_{m-1,t} + \int_0^t |||\partial_t f(t')|||_{m-1}\mathrm{d}t'\Big). \end{aligned}$$

Remark 4 Note that the inductive procedure presented in §3.1.2 to derive the compatibility conditions can be used to control $|||u(0)|||_m$ in terms of $\|u^{\rm in}\|_{H^m}$. A refinement of the energy estimate given in Theorem 2 can also be found in [18].

3.1.4 Sketch of the proof of Theorem 2

We sketch here the proof of [18] which fully exploits the specificities of the one-dimensional cases (the multi-dimensional case is treated in the classical references [20, 28, 27, 16, 29, 5]); in order to insist on the main idea, several simplifying assumptions are made, and we refer the readers to [18] for the full proof (and also for some improvements of the energy estimate provided in the theorem).
The first step is to derive an a priori L^2-energy estimate; in this first step, it is assumed that there exists a Kreiss symmetrizer, that is, a symmetrizer that provides some dissipativity on the boundary condition. In the second step, this L^2-estimate is used to derive L^2-estimates on the time derivatives (time derivatives are convenient since, contrary to space derivatives, they can be used to differentiate the boundary condition $\nu \cdot u_{|x=0}(t) = g(t)$). In step 3, an ellipticity property of the system is used to control space derivatives in terms of time derivatives. The fourth step is to actually prove the existence of the solution, using the compatibility conditions. Finally, it is shown in Step 5 that the existence of a Kreiss symmetrizer, assumed in Step 1, is a consequence of the uniform Kreiss-Lopatinski condition.

Step 1. A priori L^2-estimate. For initial value problems, an L^2-estimate is obtained easily by using a Friedrichs symmetrizer, i.e. a symmetric matrix S satisfying the two properties

$$
\begin{aligned}
&(i) \qquad \exists \alpha_0, \beta_0 > 0, \qquad \forall (v,t,x) \in \mathbb{R}^2 \times \Omega_T, \qquad \alpha_0 |v|^2 \le v^{\mathrm{T}} S(t,x) v \le \beta_0 |v|^2 \\
&(ii) \qquad \exists \beta_2 > 0, \qquad \|\partial_t S + \partial_x (SA) - 2SB\|_{L^2(\Omega_T) \to L^2(\Omega_T)} \le \beta_2.
\end{aligned}
$$

(such symmetrizers are easy to construct, see for instance chapter 16 in [33]). Multiplying the first equation of (24) by S and taking the $L^2(\Omega_t)$ scalar product with $e^{-2\gamma t} u$, we get after integration by parts,

$$
\begin{aligned}
& e^{-2\gamma t} (Su(t), u(t))_{L^2} + 2\gamma \int_0^t e^{-2\gamma t'} (Su, u)_{L^2} \mathrm{d}t' - \int_0^t e^{-2\gamma t'} (SAu \cdot u)_{|x=0} \mathrm{d}t' \\
&= (S_{|t=0} u^{\mathrm{in}}, u^{\mathrm{in}})_{L^2} + \int_0^t e^{-2\gamma t'} ((\partial_t S + \partial_x (SA) - 2SB) u + 2Sf, u)_{L^2} \mathrm{d}t'.
\end{aligned}
$$

Using assumptions (i) and (ii), this yields

$$
\begin{aligned}
& \alpha_0 e^{-2\gamma t} \|u(t)\|_{L^2}^2 + (2\alpha_0 \gamma - \beta_2) \int_0^t e^{-2\gamma' t} \|u(t')\|_{L^2}^2 \mathrm{d}t' - \int_0^t e^{-2\gamma t'} (SAu \cdot u)_{|x=0} \mathrm{d}t' \\
& \le \beta_0^{\mathrm{in}} \|u^{\mathrm{in}}\|_{L^2}^2 + 2\beta_0 \int_0^t e^{-2\gamma t'} \|f(t')\|_{L^2} \|u(t')\|_{L^2} \mathrm{d}t'.
\end{aligned}
$$

For initial boundary value problems, the term $-\int_0^t e^{-2\gamma t'} (SAu \cdot u)_{|x=0} \mathrm{d}t'$ in the left-hand-side is not present, and provided that γ is chosen large enough, one can easily derive an estimate similar to the one of Theorem 2 by a Gronwall type argument. The specificity of initial boundary value problems is therefore the analysis of this boundary term: the trace $(SAu \cdot u)$ at $x = 0$ *cannot* be controlled by the L^2-norm of

u. Obviously, the sign of $(SAu \cdot u)_{|x=0}$ is also important: it is only when it is positive that an extra argument is needed to control it. This argument is that the symmetrizer S has one extra property, namely that there are $\alpha_1 > 0$ and $\beta_1 > 0$ such that

$$(iii) \qquad \forall (v,t) \in \mathbb{R}^2 \times (0,T) \qquad v^{\mathrm{T}} S(t,0)A(t,0)v \leq -\alpha_1 |v|^2 + \beta_1 |\nu(t)\cdot v|^2;$$

such symmetrizers are called Kreiss symmetrizers [20]. If S is such a symmetrizer, then the term $-(SAu \cdot u)_{|x=0}$ has the good sign and controls $|u(t)_{|x=0}|$ up to terms that depends only on $\nu(t)\cdot u(t)_{|x=0} = g(t)$. One therefore has

$$\begin{aligned}
&\alpha_0 e^{-2\gamma t}\|u(t)\|_{L^2}^2 + (2\alpha_0\gamma - \beta_2)\int_0^t e^{-2\gamma' t}\|u(t')\|_{L^2}^2 \mathrm{d}t' + \alpha_1 e^{-2\gamma t}|u_{|x=0}|_{L^2(0,t)}^2 \\
&\leq \beta_0^{\mathrm{in}}\|u^{\mathrm{in}}\|_{L^2}^2 + \beta_1 e^{-2\gamma t}|g|_{L^2(0,t)}^2 + 2\beta_0 \int_0^t e^{-2\gamma t'}\|f(t')\|_{L^2}\|u(t')\|_{L^2}\mathrm{d}t'.
\end{aligned}$$

Using a Gronwall type argument, one then gets

$$\begin{aligned}
&\|u(t)\|_{L^2} + |u_{|x=0}|_{L^2(0,t)} \\
&\leq C(K_0)e^{C(K)t}\Big(\|u(0)\|_{L^2} + |g|_{L^2(0,t)} + \int_0^t \|f(t')\|_{L^2}\mathrm{d}t'\Big).
\end{aligned} \tag{32}$$

Step 2. Control of time derivatives. For the sake of clarity, we only treat the case where A has constant coefficients, $B = 0$ and $\nu(t) = \underline{\nu}$ does not depend on time. The general case requires the control of commutator terms, and is technically more involved, but the main steps of the proof are the same (see [18]).
Let $u_m = \partial_t^m u$. Then, u_m solves

$$\begin{cases}
\partial_t u_m + A\partial_x u_m = f_m & \text{in} \quad \Omega_T, \\
u_{m\,|t=0} = (\partial_t^m u)_{|t=0} & \text{on} \quad \mathbb{R}_+, \\
\underline{\nu}\cdot u_{m\,|x=0} = g_m(t) & \text{on} \quad (0,T),
\end{cases}$$

where

$$f_m = \partial_t^m f, \qquad g_m = \partial_t^m g.$$

Applying the inequality (32) derived in the previous step, we obtain

$$\begin{aligned}
&\|u_m(t)\|_{L^2} + |u_{m\,|x=0}|_{L^2(0,t)} \\
&\leq C(K_0)e^{C(K)t}\Big(|||u(0)|||_m + |g|_{H^m(0,t)} + \int_0^t \|f(t')\|_{H^m}\mathrm{d}t'\Big).
\end{aligned}$$

Step 3. Control of the space derivatives. Under the same simplifying assumptions as in Step 2, let k and l be nonnegative integers satisfying $k + l \leq m - 1$. Applying $\partial_t^k \partial_x^l$ to the equation, we get

$$\partial_t^{k+1}\partial_x^l u + A\partial_t^k\partial_x^{l+1} u = \partial_t^k\partial_x^l f =: f_{k,l}.$$

We have now the relation $\partial_t^k\partial_x^{l+1}u = A^{-1}(f_{k,l} - \partial_t^{k+1}\partial_x^l u)$ that allows one to trade one space derivative for one time derivative. The control of $\partial_t^k\partial_x^{l+1}u$ can therefore be deduced inductively from Step 2.

Step 4. Existence of a solution. The existence and uniqueness of a solution $u \in \mathbb{W}^m(T)$ to (24) can be deduced from the energy estimates and the compatibility condition along classical lines (see for instance [29, 30, 5]).

Step 5. Existence of a Kreiss symmetrizer. The only step left to prove is that there exists a symmetrizer with the properties $(i)-(iii)$ assumed in Step 1. Here again, for the sake of clarity, we do not take into consideration the dependance on time and space of A and $\nu = \underline{\nu}$. One first has to notice that the Kreiss-Lopatinskiĭ condition $\underline{\nu}\cdot\mathbf{e}_+ \neq 0$ is equivalent to the condition $\pi_-\underline{\nu}^\perp \neq 0$ where $\mathbf{e}_+$ is a unit eigenvector associated with the positive eigenvalue λ_+ of A, and $\pi_\pm$ denote the eigenprojectors associated with the eigenvalues $\pm\lambda_\pm$ of A. Then we have to prove that the condition $\pi_-\underline{\nu}^\perp \neq 0$ implies the property (iii) of Step 1. We want to prove that if M is chosen large enough, then

$$S = \pi_+^{\mathrm{T}}\pi_+ + M\pi_-^{\mathrm{T}}\pi_-,$$

is a suitable symmetrizer. By the definition of $\pi_\pm$, we compute indeed that

$$SA = \lambda_+\pi_+^{\mathrm{T}}\pi_+ - M\lambda_-\pi_-^{\mathrm{T}}\pi_-,$$

which is obviously symmetric. We also remark that

$$v^{\mathrm{T}}SAv = \lambda_+|\pi_+v|^2 - M\lambda_-|\pi_-v|^2.$$

We need to show that this quantity is negative on the kernel $\mathbb{R}\underline{\nu}^\perp$ of the boundary condition. Assuming without loss of generality that $|\underline{\nu}| = 1$, we see that

$$\begin{aligned}
-|\pi_-v|^2 &= -|(\underline{\nu}^\perp\cdot v)\pi_-\underline{\nu}^\perp + (\underline{\nu}\cdot v)\pi_-\underline{\nu}|^2\\
&\leq -\frac{1}{2}|\underline{\nu}^\perp\cdot v|^2|\pi_-\underline{\nu}^\perp|^2 + |\underline{\nu}\cdot v|^2|\pi_-\underline{\nu}|^2\\
&\leq -\frac{1}{2}|\pi_-\underline{\nu}^\perp|^2|v|^2 + (|\pi_-\underline{\nu}|^2 + |\pi_-\underline{\nu}^\perp|^2)|\underline{\nu}\cdot v|^2
\end{aligned}$$

and that

$$\begin{aligned}
|\pi_+v|^2 &= |(\underline{\nu}^\perp\cdot v)\pi_+\nu^\perp + (\underline{\nu}\cdot v)\pi_+\underline{\nu}|^2\\
&\leq 2|\pi_+\underline{\nu}^\perp|^2|\underline{\nu}^\perp\cdot v|^2 + 2|\pi_+\underline{\nu}|^2|\underline{\nu}\cdot v|^2\\
&\leq 4|\pi_+\underline{\nu}^\perp|^2|v|^2 + 4(|\pi_+\underline{\nu}^\perp|^2 + |\pi_+\underline{\nu}|^2)|\underline{\nu}\cdot v|^2.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
v^{\mathrm{T}}SAv \leq &-\lambda_-|\pi_-\underline{\nu}^\perp|^2\Big(\frac{M}{2} - 4\frac{\lambda_+}{\lambda_-}\frac{|\pi_+\underline{\nu}^\perp|^2}{|\pi_-\underline{\nu}^\perp|^2}\Big)|v|^2\\
&+\big\{\lambda_-M(|\pi_-\underline{\nu}|^2 + |\pi_-\underline{\nu}^\perp|^2) + 4\lambda_+(|\pi_+\underline{\nu}^\perp|^2 + |\pi_+\underline{\nu}|^2)\big\}|\underline{\nu}\cdot v|^2
\end{aligned}$$

and the result follows for M large enough (note that assumptions (i) and (ii) in Step 1 are also satisfied).

3.2 Quasilinear initial boundary value problems

The nonlinear shallow water in their formulation (1) or (2) are quasilinear systems and we therefore need to generalize the results of the previous section to such systems. More precisely, let us consider

$$\begin{cases} \partial_t u + A(u)\partial_x u + B(t,x)u = f(t,x) & \text{in} \quad \Omega_T, \\ u_{|t=0} = u^{\text{in}}(x) & \text{on} \quad \mathbb{R}_+, \\ \Phi(t, u_{|x=0}) = g(t) & \text{on} \quad (0,T), \end{cases} \tag{33}$$

where u, u^{in}, and f are $\mathbb{R}^2$-valued functions, g and Φ are real-valued functions, while A and B take their values in the space of 2×2 real-valued matrices. We also make the following assumption on the hyperbolicity of the system and on the boundary condition.

Assumption 2 Let $\mathcal{U}$ be an open set in $\mathbb{R}^2$ and $A \in C^\infty(\mathcal{U})$. Then

i. For any $u \in \mathcal{U}$, the matrix $A(u)$ has eigenvalues $\lambda_+(u)$ and $-\lambda_-(u)$ satisfying

$$\lambda_\pm(u) > 0.$$

ii. There exist a diffeomorphism $\Theta : \mathcal{U} \to \Theta(\mathcal{U}) \subset \mathbb{R}^2$ and $\nu \in C([0,T])$ such that for any $t \in [0,T]$ and any $u \in \mathcal{U}$ we have

$$\Phi(t,u) = \nu(t) \cdot \Theta(u) \quad \text{and} \quad |\nabla_u \Phi(t,u) \cdot \mathbf{e}_+(u)| > 0,$$

where $\mathbf{e}_+(u)$ is a unit eigenvector associated with the eigenvalue $\lambda_+(u)$ of $A(u)$. □

Remark 5 If $\Phi(t,u) = \Phi(u)$ is independent of t and if for some u^0 we have $|\nabla_u \Phi(t,u^0) \cdot \mathbf{e}_+(u^0)| > 0$, then by the inverse function theorem and up to shrinking $\mathcal{U}$ to a sufficiently small neighborhood of u^0, the existence of a diffeomorphism Θ satisfying the properties of point **ii** is automatic.

Example 2 For the nonlinear shallow water equations

$$\partial_t u + A(u)\partial_x u = 0,$$

with $u = (\zeta, q)^{\text{T}}$ and $A(u)$ as given by (31), whose linear version has been considered in Example 1, the first two points of the assumption are equivalent to

$$h > 0, \qquad \sqrt{\mathtt{g}h} \pm \frac{q}{h} > 0 \qquad (\text{with } h = h_0 + \zeta).$$

The condition **ii** of the assumption depends of course on the boundary condition under consideration. Let us consider here two important examples:

- Boundary condition on the horizontal water flux, that is, $q_{|x=0} = g$. This corresponds to $\Phi(t,u) = \nu \cdot u$ with $\nu = (0,1)^{\mathrm{T}}$; as seen in Example 1 the condition **ii** of the assumption is satisfied.
- Boundary condition on the outgoing Riemann invariant, that is, $2(\sqrt{\mathtt{g}h} - \sqrt{\mathtt{g}h_0}) + q/h = g$ (if we take $g = 0$, this correspond to a *transparent* boundary condition at $x = 0$) We then have $\Phi(t,u) = \Phi(u) = 2(\sqrt{\mathtt{g}h} - \sqrt{\mathtt{g}h_0}) + q/h$ and we can take the diffeomorphism defined on $\mathcal{U} = \{(h,q) \in \mathbb{R}^2\,;\ h > 0\}$ by
$$\Theta(h,q) = \big(2(\sqrt{\mathtt{g}h} - \sqrt{\mathtt{g}h_0}) + q/h, 2(\sqrt{\mathtt{g}h} - \sqrt{\mathtt{g}h_0}) - q/h\big)^{\mathrm{T}},$$
where $2(\sqrt{\mathtt{g}h} - \sqrt{\mathtt{g}h_0}) - q/h$ is the incoming Riemann invariant. Then, $\Phi(u) = \nu \cdot \Theta(u)$ with $\nu = (1,0)^{\mathrm{T}}$; moreover, we compute $\nabla_u \Phi = (1/h)(\sqrt{gh} - q/h, 1)^{\mathrm{T}}$ so that all the conditions of the secod point of the assumption are satisfied.

The main result is the following. For the sake of conciseness, we do not derive explicitly the compatibility conditions involved in the statement of the theorem; they are derived following the same principles as in §3.1.2 (see [18] for details). Note that the statement on the regularity of the trace of the solution at the boundary follows from the control of the trace in the energy estimate of Theorem 2.

Theorem 3 *Let $m \geq 2$ be an integer, $B \in L^\infty(\Omega_T)$, $\partial B \in \mathbb{W}^{m-1}(T)$, and assume that Assumption 2 is satisfied with $\Theta \in C^\infty(\mathcal{U})$ and $\nu \in W^{m,\infty}(0,T)$. If $u^{\mathrm{in}} \in H^m(\mathbb{R}_+)$ takes its values in a compact and convex set $\mathcal{K}_0 \subset \mathcal{U}$ and if the data u^{in}, $f \in H^m(\Omega_T)$, and $g \in H^m(0,T)$ satisfy the compatibility conditions up to order $m-1$, then there exist $T_1 \in (0,T]$ and a unique solution $u \in \mathbb{W}^m(T_1)$ to the initial boundary value problem* (33). *Moreover, the trace of u at the boundary $x = 0$ belongs to $H^m(0,T_1)$ and $|u_{|x=0}|_{m,T_1}$ is finite.*

3.2.1 Sketch of the proof of Theorem 3

Here again, we sketch the proof of [18], to which we refer for more details. *For the sake of clarity, we take $B = 0$ throughout this proof.*
In Step 1, through a nonlinear change of variables, we reduce the problem to the case of a linear boundary condition. In Step 2, we construct an iterative scheme that requires, at each step, to solve a linear initial boundary value problem. Particular attention must be paid to the choice of the first iterate to ensure that the compatibility conditions are satisfied at each step. In Step 3, using Theorem 2, we can derive uniform bounds on the sequence of approximate solutions constructed with this iterative scheme. The last step is then to prove that this sequence converges; this does not raise any additional difficulty with respect to the case without boundary.
Step 1. Linearization of the boundary condition. Owing to Assumption 2, we can introduce, denoting by $d_v\Theta^{-1}$ the derivative at v of the mapping Θ^{-1},

$$v = \Theta(u), \qquad J(v) = d_v(\Theta^{-1}), \quad \text{and} \quad A^\sharp(v) = J(v)^{-1}A(\Theta^{-1}(v))J(v).$$

Then, u is a classical solution to (33) if and only if v is a classical solution of

$$\begin{cases} \partial_t v + A^\sharp(v)\partial_x v = J(v)^{-1}f(t,x) & \text{in} \quad \Omega_T, \\ v_{|_{t=0}} = \Theta(u^{\rm in}(x)) & \text{on} \quad \mathbb{R}_+, \\ \nu(t)\cdot v_{|_{x=0}} = g(t) & \text{on} \quad (0,T) \end{cases} \tag{34}$$

with $\nu(t)$ as in Assumption 2.

Step 2. Construction of a solution using an iterative scheme

$$\begin{cases} \partial_t v^{n+1} + A^\sharp(v^n)\partial_x v^{n+1} = f^n & \text{in} \quad \Omega_T, \\ v^{n+1}{}_{|_{t=0}} = \Theta(u^{\rm in}(x)) & \text{on} \quad \mathbb{R}_+, \\ \nu(t)\cdot v^{n+1}{}_{|_{x=0}} = g(t) & \text{on} \quad (0,T), \end{cases} \tag{35}$$

for all $n \in \mathbb{N}$ and with

$$f^n(t,x) = J(v^n)^{-1}f(t,x).$$

Contrary to what happens for initial boundary value problems, the choice of the first iterate u^0 is important. We choose a function $u^0 \in H^{m+1/2}(\mathbb{R}\times\mathbb{R}_+)$ such that

$$(\partial_t^k u^0)_{|_{t=0}} = u_k^{\rm in} \quad \text{for} \quad k = 0,1,\dots,m$$

with $u_k^{\rm in}$ with a procedure similar as the one that led to (29). Such a choice ensures along a classical procedure [29, 30] that the data $(\Theta(u^{\rm in}), f^n, g)$ are compatible for the linear initial boundary value problem (35) in the sense of Definition 1.

Step 3. Uniform bounds. One can check that $|||v^n(0)|||_m$ is independent of n (one can inductively express the time derivatives in terms of space derivatives using the same procedure as in §3.1.2 for the compatibility conditions, and control $|||v^n(0)|||_m$ in terms of $\|u^{\rm in}\|_{H^m}$), and that there exists therefore K_0 such that

$$\frac{1}{c_0}, |||v^n(0)|||_m, \|A^\sharp(v^n)\|_{L^\infty(\Omega_{T_1})}, \|A^\sharp(v^n)^{-1}\|_{L^\infty(\Omega_{T_1})} \le K_0,$$

as long as v^n satisfies $\|v^n(t) - \Theta(u^{\rm in})\|_{L^\infty} \le \delta_0$ for $0 \le t \le T_1$. We can also prove by induction that for M large enough and T_1 small enough, for any $n \in \mathbb{N}$ we have

$$\begin{cases} \|v^n\|_{\mathbb{W}^m(T_1)} + |v^n{}_{|_{x=0}}|_{m,T_1} \le M, \\ \|v^n(t) - \Theta(u^{\rm in})\|_{L^\infty} \le \delta_0 \quad \text{for} \quad 0 \le t \le T_1. \end{cases} \tag{36}$$

This is an easy consequence of Theorem 2 applied to (35) after remarking that this latter system satisfies Assumption 1 (just notice that $\nabla_u\Phi(t,u)\cdot\mathbf{e}_+(u) = \nu(t)\cdot J(v)^{-1}\mathbf{e}_+(\Theta^{-1}(v))$).

Step 4. Convergence. Using the same arguments as in the case without boundary, one proves that the sequence $(v^n)_n$ is convergent in L^2 and that the limit belongs to $\mathbb{W}^m(T)$.

3.3 Linear initial boundary value problems on moving domains

Before considering free boundary problems, a necessary step is to consider initial boundary value problems on domains whose boundaries moves with a prescribed motion. Instead of $\mathbb{R}_+$, we therefore consider a domain $(\underline{x}(t),+\infty)$, where the left boundary $\underline{x}(t)$ is a given time dependent function. Let us consider systems of the form

$$\begin{cases} \partial_t U + A(\underline{U})\partial_x U + \mathtt{B}U = F & \text{in} \quad (\underline{x}(t),\infty) \quad \text{for} \quad t \in (0,T), \\ U_{|_{t=0}} = u^{\mathrm{in}}(x) & \text{on} \quad (0,\infty), \\ \nu(t)\cdot U_{|_{x=\underline{x}(t)}} = g(t) & \text{on} \quad (0,T), \end{cases} \tag{37}$$

where without loss of generality we assume $\underline{x}(0) = 0$.

3.3.1 Reformulation of the problem on a fixed domain

The first thing to do is of course to transform this initial boundary value problem on a moving domain into another one cast on a fixed domain, say, $\mathbb{R}_+$. This is done through a diffeomorphism $\varphi(t,\cdot)$ that maps at all times $\mathbb{R}_+$ onto $(\underline{x}(t),\infty)$ and such that for any t, we have $\varphi(t,0) = \underline{x}(t)$. Several choices are possible for φ and shall be discussed later. At this point, we just assume that $\varphi \in C^1(\Omega_T)$ and that $\varphi(0,x) = x$. Composing the interior equation in (37) with the diffeomorphism φ to work on the fixed domain $(0,\infty)$, introducing the notations

$$u = U\circ\varphi, \qquad \underline{u} = \underline{U}\circ\varphi, \qquad \partial_t^\varphi u = (\partial_t U)\circ\varphi, \qquad \partial_x^\varphi u = (\partial_x U)\circ\varphi,$$

so that, in particular,

$$\partial_x^\varphi = \frac{1}{\partial_x\varphi}\partial_x, \qquad \partial_t^\varphi = \partial_t - \frac{\partial_t\varphi}{\partial_x\varphi}\partial_x, \tag{38}$$

and writing $B = \mathtt{B}\circ\varphi$ and $f = F\circ\varphi$, we obtain the following equation for u

$$\partial_t^\varphi u + A(\underline{u})\partial_x^\varphi u + B(t,x)u = f(t,x). \tag{39}$$

The initial boundary value problem on a moving domain (37) can therefore be recast as an initial boundary value problem on a fixed domain

$$\begin{cases} \partial_t u + \mathcal{A}(\underline{u},\partial\varphi)\partial_x u + B(t,x)u = f(t,x) & \text{in} \quad \Omega_T, \\ u_{|_{t=0}} = u^{\mathrm{in}}(x) & \text{on} \quad \mathbb{R}_+, \\ \nu(t)\cdot u_{|_{x=0}} = g(t) & \text{on} \quad (0,T), \end{cases} \tag{40}$$

with

$$\mathcal{A}(\underline{u},\partial\varphi) = \frac{1}{\partial_x\varphi}\big(A(\underline{u}) - (\partial_t\varphi)\mathrm{Id}\big).$$

3.3.2 Statement of the main result

If we want to apply Theorem 2 to construct solutions to (40), it is necessary to get some information on the regularity of φ, which is of course related to the properties of the boundary coordinate $\underline{x}(t)$. A direct application of Theorem 2 requires that $\partial\varphi$ be in $\mathbb{W}^m(T)$ in order to get solutions u in $\mathbb{W}^m(T)$. Using Alinhac's good unknown [1], it is however possible to obtain refined regularity estimates, as shown in the following theorem which requires only the following assumption.

Assumption 3 We have $\underline{u} \in W^{1,\infty}(\Omega_T)$, $\underline{x} \in C^1([0,T])$, $\underline{x}(0) = 0$, and the diffeomorphism φ is in $C^1(\Omega_T)$. Moreover, there exists a constant $c_0 > 0$ such that the following three conditions hold.

i. There exists an open set $\mathcal{U} \subset \mathbb{R}^2$ such that $A \in C^\infty(\mathcal{U})$ and that for any $u \in \mathcal{U}$, the matrix $A(u)$ has eigenvalues $\lambda_+(u)$ and $-\lambda_-(u)$. Moreover, $\underline{u}$ takes its values in a compact set $\mathcal{K}_0 \subset \mathcal{U}$ and for any $(t,x) \in \Omega_T$ we have

$$\lambda_\pm(\underline{u}(t,x)) \mp \partial_t\varphi(t,x) \geq c_0 \quad \text{and} \quad \lambda_\pm(\underline{u}(t,x)) \geq c_0.$$

ii. Denoting by $\mathbf{e}_+(u)$ a unit eigenvector associated with the eigenvalue $\lambda_+(u)$ of $A(u)$, for any $t \in [0,T]$ we have

$$|\nu(t) \cdot \mathbf{e}_+(\underline{u}(t,0))| \geq c_0.$$

iii. The Jacobian of the diffeomorphism is uniformly bounded from below and from above, that is, for any $(t,x) \in \Omega_T$ we have

$$c_0 \leq \partial_x\varphi(t,x) \leq \frac{1}{c_0}.$$

Example 3 Considering as in Example 1 the linearized shallow water equations, but this time on a moving domain, Assumption 3 reduces to the conditions $\underline{h}, \underline{q} \in W^{1,\infty}(\Omega_T)$ and

$$\underline{h}(t,x) \geq c_0, \quad \sqrt{\mathtt{g}\underline{h}(t,x)} \pm \Big(\frac{\underline{q}(t,x)}{\underline{h}(t,x)} - \partial_t\varphi(t,x)\Big) \geq c_0, \quad \sqrt{\mathtt{g}\underline{h}(t,x)} \pm \frac{\underline{q}(t,x)}{\underline{h}(t,x)} \geq c_0$$

with some positive constant c_0 independent of $(t,x) \in \Omega_T$.

For the sake of brevity, we do not derive explicitly the compatibility conditions, the procedure to obtain them being absolutely similar to the one described in §3.1.2. We also insist on the fact that a direct application of Theorem 2 to solve (40) would require $\partial\tilde{\varphi}$ be in $\mathbb{W}^m(T)$ (with $\widetilde{\varphi}(t,x) = \varphi(t,x) - x$), while it is only assumed that $\partial\tilde{\varphi}$ be in $\mathbb{W}^{m-1}(T)$ in the theorem below. This gain of one derivative is due to the use of Alinhac's good unknown. Note also that, as pointed out in Remark 4, the quantity $|||u(0)|||_m$ that appears in the energy estimate can be controled in terms of $\|u^{\rm in}\|_{H^m}$.

Theorem 4 *Let $m \geq 1$ be an integer, $T > 0$, and assume that Assumption 3 is satisfied for some $c_0 > 0$. Assume moreover that there are two constants $0 < K_0 \leq K$ such that*

$$\begin{cases} \frac{1}{c_0}, |||\partial\widetilde{\varphi}(0)|||_{m-1}, |\nu|_{L^\infty(0,T)}, \|\partial\varphi\|_{L^\infty(\Omega_T)}, \|A\|_{L^\infty(\mathcal{K}_0)} \leq K_0, \\ \|\partial\widetilde{\varphi}\|_{\mathbb{W}^{m-1}(T)}, \|\partial_t\varphi\|_{H^m(\Omega_T)}, |(\partial^m\varphi)_{|x=0}|_{L^\infty(0,T)} \leq K, \\ \|\underline{u}\|_{W^{1,\infty}(\Omega_T)\cap\mathbb{W}^m(T)}, \|B\|_{W^{1,\infty}(\Omega_T)}, \|\partial B\|_{\mathbb{W}^{m-1}(T)}, |\nu|_{W^{1,\infty}\cap W^{m-1,\infty}(0,T)}, |\partial_t^m\nu|_{L^2(0,T)} \leq K, \end{cases}$$

where $\widetilde{\varphi}(t,x) = \varphi(t,x) - x$. Then, for any data $u^{\rm in} \in H^m(\mathbb{R}_+)$, $f \in H^m(\Omega_T)$, and $g \in H^m(0,T)$ satisfying the compatibility conditions up to order $m-1$ (see §3.1.2), there exists a unique solution $u \in \mathbb{W}^m(T)$ to (40)*. Moreover, the following estimate holds for any $t \in [0,T]$*

$$|||u(t)|||_m + |u_{|x=0}|_{m,t} \leq C(K_0)e^{C(K)t} \times \Big((1 + |\partial_t^m\nu|_{L^2(0,t)})|||u(0)|||_m + |g|_{H^m(0,t)} + |f_{|x=0}|_{m-1,t} + \int_0^t |||f(t')|||_m {\rm d}t' \Big).$$

3.3.3 Sketch of the proof of Theorem 4

A direct estimate in $\mathbb{W}^m(T)$ for the solution of (40) through Theorem 2 is not possible because it would require that $\partial^2\varphi \in \mathbb{W}^{m-1}(T)$ while, under the assumptions made in the statement of the theorem, we only have $\partial^2\varphi \in \mathbb{W}^{m-2}(T)$. We can however get an estimate in $\mathbb{W}^{m-1}(T)$; this is done in Step 1. In Step 2, we show that $\partial_t^\varphi u = \partial_t u - (\partial_t\varphi)\partial_x^\varphi u$ can also be estimated in $\mathbb{W}^{m-1}(T)$; this quantity is called Alinhac's good unknown and plays a crucial role here. From these two estimates, we deduce in Step 3 a bound on the solution in $\mathbb{W}^m(T)$.
Step 1. Existence and bounds on a solution in $u \in \mathbb{W}^{m-1}(T)$ to (40). This is granted by a direct application of Theorem 2.
Step 2. Bounds on $\partial_t^\varphi u$ in $\mathbb{W}^{m-1}(T)$. This is the key point of the proof; the quantity $\partial_t^\varphi u$ is called Alinhac's good unknown [1], and using it instead of $\partial_t u$ removes the most singular term in φ. Indeed, differentiating with respect to time the interior equation in (40), and writing $\dot{u} = \partial_t u$, $\dot{f} = \partial_t f$, etc., we get

$$\partial_t\dot{u} + \mathcal{A}(\underline{u},\partial\varphi)\partial_x\dot{u} + A'(\underline{u})[\dot{\underline{u}}]\partial_x^\varphi u + \mathcal{M}(\underline{u},\partial\varphi,\partial_x u)\partial\dot{\varphi} + B\dot{u} = \dot{f} - \dot{B}u \qquad (41)$$

with

$$\mathcal{M}(u,\partial\varphi,\partial_x u)\partial\dot{\varphi} = -\big((\partial_x\dot{\varphi})\mathcal{A}(\underline{u},\partial\varphi) + (\partial_t\dot{\varphi}){\rm Id}\big)\partial_x^\varphi u.$$

Obviously, the term $\mathcal{M}(\underline{u},\partial\varphi,\partial_x u)\partial\dot{\varphi}$ is responsible for the loss of one derivative, in the sense that a control of φ in $\mathbb{W}^{m+1}(T)$ is required to control the $\mathbb{W}^m(T)$ norm of u. This singular dependence is removed by working with Alinhac's good unknown

$$\dot{u}^\varphi = \dot{u} - \dot{\varphi}\partial_x^\varphi u \qquad (42)$$

instead of $\dot u$. The notations $\dot f^\varphi$ and $\dot B^\varphi$ are defined similarly. The following lemma is due to Alinhac [1] and can be checked by simple computations.

Lemma 1 *With $\dot u^\varphi = \dot u - \dot\varphi \partial_x^\varphi u$, the equation* (41) *can be rewritten under the form*

$$\partial_t \dot u^\varphi + \mathcal{A}(\underline{u}, \partial\varphi)\partial_x \dot u^\varphi + A'(\underline{u})[\underline{\dot u^\varphi}]\partial_x^\varphi u + B\dot u^\varphi = \dot f^\varphi - \dot B^\varphi u.$$

As done previously, *we now restrict to the case $B = 0$, $A(\underline{u}) = A$ a constant coefficient matrix, and $\nu(t) = \underline{\nu}$ a time independent vector* since these simplifications do not hide the key arguments (see [18] for the general case). We can use (39) to write

$$\partial_x^\varphi u = A^{-1}(f - \dot u^\varphi),$$

so that the lemma yields

$$\partial_t \dot u^\varphi + \mathcal{A}(\underline{u}, \partial\varphi)\partial_x \dot u^\varphi = \dot f^\varphi =: f_{(1)}.$$

Therefore, $\dot u^\varphi = \partial_t^\varphi u$ solves an interior equation similar to those considered in Theorem 2. Let us now consider the initial and boundary conditions for $\dot u^\varphi$. For the initial condition, we have

$$(\dot u^\varphi)_{|t=0} = u^{\rm in}_{(1)} \quad \text{with} \quad u^{\rm in}_{(1)} = (\partial_t u)_{|t=0} - (\partial_t \varphi)_{|t=0}\partial_x u^{\rm in}.$$

For the boundary condition, let us differentiate with respect to time the boundary condition in (40) to obtain $\underline{\nu} \cdot \partial_t u_{|x=0} = \partial_t g$ or equivalently

$$\underline{\nu} \cdot (\dot u^\varphi + \underline{\dot x}\partial_x^\varphi u)_{|x=0} = \partial_t g.$$

Using (39), this yields

$$\underline{\nu} \cdot \big((\mathrm{Id} - \dot x A^{-1})\dot u^\varphi\big)_{|x=0} = \partial_t g - \dot x \underline{\nu} \cdot A^{-1} f_{|x=0}.$$

It follows that $\dot u^\varphi$ satisfies an initial boundary value problem of the form (24), namely,

$$\begin{cases} \partial_t \dot u^\varphi + \mathcal{A}(\underline{u}, \partial\varphi)\partial_x \dot u^\varphi = f_{(1)} & \text{in} \quad \Omega_T, \\ \dot u^\varphi_{|t=0} = u^{\rm in}_{(1)} & \text{on} \quad \mathbb{R}_+, \\ \nu_{(1)}(t) \cdot \dot u^\varphi_{|x=0} = g_{(1)} & \text{on} \quad (0, T), \end{cases} \tag{43}$$

where

$$\begin{cases} g_{(1)} = \partial_t g - \underline{\dot x}\nu \cdot A^{-1} f_{|x=0}, \\ \nu_{(1)} = (\mathrm{Id} - \underline{\dot x} A(\underline{u}_{|x=0})^{-1})^{\rm T}\underline{\nu}. \end{cases} \tag{44}$$

With some simple linear algebra, one then proves that the boundary condition satisfies condition **iii** in Assumption 1, and Theorem 2 can therefore be used to get a bound of $\partial_t^\varphi u$ in $\mathbb{W}^{m-1}(T)$.

Step 3. Deduce a bound on u in $\mathbb{W}^m(T)$ from the bounds on u and $\partial_t^\varphi u$ in $\mathbb{W}^{m-1}(T)$. For $m = 1$ we can first use (39) to write

$$\partial_x u = (\partial_x \varphi) A^{-1}(f - \dot{u}^\varphi).$$

which provides a control of $\partial_x u$ in terms of $\partial_t^\varphi u$. We then control $\partial_t u$ through the relation

$$\partial_t u = \dot{u}^\varphi - \frac{\partial_t \varphi}{\partial_x \varphi}\partial_x u.$$

We proceed to consider the case $m \geq 2$. Applying ∂^α with a multi-index α satisfying $|\alpha| \leq m-1$ to (39) and using the identity

$$\partial_x^\varphi \partial^\alpha u = \partial^\alpha \partial_x^\varphi u + (\partial_x^\varphi \partial^\alpha \varphi)\partial_x^\varphi u + (\partial_x \varphi)^{-1}[\partial^\alpha; \partial_x \varphi, \partial_x^\varphi u] \tag{45}$$

with the symmetric commutator defined as $[\partial^\alpha; v, w] = \partial^\alpha(vw) - (\partial^\alpha v)w - v(\partial^\alpha w)$, we obtain

$$\begin{aligned} A\partial_x^\varphi \partial^\alpha u + \partial^\alpha \dot{u}^\varphi &= \partial^\alpha f + A((\partial_x^\varphi \partial^\alpha \varphi)\partial_x^\varphi u + (\partial_x \varphi)^{-1}[\partial^\alpha; \partial_x \varphi, \partial_x^\varphi u]) \\ &=: f_{1,\alpha}. \end{aligned}$$

The quantity $f_{1,\alpha}$ can be controlled through classical commutator estimates so that writing the equation under the form

$$\partial^\alpha \partial_x u = (\partial_x \varphi) A(\underline{u})^{-1}(f_{1,\alpha} - \partial^\alpha \dot{u}^\varphi),$$

provides the needed control on $\partial_x u$. which will be used to evaluate $\partial_x u$. Applying ∂^α to the identity $\partial_t u = \dot{u}^\varphi + (\partial_t \varphi)\partial_x^\varphi u$ and using (45) we obtain

$$\begin{aligned} &\partial^\alpha \partial_t u - \partial^\alpha \dot{u}^\varphi - (\partial_t \varphi)(\partial_x \varphi)^{-1}\partial^\alpha \partial_x u \\ &= (\partial^\alpha \partial_t \varphi)\partial_x^\varphi u + [\partial^\alpha; \partial_t \varphi, \partial_x^\varphi u] - (\partial_t \varphi)(\partial_x \varphi)^{-1}((\partial^\alpha \partial_x \varphi)\partial_x^\varphi u + [\partial^\alpha; \partial_x \varphi, \partial_x^\varphi u]) \\ &=: f_{2,\alpha}. \end{aligned}$$

Here again, $f_{2,\alpha}$ can be controlled with commutator estimates so that writing the equation under the form

$$\partial^\alpha \partial_t u = \partial^\alpha \dot{u}^\varphi + (\partial_t \varphi)(\partial_x \varphi)^{-1}\partial^\alpha \partial_x u + f_{2,\alpha},$$

provides the control we need on $\partial_t u$.

3.4 Application to free boundary problems with a boundary equation of “kinematic” type

We investigate here a general class of free boundary problems (to which the hydraulic piston problem of §2.1.4 belongs, up to minor adaptations). We consider a quasilinear hyperbolic system cast on a moving domain $(\underline{x}(t), \infty)$,

$$\begin{cases} \partial_t U + A(U)\partial_x U = 0 & \text{in} \quad (\underline{x}(t), \infty) \quad \text{for} \quad t \in (0,T), \\ U_{|t=0} = u^{\text{in}}(x) & \text{on} \quad (\underline{x}(0), \infty), \\ \underline{\nu} \cdot U_{|x=\underline{x}(t)} = g(t) & \text{on} \quad (0,T), \end{cases} \tag{46}$$

and assume that the evolution of the boundary is governed by a nonlinear equation of the form

$$\dot{\underline{x}} = \mathcal{X}(U_{|x=\underline{x}(t)}), \tag{47}$$

for some smooth function $\mathcal{X}$. The set of equations (46)–(47) is a free boundary problem. In the following, without loss of generality we assume $\underline{x}(0) = 0$.

3.4.1 Reformulation of the problem on a fixed domain

Using as in §3.3 a diffeomorphism $\varphi(t,\cdot) : \mathbb{R}_+ \to (\underline{x}(t), \infty)$, and recalling the notations

$$u = U \circ \varphi, \qquad \partial_x^\varphi = \frac{1}{\partial_x \varphi}\partial_x, \qquad \partial_t^\varphi = \partial_t - \frac{\partial_t \varphi}{\partial_x \varphi}\partial_x,$$

the free boundary problem (46)–(47) can therefore be recast as an initial boundary value problem on a fixed domain,

$$\begin{cases} \partial_t u + \mathcal{A}(u, \partial\varphi)\partial_x u = 0 & \text{in} \quad \Omega_T, \\ u_{|t=0} = u^{\text{in}}(x) & \text{on} \quad \mathbb{R}_+, \\ \underline{\nu} \cdot u_{|x=0} = g(t) & \text{on} \quad (0,T), \end{cases} \tag{48}$$

where $\underline{\nu} \in \mathbb{R}^2$ is a constant vector and

$$\mathcal{A}(u, \partial\varphi) = \frac{1}{\partial_x \varphi}\big(A(u) - (\partial_t \varphi)\mathrm{Id}\big),$$

complemented by the evolution equation

$$\dot{\underline{x}} = \mathcal{X}(u_{|x=0}), \qquad \underline{x}(0) = 0. \tag{49}$$

As shown in §3.3, the regularity of φ plays an important role in the analysis of the initial boundary value problem (48). It is therefore important to make an appropriate choice for the diffeomorphism. For a boundary equation of the form (49) which is of “kinematic” type, a “Lagrangian” diffeomorphism is appropriate. In particular, in the second point of the lemma, the structure of φ allows the control of $\partial_t \varphi$ in $\mathbb{W}^m(T)$ (which involves $m+1$ derivatives of φ) by u in $\mathbb{W}^m(T)$ (which involves only m derivative of u). This diffeomorphism is well defined for t small enough by

$$\varphi(t,x) = x + \int_0^t \mathcal{X}(u(t',x))\mathrm{d}t'. \tag{50}$$

3.4.2 Statement of the main result

We can now state the main result of this section, which holds under the following assumption.

Assumption 4 Let $\mathcal{U}$ be an open set in $\mathbb{R}^2$. The following conditions hold.

i. $A, \mathcal{X} \in C^\infty(\mathcal{U})$, $\mathcal{X}(0) = 0$.

ii. For any $u \in \mathcal{U}$, the matrix $A(u)$ has eigenvalues $\lambda_+(u)$ and $-\lambda_-(u)$ satisfying

$$\lambda_\pm(u) > 0 \quad \text{and} \quad \lambda_\pm(u) \mp \mathcal{X}(u) > 0.$$

iii. Denoting by $\mathbf{e}_+(u)$ a unit eigenvector associated with the eigenvalue $\lambda_+(u)$ of $A(u)$, for any $u \in \mathcal{U}$ we have

$$|\underline{\nu} \cdot \mathbf{e}_+(u)| > 0.$$

The following theorem provides a local existence result. It can easily be adapted (see [18]) to solve the hydraulic piston problem presented in §2.1.4. Once again we do not make the compatibility conditions precise, and refer to §3.1.2 for the procedure to derive them.

Theorem 5 *Let $m \geq 2$ be an integer. Suppose that Assumption 4 is satisfied. If $u^{\rm in} \in H^m(\mathbb{R}_+)$ takes its values in a compact and convex set $\mathcal{K}_0 \subset \mathcal{U}$ and if the data $u^{\rm in}$ and $g \in H^m(0,T)$ satisfy the compatibility conditions up to order $m-1$. Then there exist $T_1 \in (0,T]$ and a unique solution $(u, \underline{x})$ to* (48)–(49) *with $u \in \mathbb{W}^m(T_1)$, $\underline{x} \in H^{m+1}(0,T_1)$, and φ given by* (50).

3.4.3 Sketch of the proof of Theorem 5

As in the proof of Theorem 3, an iterative scheme must be used; the novelty is that the diffeomorphism used to transform the problem into an initial boundary value problem with fixed boundary is also part of the iterative scheme. This is discussed in Step 1. The remaining of the proof is then quite similar to the proof of Theorem 3: uniform bounds on the sequence of approximations (Step 2), and convergence (Step 3). We again refer to [18] for more details.

Step 1. Choice of an iterative scheme. The solution is classically constructed using the iterative scheme

$$\varphi^n(t,x) = x + \int_0^t \mathcal{X}(u^n(t',x))\mathrm{d}t' \tag{51}$$

and

$$\begin{cases} \partial_t u^{n+1} + \mathcal{A}(u^n, \partial\varphi^n)\partial_x u^{n+1} = 0 & \text{in} \quad \Omega_T, \\ u^{n+1}{}_{|t=0} = u^{\rm in}(x) & \text{on} \quad \mathbb{R}_+, \\ \underline{\nu} \cdot u^{n+1}{}_{|x=0} = g(t) & \text{on} \quad (0,T), \end{cases} \tag{52}$$

for all $n \in \mathbb{N}$. As for the proof of Theorem 3, the choice of the first iterate u^0 is important; we choose a function $u^0 \in H^{m+1/2}(\mathbb{R} \times \mathbb{R}_+)$ such that $(\partial_t^k u^0)_{|_{t=0}} = u_k^{\rm in}$ for $0 \leq k \leq m$ with $u_k^{\rm in}$ defined with a procedure similar as the one that led to (29). Then, for the initial boundary value problem (52), the data $(u^{\rm in}, g)$ satisfy the compatibility conditions up to order $m-1$ in the sense of Definition 1.
Step 2. Uniform bounds. We can apply Theorem 4 to get bounds on u^{n+1} and get uniform bounds by induction. One must be careful though since Theorem 4 requires a control of $\partial_t \varphi$ in $\mathbb{W}^m(T_1)$, and therefore the control of $m+1$-derivatives of φ. Fortunately, there is at least one time derivative among these $m+1$ derivatives, so that using the definition (50) of φ, the control of of $\partial_t \varphi$ in $\mathbb{W}^m(T_1)$ only requires the control of m derivatives of u^n and the bounds can be propagated by induction.
Step 3. Convergence. One classically shows that the sequence of approximate solutions $\{(u^n, \varphi^n)\}_n$ is a Cauchy sequence and therefore converges to the solution (u, φ) to (48)–(49), which belongs to $u \in \mathbb{W}^m(T_1)$ and $\underline{x} = \varphi_{|_{x=0}} \in H^{m+1}(0, T_1)$.

3.5 Application to free boundary problems with a fully nonlinear boundary equation

We consider here a second class of hyperbolic free boundary value problems that turns out to be more singular than the class investigated in §3.4. The floating body problem presented in §2.1.5 can easily be related to this class (see [18]).
We consider a 2×2 quasilinear hyperbolic system on a moving domain $(\underline{x}(t), \infty)$:

$$\partial_t U + A(U)\partial_x U = 0 \quad \text{in} \quad (\underline{x}(t), \infty) \tag{53}$$

with a fully nonlinear boundary condition

$$U = U_{\rm i} \quad \text{on} \quad x = \underline{x}(t), \tag{54}$$

where $U_{\rm i} = U_{\rm i}(t, x)$ is a given $\mathbb{R}^2$-valued function, whereas $\underline{x}(t)$ is an unknown function.

It is important to notice that (54) seems to provide *two* boundary conditions for the hyperbolic problem (53). According to §3.1.1 and Theorem 3, this would be an overdetermined problem if the boundary were fixed. The difference is that the location $\underline{x}(t)$ is now free to move, and this removes this overdetermination: one must not consider (54) as two boundary conditions for (53) but as one boundary condition and one equation for the evolution of the free boundary.

The difference with the free boundary problem (46)–(47) is that the evolution equation of the boundary is implicit. In order to make it explicit, we can differentiate the boundary condition $U(t, \underline{x}(t)) = U_{\rm i}(t, \underline{x}(t))$ with respect to t and take the Euclidean inner product of the resulting equation with $\partial_x U - \partial_x U_{\rm i}$, to obtain

$$\dot{\underline{x}} = \chi((\partial U)_{|_{x=\underline{x}}}, (\partial U_{\rm i})_{|_{x=\underline{x}}}), \tag{55}$$

where

$$\chi(\partial U, \partial U_{\mathrm{i}}) = -\frac{(\partial_x U - \partial_x U_{\mathrm{i}}) \cdot (\partial_t U - \partial_t U_{\mathrm{i}})}{|\partial_x U - \partial_x U_{\mathrm{i}}|^2}.$$

This evolution equation is more singular than the boundary equation (47) of kinematic type. Indeed, this latter gave a formula for $\underline{\dot{x}}(t)$ in terms of the trace of U at the boundary while (55) also involves the traces of its derivatives $\partial_t U$ and $\partial_x U$.

3.5.1 Reformulation of the problem on a fixed domain

As in §3.4 and with the same notations, we use a diffeomorphism $\varphi(t,\cdot) : \mathbb{R}_+ \to (\underline{x}(t), \infty)$ and put $u = U \circ \varphi$ and $u_{\mathrm{i}} = U_{\mathrm{i}} \circ \varphi$. Then, the free boundary problem (53)–(54) is recast as a problem on the fixed domain:

$$\begin{cases} \partial_t^\varphi u + A(u)\partial_x^\varphi u = 0 & \text{in} \quad \Omega_T, \\ u_{|_{x=0}} = u_{\mathrm{i}}{}_{|_{x=0}} & \text{on} \quad (0,T). \end{cases} \tag{56}$$

We impose an initial condition of the form

$$u_{|_{t=0}} = u^{\mathrm{in}}(x) \quad \text{on} \quad \mathbb{R}_+, \qquad \underline{x}(0) = 0. \tag{57}$$

We also note that the equation (55) for the free boundary is then reduced to

$$\underline{\dot{x}} = \chi((\partial^\varphi u)_{|_{x=0}}, (\partial^\varphi u_{\mathrm{i}})_{|_{x=0}}). \tag{58}$$

The different nature of the boundary equation (55) (when compared to the evolution equation of kinematic type (47)) also leads us to choose a different kind of diffeomorphism. Let $\psi \in C_0^\infty(\mathbb{R})$ be a cut-off function such that $\psi(x) = 1$ for $|x| \le 1$ and $= 0$ for $|x| \ge 2$. We define the diffeomorphism by

$$\varphi(t,x) = x + \psi\Big(\frac{x}{\varepsilon}\Big)\underline{x}(t), \tag{59}$$

with $\varepsilon > 0$ small enough.

This diffeomorphism behaves differently than the Lagrangian diffeomorphism studied in the previous section; in particular, the latter has a better time regularity, while the former has a better space regularity. More precisely, with the Lagrangian diffeomorphism, it is possible to control $\partial_t\varphi$ in $\mathbb{W}^m(T)$ using only m derivatives of u, but $m+1$ derivatives of u are needed to control $\partial_x\varphi$ in $\mathbb{W}^m(T)$. With the diffeomorphism (59), this is the reverse: one can control $\partial_x\varphi$ in $\mathbb{W}^m(T)$ using only m derivatives of u, but $m+1$ derivatives of u are needed to control $\partial_t\varphi$ in $\mathbb{W}^m(T)$. For technical reasons, it is more convenient to work with (59) here.

3.5.2 Statement of the main result

We naturally make the following assumption.

Assumption 5 Let $\mathcal{U}$ be an open set in $\mathbb{R}^2$.

i. $A \in C^\infty(\mathcal{U})$.
ii. There exists $c_0 > 0$ such that for any $u \in \mathcal{U}$, the matrix $A(u)$ has eigenvalues $\lambda_+(u)$ and $-\lambda_-(u)$ satisfying $\lambda_\pm(u) \geq c_0$. □

As before, this condition ensures that the system is strictly hyperbolic. We denote by $\mathbf{e}_\pm(u)$ normalized eigenvectors associated with the eigenvalues $\pm\lambda_\pm(u)$ of $A(u)$. They are uniquely determined up to a sign. Since both eigenvalues are simple, we have $\lambda_\pm, \mathbf{e}_\pm \in C^\infty(\mathcal{U})$ under an appropriate choice of the sign of $\mathbf{e}_\pm$. A quick look at (55) shows that a discontinuity of $\partial_x U$ at the free boundary is necessary, which leads us to work in a class of solutions satisfying

$$|(\partial_x^\varphi u - \partial_x^\varphi u_i)_{|x=0}| \geq c_0, \tag{60}$$

for some positive constant c_0. For technical reasons, we also need to make a slightly stronger assumption than the second point of Assumption 4, and we assume that

$$\lambda_\pm(u) \mp \partial_t \varphi \geq c_0 \quad \text{in} \quad (0,T) \times \mathbb{R}_+. \tag{61}$$

As we will see below, with the diffeomorphism constructed in the previous section, (61) is satisfied if the solution satisfies

$$\lambda_\pm(u_{|x=0}) \mp \dot{\underline{x}} \geq 2c_0 \quad \text{on} \quad (0,T). \tag{62}$$

Our main result is stated below: it can be used with minor adaptation to cover the floating body problem of §2.1.5 (see [18]). Once again, the derivation of the formula for the compatibility conditions is left to the reader.

Theorem 6 *Let $m \geq 2$ be an integer. Suppose that Assumption 5 is satisfied. If $u^{\rm in} \in H^m(\mathbb{R}_+)$ takes its values in a compact and convex set $\mathcal{K}_0 \subset \mathcal{U}$ and if the data $U_{\rm i} \in W^{m,\infty}((0,T) \times (-\delta,\delta))$ (with $\delta > 0$) and $u^{\rm in}$ satisfy*

i. $\lambda_\pm({u^{\rm in}}_{|x=0}) \mp \underline{x}_1^{\rm in} > 0$,
ii. $(\partial_x u^{\rm in})_{|x=0} - (\partial_x U_{\rm i})_{|t=x=0} \neq 0$,
iii. $((\partial_x u^{\rm in})_{|x=0} - (\partial_x U_{\rm i})_{|t=x=0})^\perp \cdot \mathbf{e}_+({u^{\rm in}}_{|x=0}) \neq 0$

(where $\underline{x}_1^{\rm in} = (\partial_t \underline{x})_{|t=0}$ can be expressed in terms of the initial data using the equations), and the compatibility conditions up to order $m-1$, then there exist $T_1 \in (0,T]$ and a unique solution $(u,\underline{x})$ to (56)–(57) *with $u, \partial_x u \in \mathbb{W}^{m-1}(T_1)$, $\underline{x} \in H^m(0,T_1)$, and φ given by* (59).

3.5.3 Sketch of the proof of Theorem 6

The main difficulty of the proof is the full nonlinearity of the boundary condition. In order to make it more apparent, we make in this proof the *simplifying assumption that A has constant coefficients, with eigenvalues $\pm\lambda_\pm$ and corresponding eigenvectors* $\mathbf{e}_\pm$. We refer to [18] for the complete proof.
In Step 1, we transform the problem into a quasilinear initial boundary value problem. This is achieved by differentiating the equations and introducing a second order good unknown $u_{(2)} = (\partial_t^\varphi)^2 u$ generalizing Alinhac's good unknown. The main difficulty is to derive the boundary condition satisfied by this new quantity. The extended system satisfied by u and $u_{(2)}$ turns out to have the expected quasilinear structure. It is then possible to prove in Step 2 that there exists a unique solution for this extended free boundary value problem; the main difficulty is to check that the boundary condition for $u_{(2)}$ satisfies the Kreiss-Lopatinskiĭ conditions necessary to apply the result of the previous sections. In Step 3, we show that the solution $(u, u_{(2)})$ constructed in Step 2 is actually a solution of the original system (one must check that if u and $u_{(2)}$ are provided by Step 2 then necessarily $u_{(2)} = (\partial_t^\varphi)^2 u$). Finally, optimal regularity is achieved in Step 4.

Step 1. Reduction to a system with quasilinear boundary conditions. We have already seen that the overdetermined boundary condition (56) provides the evolution equation (58) for the free boundary $\underline{x}(t)$. This equation is a fully nonlinear boundary condition in the sense that it involves a nonlinear expression of the derivatives of u. When dealing with fully nonlinear problems, one typically writes the equations for the derivatives of the solution in order to recover a quasilinear structure. This approach is complicated here by the fact that we are dealing with a free boundary problem. Even in the "quasilinear" case investigated in §3.4 we had to write the equation satisfied by Alinhac's good unknown $u_{(1)} := \partial_t^\varphi u$. In the situation considered here of a fully nonlinear problem, we need to introduce a *second order Alinhac* unknown, namely,

$$u_{(2)} = \partial_t^\varphi \partial_t^\varphi u. \tag{63}$$

Our goal is to derive a system for u and $u_{(2)}$ with quasilinear boundary conditions together with a quasilinear evolution equation for $\underline{x}$. Applying differential operators ∂_t^φ and ∂_x^φ to the first equation in (56), and noting that ∂_t^φ and ∂_x^φ commute, we can express $\partial_t^\varphi \partial_x^\varphi u$ and $\partial_x^\varphi \partial_x^\varphi u$ in terms of $u_{(2)}$, u, and $\partial^\varphi u$ as

$$\partial_t^\varphi \partial_x^\varphi u = (-A^{-1})u_{(2)}, \qquad \partial_x^\varphi \partial_x^\varphi u = (-A)^2 u_{(2)}. \tag{64}$$

Applying $\partial_t^\varphi \partial_t^\varphi$ to the first equation in (56) and using the above relations, we obtain

$$\partial_t^\varphi u_{(2)} + A\partial_x^\varphi u_{(2)} = 0,$$

This is an equation for $u_{(2)}$. We proceed to derive a boundary condition for $u_{(2)}$ and an evolution equation for $\underline{x}$. Differentiating the boundary condition $u = u_{\rm i}$ on $x = 0$ with respect to t twice and using the relation $\partial_t = \partial_t^\varphi + (\partial_t\varphi)\partial_x^\varphi$, we have

$$\partial_t^\varphi \partial_t^\varphi u + 2\underline{\dot{x}}\partial_t^\varphi \partial_x^\varphi u + \underline{\dot{x}}^2 \partial_x^\varphi \partial_x^\varphi u + \underline{\ddot{x}}\partial_x^\varphi u = \partial_t^\varphi \partial_t^\varphi u_{\rm i} + 2\underline{\dot{x}}\partial_t^\varphi \partial_x^\varphi u_{\rm i} + \underline{\dot{x}}^2 \partial_x^\varphi \partial_x^\varphi u_{\rm i} + \underline{\ddot{x}}\partial_x^\varphi u_{\rm i}$$

on $x = 0$, where we used $\partial_t \varphi(t, 0) = \underline{\dot{x}}(t)$. This together with (64) implies

$$(\mathrm{Id} - \underline{\dot{x}}A^{-1})^2 u_{(2)} + \underline{\ddot{x}}(\partial_x^\varphi u - \partial_x^\varphi u_{\rm i}) = g_1(\underline{\dot{x}}, \partial^\varphi \partial^\varphi u_{\rm i}),$$

where

$$g_1(\underline{\dot{x}}, \partial^\varphi \partial^\varphi u_{\rm i}) = \partial_t^\varphi \partial_t^\varphi u_{\rm i} + 2\underline{\dot{x}}\partial_t^\varphi \partial_x^\varphi u_{\rm i} + \underline{\dot{x}}^2 \partial_x^\varphi \partial_x^\varphi u_{\rm i}.$$

Decomposing this relation into the direction $\partial_x^\varphi u - \partial_x^\varphi u_{\rm i}$ and its perpendicular direction, we obtain an evolution equation for $\underline{x}$ as

$$\underline{\ddot{x}} = \chi(\underline{\dot{x}}, u_{(2)}, \partial^\varphi u, \partial^\varphi u_{\rm i}, \partial^\varphi \partial^\varphi u_{\rm i}),$$

where

$$\begin{aligned} &\chi(\underline{\dot{x}}, u_{(2)}, \partial^\varphi u, \partial^\varphi u_{\rm i}, \partial^\varphi \partial^\varphi u_{\rm i}) \\ &\quad = \frac{(\partial_x^\varphi u - \partial_x^\varphi u_{\rm i}) \cdot \big(g_1(\underline{\dot{x}}, \partial^\varphi \partial^\varphi u_{\rm i}) - (\mathrm{Id} - \underline{\dot{x}}A^{-1})^2 u_{(2)}\big)}{|\partial_x^\varphi u - \partial_x^\varphi u_{\rm i}|^2} \end{aligned}$$

and a boundary condition for $u_{(2)}$ as

$$\nu_{(2)} \cdot u_{(2)} = g_{(2)},$$

where $\nu_{(2)} = \nu_{(2)}(\underline{\dot{x}}, \partial_x^\varphi u, \partial_x^\varphi u_{\rm i})$ and $g_{(2)} = g_{(2)}(\underline{\dot{x}}, \partial^\varphi u, \partial^\varphi u_{\rm i}, \partial^\varphi \partial^\varphi u_{\rm i})$ are defined by

$$\begin{cases} \nu_{(2)} = ((\mathrm{Id} - \underline{\dot{x}}A^{-1})^2)^{\rm T}((\partial_x^\varphi u - \partial_x^\varphi u_{\rm i})^\perp), \\ g_{(2)} = (\partial_x^\varphi u - \partial_x^\varphi u_{\rm i})^\perp \cdot g_1(\underline{\dot{x}}, \partial^\varphi u, \partial^\varphi \partial^\varphi u_{\rm i}). \end{cases} \tag{65}$$

Concerning a boundary condition for u, we have already said that the condition $u = u_{\rm i}$ on $\{x = 0\}$ should be considered as the union of a scalar boundary condition and of an evolution equation for the free boundary $\underline{x}$. In order to be able to use previous results, we would like to write the scalar boundary condition in the form $\nu \cdot u = g$. However, we have a high degree of freedom for choosing the vector ν. From the considerations of §3.1.1, the most convenient choice is $\nu = \underline{\nu}$, where $\underline{\nu} = \mathbf{e}_+$.
As before, we introduce the matrix $\mathcal{A}(\partial\varphi) = (\partial_x \varphi)^{-1}(A - (\partial_t \varphi)\mathrm{Id})$. The eigenvalues of this matrix are $(\partial_x \varphi)^{-1}(\pm\lambda_\pm - \partial_t \varphi)$, whereas the corresponding eigenvectors are $\mathbf{e}_\pm$ which do not depend on $\partial\varphi$. Summarizing the above arguments, the initial value problem (56)–(57) yields the following:

$$\begin{cases} \partial_t u + \mathcal{A}(\partial\varphi)\partial_x u = 0 & \text{in} \quad \Omega_T, \\ u_{|_{t=0}} = u^{\rm in}(x) & \text{on} \quad \mathbb{R}_+, \\ \underline{\nu} \cdot u_{|_{x=0}} = \underline{\nu} \cdot u_{\rm i}{}_{|_{x=0}} & \text{on} \quad (0, T), \end{cases} \tag{66}$$

together with

$$\begin{cases} \partial_t u_{(2)} + \mathcal{A}(\partial\varphi)\partial_x u_{(2)} = 0 & \text{in} \quad \Omega_T, \\ u_{(2)|_{t=0}} = u^{\text{in}}_{(2)}(x) & \text{on} \quad \mathbb{R}_+, \\ \nu_{(2)} \cdot u_{(2)|_{x=0}} = g_{(2)|_{x=0}} & \text{on} \quad (0,T), \end{cases} \tag{67}$$

and an equation for the evolution of the free boundary given by

$$\begin{cases} \ddot{\underline{x}} = \chi(\dot{\underline{x}}, u_{(2)}, \partial^\varphi u, \partial^\varphi u_{\text{i}}, \partial^\varphi \partial^\varphi u_{\text{i}})_{|_{x=0}} & \text{for} \quad t \in (0,T), \\ \underline{x}(0) = 0, \quad \dot{\underline{x}}(0) = x^{\text{in}}_{(1)}, \end{cases} \tag{68}$$

where the initial data $u^{\text{in}}_{(2)}$ and $x^{\text{in}}_{(1)}$ should be chosen appropriately for the equivalence of (66)–(68) with (56)–(57) and will be given in the next subsection.

Remark 6 It is essential to differentiate (56) twice in time to derive a system with quasilinear boundary conditions. For example, the first derivative $u_{(1)} = \partial_t^\varphi u$ satisfies a boundary condition

$$(A^{-1}u_{(1)} + \partial_x^\varphi u_{\text{i}})^\perp \cdot (u_{(1)} - \partial_t^\varphi u_{\text{i}})_{|_{x=0}} = 0 \quad \text{on} \quad (0,T),$$

which is still nonlinear in $u_{(1)}$.

Step 2. Existence of the solution $(u, u_{(2)}, \underline{x})$ to the reduced system (66)–(68) with the diffeomorphism φ given by (59) under an additional assumption $m \geq 4$. This can be obtained through an iterative scheme similar to the one used to prove Theorem 5. One must check however that the vector $\nu_{(2)}$ is such that **ii** in Assumption 1 is satisfied. It is actually possible to show with some linear algebra [18] that

$$|\nu_{(2)} \cdot \mathbf{e}_+| = \frac{(\lambda_+ - \dot{\underline{x}})^3}{\lambda_+^2} \frac{|\partial_x^\varphi u - \partial_x^\varphi u_{\text{i}}|}{|(\dot{\underline{x}}\text{Id} - A)^{\text{T}}\mu|} |\mu \cdot \mathbf{e}_+|.$$

so that this property is essentially equivalent to the positivity of $|\mu \cdot \mathbf{e}_+|$, where μ is a unit vector in the unique direction along which the quantity $\partial_t^\varphi u + A\partial_x^\varphi u$ is continuous across the boundary.

Step 3. The solution $(u, u_{(2)}, \underline{x})$ to (66)–(68) constructed in Step 2 is in fact a solution to (56)–(57) and satisfies $\partial_t^\varphi \partial_t^\varphi u = u_{(2)}$. Denoting $\widetilde{u}_{(2)} = \partial_t^\varphi \partial_t^\varphi u$, one can prove that $v_{(2)} = \widetilde{u}_{(2)} - u_{(2)}$ is a solution to the initial boundary value problem

$$\begin{cases} \partial_t v_{(2)} + \mathcal{A}(\partial\varphi)\partial_x v_{(2)} = 0 & \text{in} \quad \Omega_{T_1}), \\ v_{(2)|_{t=0}} = 0 & \text{on} \quad \mathbb{R}_+, \\ \widetilde{\nu}_{(2)} \cdot v_{(2)|_{x=0}} = 0 & \text{on} \quad (0,T_1), \end{cases}$$

where $\widetilde{\nu}_{(2)} = ((\text{Id} - \dot{\underline{x}}A^{-1})^2)^{\text{T}}\underline{\nu}$. Here, we have

$$\widetilde{\nu}_{(2)} \cdot \mathbf{e}_+ = \Big(1 - \frac{\dot{\underline{x}}}{\lambda_+}\Big)\mathbf{e}_+ \cdot \mathbf{e}_+,$$

which is not zero. Therefore, we can apply Theorem 4 to the above problem and the uniqueness of the solution gives $v_{(2)} = 0$, that is, $\widetilde{u}_{(2)} = u_{(2)}$.

We proceed to show the boundary condition in (56). Putting $w(t) = (u - u_{\rm i})_{|x=0}$ we have

$$\ddot{w} = \big((\mathrm{Id} - \underline{\dot{x}}A^{-1})^2\widetilde{u}_{(2)} + \underline{\ddot{x}}(\partial_x^\varphi u - \partial_x^\varphi u_{\rm i}) - g_1(\underline{\dot{x}}, \partial^\varphi\partial^\varphi u_{\rm i})\big)_{|x=0} = 0.$$

The compatibility conditions implies $w_{|t=0} = \dot{w}_{|t=0} = 0$. Therefore, we obtain $w = 0$, that is, $u = u_{\rm i}$ on $x = 0$, so that $(u, \underline{x})$ is in fact the solution to (56)–(57). Uniqueness of the solution follows from that of the reduced problem (66)–(68).

Step 4. One can weaken the assumption $m \geq 4$ and replace it by $m \geq 2$. We will not go here through this technical step, the main idea is that we can use again the reduced system (66)–(68), but use the fact that we now know that $\partial_t^\varphi\partial_t^\varphi u = u_{(2)}$ and use this relation to obtain an additional regularity of u.

4 Mathematical analysis of some initial boundary value problems for dispersive perturbations of hyperbolic systems

As seen in the previous section, there is a general theory to address hyperbolic initial boundary value problems. Unfortunately, if we add a dispersive perturbation to these systems, this theory cannot be used anymore. At this day, there is no general theory to handle such dispersive perturbations of hyperbolic systems and only some particular cases have been considered, such as in [34] with homogeneous boundary conditions, [10, 2] for a particular class of Boussinesq systems (the Bona-Smith family) where a regularizing dispersion is also present in the first equation, or [24] for the shoreline problem (vanishing depth). Of particular interest is [3] where the author proposes a generalization of Kreiss symmetrizers to cover certain linear dispersive equations (that do not contain the linear version of the Boussinesq systems considered here). Transparent boundary conditions have also been considered for the linear problem for scalar equations (such as KdV or BBM) as well as for the linearization of a Boussinesq system around the rest state [7, 6]. Here, we will describe the approach developed in [25] and [13] and restrict our attention to two specific variants of Boussinesq systems. The first one is the standard Boussinesq-Abbott system and is considered in §4.1 below, where the local well-posedness is established. A drawback of this result is that the existence time provided shrinks to zero in the dispersionless limit. We address this issue in §4.2 where a wave-structure interaction problem is considered for another Boussinesq system. A precise analysis of the role of *dispersive boundary layers* allows one to obtain a uniform existence time similar to the one classically obtained in the case without boundary. Throughout this section, we try to point out the main differences with the hyperbolic case treated in the previous section.

4.1 The initial boundary value problem for the Boussinesq-Abbott system

We shall consider in this section the initial boundary value problem formed by the Boussinesq-Abbott equations on $t > 0$, $x > 0$,

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t q + \partial_x\big(\varepsilon \frac{1}{h} q^2 + \frac{1}{2\varepsilon} h^2\big) = 0, \end{cases} \tag{69}$$

and a boundary condition at $x = 0$. We shall either consider a boundary condition on q,

$$q_{|x=0}(t) = g(t), \tag{70}$$

or a boundary condition on ζ (also referred to "generating boundary condition" because it is commonly used to generate an incoming swell at the entrance of the numerical model),

$$\zeta_{|x=0}(t) = g(t); \tag{71}$$

finally, initial data are provided for ζ and q at $t = 0$,

$$(\zeta, q)_{|t=0} = (\zeta^{\rm in}, q^{\rm in}). \tag{72}$$

Following [25] (where a numerical implementation of the method is also performed), we first reformulate in §4.1.1 the problem as an evolution equation for ζ and q that contains a source term that can be understood as a a component of a dispersive boundary layer, and whose size is the time derivative of the boundary value of q. When the boundary condition is (70), this quantity is straightforwardly given by $\dot{g}$, but when the boundary condition is (71), the problem is more complex and we address it in §4.1.2. We then show in §4.1.3 that this reformulation of the problem has the structure of an ordinary differential equation and deduce a well-posedness result. The differences between the initial boundary value problems (69)-(72) and the corresponding ones for the nonlinear shallow water equations (formally obtained by setting $\mu = 0$) are then commented in §4.1.4.

4.1.1 Inverting the operator $(1 - \frac{\mu}{3}\partial_x^2)$ on the half line and the dispersive boundary later

Solving the equations (69) on the full line requires the inversion of the operator $(1 - \frac{\mu}{3}\partial_x^2)$, which does not raise any difficulty. The situation is different here since we need to invert this operator on the half-line $(0, \infty)$. We shall denote

$$\underline{q}(t) = q(t, x = 0),$$

and we also need to define the Dirichlet and Neumann inverses of the operator $(1 - \frac{\mu}{3}\partial_x^2)$.

Definition 3 We denote by R_0 and R_1 the inverse of the operator $(1 - \frac{\mu}{3}\partial_x^2)$ with homogeneous Dirichlet and Neumann boundary conditions respectively,

$$R_0 : \begin{array}{ll} L^2(\mathbb{R}_+) & \to H^2(\mathbb{R}_+) \\ g & \mapsto u, \end{array} \qquad \text{where} \qquad \begin{cases} (1 - \frac{\mu}{3}\partial_x^2)u = g, \\ u(0) = 0, \end{cases}$$

and

$$R_1 : \begin{array}{ll} L^2(\mathbb{R}_+) & \to H^2(\mathbb{R}_+) \\ g & \mapsto v, \end{array} \qquad \text{where} \qquad \begin{cases} (1 - \frac{\mu}{3}\partial_x^2)v = g, \\ \partial_x v(0) = 0. \end{cases}$$

We also introduce the boundary operator $\underline{R}_1$ as

$$\underline{R}_1 : \begin{array}{ll} L^2(\mathbb{R}_+) & \to \mathbb{R} \\ g & \mapsto (R_1 g)_{|x=0}. \end{array}$$

Recalling that the ODE

$$Y - \frac{\mu}{3}Y'' = g, \qquad Y(0) = Y_0$$

admits a unique solution in $H^2(\mathbb{R}_+)$ given by

$$Y(x) = (R_0 g)(x) + Y_0 \exp\big(-\frac{x}{\kappa}\big) \quad \text{with} \quad \kappa = \sqrt{\frac{\mu}{3}},$$

the second equation in (69) can be written equivalently under the form

$$\partial_t q = -R_0\partial_x\big(\frac{1}{2\varepsilon}h^2 + \varepsilon\frac{1}{h}q^2\big) + \underline{\dot{q}}\exp\big(-\frac{x}{\kappa}\big). \tag{73}$$

The last term can be seen as part of *a dispersive boundary layer* and plays an important role in the understanding of the dispersionless limit $\mu \to 0$.
It follows that the Boussinesq-Abbott equations (69) can be equivalently reformulated as

$$\begin{cases} \partial_t\zeta + \partial_x q = 0, \\ \partial_t q + R_0\partial_x\big(\varepsilon\frac{1}{h}q^2 + \frac{1}{2\varepsilon}h^2\big) = \underline{\dot{q}}\exp\big(-\frac{x}{\kappa}\big). \end{cases} \tag{74}$$

The next step is therefore to determine $\underline{\dot{q}}$ in terms of the boundary data.

4.1.2 Determining the boundary data $\underline{q}$

We have seen that the reformulation (74) requires the knowledge of the boundary data $\underline{q}$, or more precisely of its time derivative $\underline{\dot{q}}$. This does not raise any difficulty when the boundary condition is imposed on q as in (70), but requires more work in the case of generating boundary condition (71).

Proposition 1 *If* (ζ, q) *are a smooth enough solution of* (69)*, then,*
i. *in the case of the boundary condition* (70) *on* q*, one has*

$$\dot{\underline{q}} = \dot{g};$$

ii. *in the case of the generating boundary condition* (71)*, one recovers* $\underline{q}$ *in terms of* g *and interior values of* (ζ, q) *by solving the ODE*

$$\dot{\underline{q}} - \frac{\varepsilon}{\kappa}\frac{\underline{q}^2}{1+\varepsilon g} = \kappa\ddot{g} + \frac{1}{\kappa}(1+\frac{\varepsilon}{2}g)g - \frac{1}{\kappa}\underline{R}_1\big(\frac{1}{2\varepsilon}(h^2-1) + \varepsilon\frac{1}{h}q^2\big),$$

where $\kappa = \sqrt{\mu/3}$ *and* $\underline{R}_1$ *is the boundary operator introduced in Definition 3.*

Proof The first point being straightforward, we just focus our attention on the second one and reproduce the proof of [25]. Differentiating (73) with respect to x, one obtains

$$\partial_t\partial_x q = -\partial_x R_0 \partial_x\big(\frac{1}{2\varepsilon}(h^2-1) + \varepsilon\frac{1}{h}q^2\big) - \frac{1}{\kappa}\dot{\underline{q}}\exp\big(-\frac{x}{\kappa}\big). \qquad (75)$$

Lemma 2 *For all* $w \in L^2(\mathbb{R}_+)$*, the following identity holds,*

$$R_0\partial_x w = \partial_x R_1 w.$$

Proof (Proof of the lemma) Just remark that if $v = R_1 w$, then one easily gets from the definition of R_1 that

$$\begin{cases}(1-\frac{\mu}{3}\partial_x^2)(\partial_x v) = \partial_x w,\\ (\partial_x v)(0) = 0,\end{cases}$$

so that, by definition of R_0, one has $\partial_x v = R_0\partial_x w$ (note that by classical variational arguments, R_0 is well defined as a mapping $\partial_x L^2(\mathbb{R}_+) \to H^1(\mathbb{R}_+)$). □

Using the first equation of (69) to substitute $\partial_t\partial_x q = -\partial_t^2\zeta$ and the lemma, one then deduces from (75) that

$$-\partial_t^2\zeta = -\partial_x^2 R_1\big(\frac{1}{2\varepsilon}(h^2-1) + \varepsilon\frac{1}{h}q^2\big) - \frac{1}{\kappa}\dot{\underline{q}}\exp\big(-\frac{x}{\kappa}\big).$$

Remarking further that $-\partial_x^2 = \frac{1}{\kappa^2}(1-\frac{\mu}{3}\partial_x^2) - \frac{1}{\kappa^2}$ and recalling that $(1-\frac{\mu}{3}\partial_x^2)R_1 = \mathrm{Id}$, we obtain that

$$\partial_t^2\zeta = \frac{1}{\kappa^2}(R_1 - \mathrm{Id})\big(\frac{1}{2\varepsilon}(h^2-1) + \varepsilon\frac{1}{h}q^2\big) + \frac{1}{\kappa}\dot{\underline{q}}\exp\big(-\frac{x}{\kappa}\big).$$

Taking the trace of this expression at $x = 0$ then yields

$$\ddot{g} + \frac{1}{\kappa^2}(1+\frac{\varepsilon}{2}g)g = \frac{1}{\kappa^2}\big[R_1\big(\frac{1}{2\varepsilon}(h^2-1) + \varepsilon\frac{1}{h}q^2\big)\big]_{|x=0} + \frac{1}{\kappa}\dot{\underline{q}} - \frac{\varepsilon}{\kappa^2}\frac{\underline{q}^2}{1+\varepsilon g},$$

from which the result follows. □

Using once again the lemma to replace $R_0\partial_x$ by $\partial_x R_1$ in (73), it follows from the above that the dimensionless Boussinesq-Abbott equations (69) with boundary conditions (70) or (71) can be equivalently written under the form

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x R_1\big(\frac{1}{2\varepsilon}(h^2-1) + \varepsilon \frac{1}{h} q^2\big) = \underline{Q}(\underline{q}, g, \dot{g}, \ddot{g}, \zeta, q) \exp\big(-\frac{x}{\kappa}\big), \end{cases} \tag{76}$$

where $\underline{q} = q_{|x=0}$ and $\underline{Q} = \underline{Q}(\underline{q}, g, \dot{g}, \ddot{g}, \zeta, q)$ given by

$$\underline{Q} = \begin{cases} \dot{g} & \text{if the boundary condition is (70),} \\ \frac{\varepsilon}{\kappa}\frac{\underline{q}^2}{1+\varepsilon g} + \kappa\ddot{g} + \frac{1}{\kappa}(1+\frac{\varepsilon}{2}g)g - \frac{1}{\kappa}\underline{R}_1\big(\frac{1}{2\varepsilon}(h^2-1) + \varepsilon\frac{1}{h}q^2\big) & \\ & \text{if the boundary condition is (71),} \end{cases} \tag{77}$$

and with the boundary condition (70) or (71). The fact that this new formulation of the equations is well-posed is addressed in the following section.

4.1.3 Well-posedness of the initial boundary value problem

Few results exist regarding the local well-posedness for the initial boundary value problem for Boussinesq systems, except in some special cases such as [2, 34]. The reformulation (76) was used in [25] to establish such a result for the Boussinesq-Abbott system with the generating boundary condition (71). The same proof extends easily to the case of the boundary condition (70) in q, as shown in the theorem below. The key point is the the initial boundary value problem can be recast as a simple ODE on (ζ, q).

Theorem 7 *Let $k \in \mathbb{N}$ and $g \in C^{k+1}(\mathbb{R}_+)$ (resp. $g \in C^{k+2}(\mathbb{R}_+)$) if the boundary condition is* (70) *(resp.* (71)*). Let also $n \in \mathbb{N}$ and $(\zeta^{\text{in}}, q^{\text{in}}) \in H^{n+1}(\mathbb{R}_+) \times H^{n+2}(\mathbb{R}_+)$ be such that $\inf(1+\varepsilon\zeta^{\text{in}}) > 0$. Then there exist $T > 0$ and a unique solution $(\zeta, q) \in C^{k+1}([0,T]; H^{n+1}(\mathbb{R}_+) \times H^{n+2}(\mathbb{R}_+))$ to* (76)-(77) *with initial data $(\zeta^{\text{in}}, q^{\text{in}})$. If moreover*

$$q^{\text{in}}_{|x=0} = g(0) \quad \textit{if the boundary condition is } (70)$$

or

$$\zeta^{\text{in}}{}_{|x=0} = g(0) \quad \textit{and} \quad -\partial_x q^{\text{in}}{}_{|x=0} = \dot{g}(0) \quad \textit{if the boundary condition is } (71),$$

then the boundary condition (70) *or* (71) *is also satisfied for all times.*

Proof Except for the case of the boundary condition (70), we reproduce here the proof of [25]. To prove the first part of the theorem, it is enough to prove that (76)-(77) is actually an ODE on $H^{n+1}(\mathbb{R}^+) \times H^{n+2}(\mathbb{R}^+)$ meeting the requirements of the Cauchy-Lipschitz theorem.
With $U = (\zeta, q)^{\text{T}}$ we can write the equations under the form

$$\partial_t U = \mathcal{F}(t,U) \quad \text{with} \quad \mathcal{F}(t,U) = \begin{pmatrix} -\partial_x q \\ -\partial_x R_1 \mathfrak{f}(\zeta,q) + \underline{Q}(\underline{q},g,\dot{g},\ddot{g},\zeta,q)\exp(-\frac{x}{\kappa}) \end{pmatrix},$$

where we denoted $\mathfrak{f}(\zeta,q) := \frac{1}{2\varepsilon}(h^2-1) + \varepsilon\frac{1}{h}q^2$. By standard product estimates, $(\zeta,q) \in H^{n+1}\times H^{n+2} \mapsto \mathfrak{f}(\zeta,q) \in H^{n+1}$ is regular in a neighborhood of $(\zeta^{\mathrm{in}},q^{\mathrm{in}})$; moreover, $\partial_x R_1$ maps H^{n+1} into H^{n+2} by definition of R_1. It follows easily that $\mathcal{F}(t,U)$ is continuous and locally Lipschitz with respect to the second variable, so that we can apply Cauchy-Lipschitz theorem.
If the boundary condition is (70), we need to prove that $q_{|x=0} = g$ if the equality is satisfied at $t=0$. This is obvious because, taking the trace of the second equation in (76) one has $\frac{d}{dt}q_{|x=0} = \dot{g}$.
If the boundary condition is (71), we now need to check that $\zeta(t,0) = g(t)$ for all time. In order to do so, one computes from the first equation in (76) that $\partial_t^2\zeta = -\partial_t\partial_x q$. Using the second equation to compute $\partial_t\partial_x q$ and taking the trace at $x=0$ one gets (proceeding as in the proof of Proposition 1) that

$$\frac{d^2}{dt^2}(\zeta_{|x=0}) = \ddot{g} + \frac{1}{\kappa^2}\Big(\mathfrak{f}(g,\underline{q}) - \mathfrak{f}(\zeta_{|x=0},\underline{q})\Big).$$

This can be seen as a second order non-autonomous ODE on $\zeta_{|x=0}$ with a right-hand side that is locally Lipschitz with respect to $\zeta_{|x=0}$. There is therefore a unique solution to this ODE satisfying the initial conditions $\zeta_{|x=0}(0) = g(0)$ and $\frac{d}{dt}(\zeta_{|x=0})(0) = -\partial_x q^{\mathrm{in}}(0) = \dot{g}(0)$. This solution is obviously given by $\zeta_{|x=0} = g$, so that the proof is complete. □

4.1.4 The influence of the dispersive term on the initial boundary value problem

As already said, the Boussinesq-Abbott equations (18) are a dispersive perturbation of the nonlinear shallow water equations (17); they only differ from this hyperbolic system by the presence of the "small" dispersive term $-\frac{1}{3}\mu\partial_x^2\partial_t q$ in the momentum equation. Yet, this perturbative term drastically changes the nature of the initial boundary value problem. We review in this section some of these major differences.

Throughout this section, we shall denote by (NSW) the initial boundary value problem associated with the nonlinear shallow water equations (17) and boundary condition

$$(i) \quad q_{|x=0} = g \quad \text{or} \quad (ii) \quad \zeta_{|x=0} = g,$$

and by (B) the initial boundary value problem associated with the Boussinesq-Abbott equations (18) and same boundary condition.

- Nature of the problem. (NSW) is a system of partial differential equations whereas the regularizing properties of the dispersive term make (B) a simple ordinary differential equation.
- Compatibility conditions. In order to have a solution in $C([0,T];H^m(\mathbb{R}_+))$ ($m \geq 2$) for (NSW), the initial and boundary data must satisfy m compatibility conditions

(see §3.1.2). For $m = 2$ there are therefore 2 compatibility conditions,

$$q^{\rm in}_{|x=0} = g(0) \quad \text{and} \quad -\partial_x \mathfrak{f}^{\rm in}_{|x=0} = \dot g(0) \quad \text{if the boundary condition is } (i),$$
$$\zeta^{\rm in}_{|x=0} = g(0) \quad \text{and} \quad -\partial_x q^{\rm in}_{|x=0} = \dot g(0) \quad \text{if the boundary condition is } (ii),$$

where $\mathfrak{f}^{\rm in} = \mathfrak{f}_{|t=0}$ and we recall that $\mathfrak{f} = \varepsilon\frac{1}{h}q^2 + \frac{h^2-1}{2\varepsilon}$. In sharp contrast, for (B), the sole compatibility condition $q^{\rm in}_{|x=0} = g(0)$ is enough to allow a solution in $C^\infty([0,T]; H^{n+1}\times H^{n+2}(\mathbb{R}_+))$ for any $n \in \mathbb{N}$ when the boundary condition is (i). When the boundary condition is (ii), the same compatibility conditions as in the hyperbolic case are needed if $m = 2$, but no additional compatibility condition is needed to allow solutions in $C^\infty([0,T]; H^{n+1}\times H^{n+2}(\mathbb{R}_+))$ for any $n \geq 1$.

- Minimal regularity. Due to its quasilinear structure, (NSW) is well-posed in $C([0,T]; H^m(\mathbb{R}_+))$ with $m \geq 2$, which is the typical integer valued quasilinear index (we recall that for symmetrizable quasilinear systems in $\mathbb{R}^d$, well posedness is possible in $H^s(\mathbb{R}^d)$ with $s > d/2 + 1$, so that in dimension $d = 1$, $m = 2$ is the smallest integer satisfying this condition). For (B), Theorem 7 shows that well-posedness is possible below this threshold, namely, for (ζ, q) in $H^{n+1}\times H^{n+2}(\mathbb{R}_+)$ and $n \in \mathbb{N}$.

These important differences raise some natural questions, and in particular on the dispersionless limit $\mu \to 0$. In the absence of a boundary (when the equations are cast on the full line $\mathbb{R}$), it is quite easy to show that the solutions of the Boussinesq-Abbott system (18) converge as $\mu \to 0$ to the solution of the shallow water equations (17) with same initial data on a time interval $[0, T/\varepsilon]$, for some $T > 0$ independent of ε and μ. Moreover, this convergence can be established in spaces of any regularity index $m \geq 2$ (provided that the initial data are smooth enough of course).

The above comments show that the dispersionless limit will be much more complex to understand in the presence of a boundary at $x = 0$,

- What is the nature of the transition from ODE to PDE as $\mu \to 0$?
- When compatibility conditions (see §3.1.2) are satisfied up to order $m-1$ only, it is not possible to construct solutions of (NSW) in $C([0,T]; H^m(\mathbb{R}_+)^2)$ while Theorem 7 provides solutions of (B) in $C^\infty([0,T]; H^m\times H^{m+1}(\mathbb{R}_+))$. What is the nature of the singularities that appear in the limit $\mu \to 0$ and that prevents the convergence of solutions to (B) towards solutions to (NSW) in spaces of regularity index higher than m ?
- What is the behavior of solutions to (B) that are below the quasilinear regularity index $m = 2$ as $\mu \to 0$?

Answering these questions requires among other things a good understanding of the role played by the dispersive boundary layer introduced in §4.1.1. Some of these issues are addressed in the next section whose motivation lies in the wave-structure interaction problem described in §2.2.3. In particular, Theorem 9 establishes an existence time of order $O(1/\varepsilon)$, uniformly with respect to μ, provided that some compatibility conditions are satisfied. These compatibility are of a new type and adapted to the presence of the dispersion, but are shown to degenerate into the hyperbolic compatibility condition as $\mu \to 0$.

4.2 Waves interacting with a partially immersed object in the Boussinesq regime

As explained in §2.2.3, the interactions of waves, described using a Boussinesq model, with a partially immersed structure can be reduced to the transmission problem (20)-(23). We present here some important steps related to the mathematical analysis of these equations and more specifically to the proof of a well-posedness result over a time scale of order $O(1/\varepsilon)$ which is, as said above, the relevant time scale for Boussinesq models without floating object or boundary.

We recall that the equations are given by

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{1}{3}\mu\partial_x^2)\partial_t q + \varepsilon\partial_x(q^2) + h\partial_x\zeta = 0 \end{cases} \quad \text{on} \quad \mathcal{E}, \tag{78}$$

with $\mathcal{E} = \mathcal{E}_- \cup \mathcal{E}_+ = (-\infty, -R) \cup (R, +\infty)$, and with transmission conditions

$$[\![q]\!] = 0, \qquad \langle q \rangle = q_{\rm i}, \tag{79}$$

and where $q_{\rm i}$ is a function of time only provided by

$$-\alpha \dot{q}_{\rm i} = -\frac{\mu}{3}\partial_t [\![\partial_x q]\!] + [\![\zeta + \varepsilon\frac{1}{2}\zeta^2]\!] \tag{80}$$

for some coefficient $\alpha > 0$. The system is complemented by the initial condition

$$(\zeta, q)_{|t=0} = (\zeta^{\rm in}, q^{\rm in}). \tag{81}$$

We first reformulate this problem in §4.2.1 in order to replace (80) by a more convenient linear equation. We then provide in §4.2.2 a local existence result which is of the same kind as Theorem 7 in the sense that the existence time it provides shrinks to zero as $\mu \to 0$. In §4.1.4, we state the main theorem of [13] establishing existence for a time interval of order $O(1/\varepsilon)$, uniformly with respect to μ, which is the typical existence time scale for the Boussinesq equations cast on a domain without boundary. A sketch of the proof of this theorem is provided in §4.2.4.

4.2.1 Reducing to a linear transmission condition

The fact that the equations (80), which is part of the boundary condition, is nonlinear complicates the analysis; we therefore rewrite the system in (θ, q) variables where

$$\theta = \zeta + \varepsilon\frac{1}{2}\zeta^2 \quad \text{or equivalently } \zeta = \theta + \varepsilon c(\theta) \quad \text{with} \quad c(\theta) = -\frac{2\theta^2}{(1+\sqrt{1+2\varepsilon\theta})^2}.$$

We thus obtain a transmission problem with linear transmission conditions, namely

$$\begin{cases} (1+\varepsilon c'(\theta))\partial_t\theta + \partial_x q = 0, \\ [1 - \frac{\mu}{3}\partial_x^2]\partial_t q + \varepsilon\partial_x(q^2) + \partial_x\theta = 0 \end{cases} \quad \text{on} \quad \mathcal{E}, \tag{82}$$

with the linear transmission conditions

$$[\![q]\!] = 0 \quad \text{and} \quad \langle q\rangle = q_{\rm i}, \tag{83}$$

where

$$-\alpha\dot{q}_{\rm i} = -\frac{\mu}{3}\partial_t[\![\partial_x q]\!] + [\![\theta]\!], \tag{84}$$

and the initial condition

$$(\theta, q)_{|_{t=0}} = (\theta^{\rm in}, q^{\rm in}), \tag{85}$$

where $\theta^{\rm in} = \zeta^{\rm in} + \varepsilon\,\frac{1}{2}(\zeta^{\rm in})^2$.

4.2.2 Reduction to an ODE and local well-posedness

Using the same approach as the one used for Theorem 7, we show that (82)–(85) can be reformulated as an ODE on some functional space, and deduce a local existence result through Cauchy-Lipschitz theorem.

Theorem 8 *For $n \geq 0$, consider initial data $(\theta^{\rm in}, q^{\rm in}) \in H^{n+1}\times H^{n+2}(\mathcal{E})$ satisfying $[\![q^{\rm in}]\!] = 0$, $\inf\{1+2\varepsilon\theta^{\rm in}\} > 0$ and $\inf\{1+\varepsilon c'(\theta^{\rm in})\} > 0$. Then for all $\varepsilon \in [0,1]$ and $\delta > 0$, there is $T > 0$ such that the system (82)–(85) has a unique solution in $C^1([0,T[; H^{n+1}\times H^{n+2}(\mathcal{E}))$, which in addition belongs to $C^\infty([0,T[; H^n\times H^{n+1}(\mathcal{E}))$. Moreover, if T^* denotes the maximal existence time and $T^* < \infty$, one has*

$$\limsup_{T\to T^*}\big\|\theta, q, \partial_x q, 1/(1+\varepsilon c'(\theta))\big\|_{L^\infty([0,T]\times\mathcal{E})} = +\infty. \tag{86}$$

Proof The proof is very similar to the proof of Theorem 7 (with boundary condition on q), so we just give the main elements. Let R_0 denote the inverse of $(1-\frac{\mu}{3}\partial_x^2)$ on $(-\infty,-R)\cup(R,\infty)$ with Dirichlet condition at $\pm R$. The second equation in (82) can be written

$$\partial_t q = -R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big) + \dot{q}_{\rm i}\exp\big(-\frac{1}{\kappa}|x|_R\big), \tag{87}$$

with $\kappa = \sqrt{\mu/3}$ and $|x|_R = (x-R)$ if $x > R$ and $-R-x$ if $x < -R$, and where, according to (83), we have $q_{|_{x=-R}} = q_{|_{x=R}} = q_{\rm i}$. Space differentiating yields

$$\partial_t\partial_x q = -\,\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big) \mp \frac{1}{\kappa}\dot{q}_{\rm i}\exp\big(-\frac{1}{\kappa}|x|_R\big),$$

so that

$$[\![\partial_t \partial_x q]\!] = -[\![\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big)]\!] - \frac{2}{\kappa}\dot{q}_{\rm i}.$$

Using (84) this gives

$$\frac{3}{\mu}[\![\theta]\!] + \frac{6R}{\mu}\dot{q}_{\rm i} = -[\![\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big)]\!] - \frac{2}{\kappa}\dot{q}_{\rm i}.$$

We therefore get the following expression for $\dot{q}_{\rm i}$,

$$\dot{q}_{\rm i} = -\frac{1}{6R + 2\kappa}\big[3[\![\theta]\!] + [\![\partial_x R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big)]\!]\big] =: q_{{\rm i},1}.$$

Plugging into (87) we have therefore the following formulation of the problem

$$\partial_t \begin{pmatrix} \theta \\ q \end{pmatrix} = \begin{pmatrix} -\partial_x q \\ R_0\big(\varepsilon\partial_x(q^2) + \partial_x\theta\big) - q_{{\rm i},1} \exp\big(-\frac{1}{\kappa}|x|_R\big) \end{pmatrix}. \qquad (88)$$

If $n \in \mathbb{N}$, it is quite easy to check that this is an ODE on $H^{n+1} \times H^{n+2}(\mathcal{E})$ so that one can get a solution in a standard way. □

4.2.3 Main result

As commented in §4.1.4 about the result of Theorem 7, the existence time provided by Theorem 8 is not uniform with respect to μ. The following theorem answers the questions raised in §4.1.4 and establishes existence on a time interval of order $O(1/\varepsilon)$ (when $\varepsilon \sim \mu$, which is the typical range of validity for the Boussinesq equations), uniformly with respect to μ. We prefer not to state explicitly at this point the compatibility conditions mentioned in the theorem because they will seem much more natural after some elements of the proof will be given.

Theorem 9 *Let $n \geq 5$. Given $M > 0$, there is $\tau > 0$ such that for all initial data $(\theta_0^{\rm in}, q_0^{\rm in}) \in H^{n+1} \times H^{n+2}(\mathcal{E})$ and parameters $\varepsilon \in [0,1]$ and $\mu \in (0,1]$ satisfying* (104) *and* (105)*, and the compatibility conditions* (103) *or their approximate version* (108) *defined below, there is a unique solution $U = (\theta, q) \in C^1([0,T]; H^{n+1} \times H^{n+2}(\mathcal{E}))$ of* (82)–(85)*, with $T = \tau/(\varepsilon + \mu)$.*

Remark 7 This theorem answers the questions raised in §4.1.4. Contrary to Theorem 8, compatibility conditions are required in the statement of Theorem 9. These compatibility conditions are of a new type in the sense that they are not algebraic conditions but *estimates*. They do not arise as in the hyperbolic case by investigating the regularity of the solution at the corner $x = t = 0$ but when one has to control the time derivatives of the solution at $t = 0$; when they are satisfied, they prevent the appearence of fast variations due to *dispersive boundary layers*. It is quite striking that they converge to the hyperbolic compatibility conditions in the dispersionless limit $\mu \to 0$. The convergence of the solution (82)–(85) towards the solutions of the corresponding hyperbolic transmission problem is then an easy corollary.

4.2.4 Sketch of the proof of Theorem 9

As seen above, it is possible to construct a solution to the transmission problem (82)-(85); however, the existence time and the bounds on the solution are not uniform when the parameters ε and μ become small. We sketch here the strategy developed in [13] to get uniform estimates:

1. Uniform L^2-estimates for the linearized equations. We use in particular the fact that the transmission conditions have some "dissipativity" property that bring some useful control in the main energy estimate.
2. Uniform bounds on the time derivative of the solution of the linearized equations. These estimates can be obtained by time differentiating the equations and using the L^2-estimates. However, the resulting estimates involve the norm of $\partial_t^j \theta$ and $\partial_t^j q$ at $t = 0$. Contrary to what happens for hyperbolic systems, it is not straightforward to control these quantities in terms of Sobolev norms of the initial data. This requires the introduction of compatibility conditions that prevent the creation of *dispersive boundary layers.*
3. Control of space derivatives. In the hyperbolic case, space derivatives can be controlled in terms of time derivatives using the equations. Here, this is not directly possible because of the dispersive terms. The solution is to derive an ODE satisfied by $\partial_x \theta$ and to control it through an analysis of this ODE.
4. Construction of a solution. Finding an appropriate functional space, one can run an iterative scheme using the previous steps.

We also recall that the parameters κ and μ are related through

$$\kappa^2 = \frac{\mu}{3}.$$

Step 1. Dissipativity of the transmission conditions and uniform L^2 estimates. For hyperbolic systems, the proof of Theorem 2 for the well-posedness of initial boundary value problems was presented in §3.1.4. The first step was an L^2 estimate under the assumption that a Kreiss symmetrizer existed. The goal of this symmetrizer was to transform the problem in such a way that the boundary conditions were "dissipative" in the rough sense that they provided some extra information in the energy estimate (in Theorem 2 this extra information was the control of the trace of the solution). This section can be seen as an equivalent result for the initial boundary value problem (actually a transmission problem) (82)–(85). One cannot use the approach developed in §3.1.4 because of the presence of the dispersive terms, but it turns out that the boundary (or transmission) conditions actually have some "dissipativity" properties. This dissipativity property is actually related to the total conservation of the fluid energy (in the exterior and interior domain). The extra information provided by this "dissipativity" is therefore the control of the energy of the fluid located under the solid, which turns out to be related to $q_{\rm i}$.

Consider the following linearized version of (82)–(84) around some reference state $(\underline{\theta}, \underline{q})$, with non-homogeneous source terms

$$\begin{cases} (1+\varepsilon c'(\underline{\theta}))\partial_t\theta + \partial_x q = \varepsilon f, \\ [1-\kappa^2\partial_x^2]\partial_t q + 2\varepsilon \underline{q}\partial_x q + \partial_x\theta = \varepsilon g \end{cases} \qquad \text{on} \quad \mathcal{E}, \tag{89}$$

with transmission conditions

$$[\![q]\!] = 0 \quad \text{and} \quad \langle q\rangle = q_{\rm i}, \tag{90}$$

where

$$\alpha\frac{d}{dt}q_{\rm i} = \kappa^2\frac{d}{dt}[\![\partial_x q]\!] - [\![\theta]\!]. \tag{91}$$

We derive here an a priori bound for the total energy associated to this linear system,

$$E^{\rm tot}_{\underline{U}}(U,q_{\rm i}) = E^{\rm ext}_{\underline{U}}(U) + \frac{1}{2}\alpha {q_{\rm i}}^2$$

where $\frac{1}{2}\alpha {q_{\rm i}}^2$ represents the (linearized) energy of the fluid under the object (up to terms that are constant since the solid is not moving), while $E^{\rm ext}_{\underline{U}}(U)$ corresponds to the full (linearized) energy of the fluid in the exterior domain,

$$E^{\rm ext}_{\underline{U}}(U) = \frac{1}{2}\int_{\mathcal{E}}\big((1+\varepsilon c'(\underline{\theta}))\theta^2 + q^2 + \kappa^2(\partial_x q)^2\big)dx.$$

Proposition 2 *Let $T > 0$ and assume that $\underline{U} \in W^{1,\infty}([0,T]\times\mathcal{E})$ satisfies $[\![\underline{q}]\!] = 0$ and that there are constants $0 < c_0 \leq C_0$ and $m > 0$ such that*

$$c_0 \leq 1+\varepsilon c'(\underline{\theta}) \leq C_0, \qquad |\underline{\theta}, \partial_t\underline{\theta}, \partial_x\underline{q}| \leq m \qquad \text{on } [0,T]\times\mathcal{E}. \tag{92}$$

Then there exists $\gamma = \gamma(m, \frac{1}{c_0})$ such that the solutions $U \in C^1([0,T]; L^2\times H^1(\mathcal{E}))$ of (89)–(91) *satisfy for $0 \leq t \leq T$*

$$E^{\rm tot}_{\underline{U}}\big(U(t), q_{\rm i}(t)\big) \leq e^{\varepsilon\gamma t}\Big(E^{\rm tot}_{\underline{U}}\big(U_{|_{t=0}}, {q_{\rm i}}_{|_{t=0}}\big) + \frac{\varepsilon}{2}\int_0^t e^{-\varepsilon\gamma s}\|(f,g)(s)\|^2_{L^2(\mathcal{E})}ds\Big).$$

Proof Multiplying the first equation of (89) by θ and the second by q, one gets

$$\partial_t e_{\underline{U}} + \partial_x\mathcal{F}_{\underline{U}} = \varepsilon f\theta + \varepsilon g q + \frac{\varepsilon}{2}\partial_t(c'(\underline{\theta}))\theta^2 + \varepsilon(\partial_x\underline{q})q^2$$

with

$$e_{\underline{U}} = \frac{1}{2}(1+\varepsilon c'(\underline{\theta}))\theta^2 + \frac{1}{2}q^2 + \frac{1}{2}\kappa^2(\partial_x q)^2 \quad \text{and} \quad \mathcal{F}_{\underline{U}} = q\big(\theta + \varepsilon\underline{q}q - \kappa^2\partial_x\partial_t q\big).$$

Integrating over $\mathcal{E}$, we then get

$$\begin{aligned}\frac{d}{dt}E^{\rm ext}_{\underline{U}}(U) - [\![\mathcal{F}_{\underline{U}}]\!] \leq &\varepsilon\|(f,g)\|_{L^2(\mathcal{E})}\|(\theta,q)\|_{L^2(\mathcal{E})} \\ &+ \frac{1}{2}\varepsilon\max\Big\{\partial_t c'(\underline{\theta}), 2\partial_x\underline{q}\Big\}\|(\theta,q)\|^2_{L^2(\mathcal{E})}\end{aligned}$$

Because $[\![q]\!] = [\![\underline{q}]\!] = 0$, (91) implies that

$$[\![\mathcal{F}_{\underline{U}}]\!] = \langle q\rangle\big([\![\theta]\!] - \kappa^2[\![\partial_t\partial_x q]\!]\big) = -\frac{1}{2}\alpha\partial_t\langle q\rangle^2.$$

Therefore (recalling that $\langle q\rangle = q_{\rm i}$),

$$\frac{d}{dt}E^{\rm tot}_{\underline{U}}(U,q_{\rm i}) \le \frac{\varepsilon}{2}\|(f,g)\|^2_{L^2(\mathcal{E})} + \frac{1}{2}\varepsilon\gamma\|U\|^2_{L^2(\mathcal{E})}. \tag{93}$$

$$\gamma = \max\{1, \|\partial_t c'(\underline{\theta}), 2\partial_x\underline{q}\|_{L^\infty}\}$$

Estimating $\|U\|^2_{L^2(\mathcal{E})}$ by $\frac{1}{c_0}E^{\rm tot}_{\underline{U}}(U,q_{\rm i})$, the result follows from Gronwall's lemma. □

Step 2. Uniform estimates for the time derivatives and compatibility conditions. Let $U \in C^\infty([0,T], H^{n+1}\times H^{n+2}(\mathcal{E}))$ be a solution of the transmission problem (82)–(84). Let also $U_j = (\theta_j, q_j) = (\partial_t^j\theta, \partial_t^j q) = \partial_t^j U$. Differentiating the equations in time yields

$$\begin{cases}(1+\varepsilon c'(\theta))\partial_t\theta_j + \partial_x q_j = \varepsilon f_{(j)},\\ [1-\kappa^2\partial_x^2]\partial_t q_j + 2\varepsilon q\partial_x q_j + \partial_x\theta_j = \varepsilon g_{(j)}\end{cases} \quad \text{on} \quad \mathcal{E} \tag{94}$$

with transmission conditions

$$[\![q_j]\!] = 0 \quad \text{and} \quad \langle q\rangle = q_{{\rm i},j}, \tag{95}$$

where

$$\alpha\frac{d}{dt}q_{{\rm i},j} = \kappa^2\frac{d}{dt}[\![\partial_x q_j]\!] - [\![\theta_j]\!] \tag{96}$$

and

$$f_{(j)} = -\sum_{k=1}^{j}\binom{k}{j}\partial_t^k(c'(\theta))\theta_{j+1-k} \quad \text{and} \quad g_{(j)} = -2\sum_{k=1}^{j}\binom{k}{j}q_k\partial_x q_{j-k}. \tag{97}$$

We can now estimate the L^2 norm of U_j applying Proposition 2 to (94); up to some technical aspects regarding the control of the commutator terms (one of the difficulties, leading to the presence of the secular term $\varepsilon\gamma t$ in the estimate provided by the Proposition below, is to control the dependence on T in the constant of the Gagliardo-Nirenberg inequality (see [13]) for details), one obtains a control of the following energy

$$\mathfrak{E}(t) = \sum_{0\le j\le n+1}\Big(\big\|\partial_t^j U\big\|^2_{L^2(\mathcal{E})} + \kappa^2\big\|\partial_t^j\partial_x q\big\|^2_{L^2(\mathcal{E})} + \alpha|\frac{d^j}{dt^j}q_{\rm i}|^2\Big). \tag{98}$$

Proposition 3 *Let* $n \in \mathbb{N}$ *and* $T > 0$*, and assume that* $U \in C^\infty([0,T]; H^{n+1}\times H^{n+2}(\mathcal{E}))$ *is a solution of* (82)–(84) *such that there are constants* $0 < c_0 \le C_0$ *and* $0 < m$ *such that*

$$c_0 \le 1 + \varepsilon c'(\theta) \le C_0 \quad \textit{and} \quad |\theta, \partial_t \theta, \partial_t q| \le m \qquad \textit{on } [0,T] \times \mathcal{E}.$$

Then there are constants $C = C(C_0, c_0)$ *and* $\gamma = \gamma(K_0, m, c_0^{-1}, C_0)$ *such that*

$$\mathfrak{E}(t) \le e^{\varepsilon\gamma t}\big(C\mathfrak{E}(0) + \varepsilon\gamma\, t\big),$$

with $K_0 = \sum_{1\le l\le n} \|\partial_t^l \theta_{|t=0}\|_{H^1(\mathcal{E})}$.

Proposition 3 provides a control on the time derivatives of the solution in terms of the L^2-norms of the time derivatives if the solution at $t = 0$ (through the quantity $\mathfrak{E}(0)$ in the right-hand side of the energy estimate). In the hyperbolic case, these quantities can be controlled in terms of Sobolev norms of the initial data following a procedure similar to the one described in §3.1.2 for the compatibility conditions (see Step 3 in the proof of Theorem 3). This procedure must be adapted here because of the presence of the dispersive term in the equations.

In order to seek conditions which ensure uniform bounds for the initial values $U_j^{\rm in}$ of the U_j, we adapt the procedure of §3.1.2 and compute them inductively. Namely one has, when $\kappa > 0$,

$$\begin{cases} \theta_{j+1}^{\rm in} = -\Phi_j^{\rm in} \\ q_{j+1}^{\rm in} = -\mathcal{R}\big(\Gamma_j^{\rm in}, [\![\theta_j^{\rm in}]\!]\big) \end{cases} \tag{99}$$

where

$$\mathcal{R}(\Gamma_j^{\rm in}, [\![\theta_j^{\rm in}]\!]) = R_0\Gamma_j^{\rm in} + \frac{1}{\alpha + 2\kappa}\big(\kappa^2 [\![\partial_x R_0 \Gamma_j^{\rm in}]\!] + [\![\theta_j^{\rm in}]\!]\big) e^{-\frac{1}{\kappa}|x|_R} \tag{100}$$

and

$$\Phi_j^{\rm in} = \sum_{k=0}^{j} C_k^j \Big[\frac{1}{1+\varepsilon c'(\theta)}\Big]_k^{\rm in} \partial_x q_{j-k}^{\rm in}, \qquad \Gamma_j^{\rm in} = \partial_x\big(\theta_j^{\rm in} + \varepsilon \sum_{k=0}^{j} C_k^j q_k^{\rm in}\, q_{j-k}^{\rm in}\big), \tag{101}$$

using systematically the notations $f_j = \partial_t^j f$, $f^{\rm in} = f_{|t=0}$, $f_j^{\rm in} = \partial_t^j f_{|t=0}$. Indeed, $\Phi_j^{\rm in}$ and $\Gamma_j^{\rm in}$ are non linear functions of $(\theta_k^{\rm in}, q_k^{\rm in})$ $(\partial_x \theta_k^{\rm in}, \partial_x q_k^{\rm in})$ for $k \le j$, so that (99) defines inductively $U_j^{\rm in}$ for all j in $H^{n+1}\times H^{n+2}(\mathcal{E})$ if $U_0^{\rm in} \in H^{n+1}\times H^{n+2}(\mathcal{E})$.

Compared to the hyperbolic case, the difficulty is that the relations (99) do not provide a uniform control (with respect to κ) of the space derivatives $\partial_x^k U_{j+1}^{\rm in}$ in terms of space derivatives of the U_j. Indeed, it follows from (99) and the definition of $\mathcal{R}$ that

$$\partial_x^k q_{j+1}^{\rm in} = -\partial_x^k R_0 \Gamma_j^{\rm in} - \frac{1}{\alpha+2\kappa}(\kappa^2 [\![\partial_x R_0 \Gamma_j^{\rm in}]\!] + [\![\theta_j^{\rm in}]\!]) \frac{d^k}{dx^k}\big(e^{-\frac{1}{\kappa}|x|_R}\big),$$

and it appears that both terms in the right-hand-side are of size $O(\kappa^{-k+1/2})$ in $L^2(\mathcal{E})$. The only way one can expect a uniform control of $\partial_x^k q_{j+1}^{\rm in}$ is that these two singular terms cancel one another; this is a necessary condition to avoid the presence of a *dispersive boundary layer* whith fast variations. After some computations (see [13] for details), one can actually deduce from the above identity that

$$\|(q_{j+1},\kappa\partial_x q_{j+1})\|_{H^{n-j}(\mathcal{E})} \leq C\|\Gamma_j^{\rm in}\|_{H^{n-j}(\mathcal{E})} + C\kappa^{j-n+1/2}\big(|A_j| + |B_j|\big), \qquad (102)$$

where

$$A_j = [\![\mathcal{D}^\pm_{n-j}\Gamma_j^{\rm in}]\!] \quad \text{and} \quad B_j = \alpha\langle\mathcal{D}^\pm_{n-j}\Gamma_j^{\rm in}\rangle - 2\kappa\langle\mathcal{P}^\pm_{n-j}\Gamma_j^{\rm in}\rangle + [\![\theta_j^{\rm in}]\!],$$

and with

$$\mathcal{D}^\pm_k f = \sum_{2l<k}(\kappa\partial_x)^{2l} f_{|_{x=\pm R}}, \qquad \mathcal{P}^\pm_k f = \sum_{2l+<k}(\pm\kappa\partial_x)^{2l+1} f_{|_{x=\pm R}}.$$

The singularity in the right-hand-side of (102) as $\kappa \to 0$ is therefore removed provided that the following *compatibility conditions* are satisfied, for some $M > 0$ and all $\kappa \in (0,1]$,

$$\begin{cases} \big|[\![\mathcal{D}^\pm_{n-j}\Gamma_j^{\rm in}]\!]\big| \leq M\kappa^{n-j-1/2} \\ \big|\alpha\langle\mathcal{D}^\pm_{n-j}\Gamma_j^{\rm in}\rangle - 2\kappa\langle\mathcal{P}^\pm_{n-j}\Gamma_j^{\rm in}\rangle + [\![\theta_j^{\rm in}]\!]\big| \leq M\kappa^{n-j-1/2} \end{cases} \quad \text{for } 0 \leq j \leq n-1. \quad (103)$$

Under such conditions, it is possible to control the $U_j^{\rm in}$ in Sobolev spaces, as shown in the following proposition (see [13] for the proof).

Proposition 4 *Given $n \in \mathbb{N}$ and $M > 0$, there is a constant C such that for all initial data $(\theta_0^{in}, q_0^{in}) \in H^{n+1} \times H^{n+2}(\mathcal{E})$ and parameters ε in $[0,1]$ and $\kappa \in (0,1]$ satisfying*

$$[\![q_0^{\rm in}]\!] = 0, \quad \|\theta_0^{\rm in}\|_{H^{n+1}(\mathcal{E})} \leq M, \quad \|(q_0^{\rm in},\kappa\partial_x q_0^{\rm in})\|_{H^{n+1}(\mathcal{E})} \leq M, \qquad (104)$$

$$1 + \varepsilon c'(\theta_0^{\rm in}) \geq M^{-1}, \qquad 1 + 2\varepsilon\theta_0^{\rm in} \geq M^{-1} \qquad (105)$$

and the conditions (103) *for $j < n$, one has*

$$\big\|(\theta_j^{\rm in}, q_j^{\rm in}, \kappa\partial_x q_j^{\rm in})\big\|_{H^{n+1-j}(\mathcal{E})} \leq C \qquad for \ \ 0 \leq j \leq n+1. \qquad (106)$$

Contrary to the hyperbolic compatibility conditions derived in §3.1.2, the compatibility conditions (103) are not easy to check, as the construction of the $U_j^{\rm in}$ involve the nonlocal operator R_0 through the operator $\mathcal{R}$ in (99). For instance, it is not clear to assess wether smooth initial data compactly supported away from the boundary satisfy the compatibility conditions (103). Taking advantage of the fact that the compatibility conditions (103) are *estimates* rather than equations, it is possible as shown in [13] to derive approximate compatibility conditions that do not involve nonlocal operator (and therefore much easier to check) and that are enough to obtain the result of Proposition 4.

We start by noticing that the second equation in (99) can be equivalently written

$$q_{j+1}^{\rm in} = -(1-\kappa^2\partial_x^2)^{-1}\Gamma_j^{\rm in}$$

where the inverse operator is associated to the boundary conditions

$$q^{\text{in}}_{j+1}\,|_{x=\pm R} = \frac{1}{\alpha+2\kappa}\big(\kappa^2[\![\partial_x R_0 \Gamma^{\text{in}}_j]\!] + [\![\theta^{\text{in}}_j]\!]\big).$$

A very naive approximation of this formula is to replace the inverse by its Neumann expansion,

$$q^{\text{in}}_{j+1} \sim -\sum_{2l<n-j} \kappa^{2l}\partial_x^{2l}\Gamma^{\text{in}}_j.$$

Replacing the second equation in (99) by this approximation leads us to approximate $\partial_x^j U^{\text{in}}_j$ by $\widehat{U}_{j,k}$ defined through the induction relation

$$\widehat{U}_{0,k} = \partial_x^k U^{\text{in}} \quad \text{and} \quad \begin{cases} \widehat{\theta}_{j+1,k} = -\widehat{\Phi}_{j,k} \\ \widehat{q}_{j+1,k} = -\displaystyle\sum_{2l<n-k-j} \kappa^{2l}\widehat{\Gamma}_{j,2l+k}\,, \end{cases} \qquad j+k<n, \tag{107}$$

with $\widehat{\Phi}_{j,k}$ and $\widehat{\Gamma}_{j,k}$ defined as $\partial_x^k\Phi_j$ and $\partial_x^k\Gamma_j$ but in terms of the $\widehat{U}_{j,k}$ rather than the $\partial_x^k U_j$. This induction allows to construct $\widehat{U}_{j,k}$ for $j+k \le n$ proceeding as follows: assuming that $\widehat{U}_{j,k}$ is known for $k < n-j+1$, we construct $\widehat{U}_{j+1,k}$ for $k < n-(j+1)+1 = n-j$ using the above relation. It is then natural to define the following approximate compatibility conditions: for some $M>0$ and all $\kappa \in [0,1]$,

$$\begin{cases} \big|[\![\widehat{q}^{\text{in}}_{j+1,0}]\!]\big| \le M\kappa^{n-j-1/2} \\ \big|\alpha\langle \widehat{q}^{\text{in}}_{j+1,0}\rangle + [\![\widehat{\theta}^{\text{in}}_{j,0}]\!] - \kappa^2[\![\widehat{q}^{\text{in}}_{j+1,1}]\!]\big| \le M\kappa^{n-j-1/2} \end{cases} \qquad \text{for } 0 \le j \le n-1. \tag{108}$$

Remark 8 One can check that these conditions converge to the standard hyperbolic compatibility conditions as $\kappa \to 0$.

Step 3. Control of space derivatives. In the proof of Theorem 2, we could control the space derivative inductively in terms of time derivatives. For instance, we had for the first space derivative

$$\partial_x u = A^{-1}(f - \partial_t u).$$

While the first equation of (82) allows us to control trade one x derivative of q in terms of a t derivative of θ (e.g. $\partial_x q = -(1+\varepsilon c'(\theta))\partial_t\theta$) in the same way as in the hyperbolic case, the situation is drastically different for the second equation of (82). Indeed, if we followed this procedure, then we would have to trade one x derivative of θ with two x derivatives and one t derivative of q (e.g. at first order, $\partial_x\theta = -(1-\mu/3\partial_x^3)\partial_t q - \varepsilon\partial_x(q^2)$).

We therefore need another technique to deal with the second equation of (82). We show here how to control $\partial_x\theta$ and refer to [13] for the control of higher order derivatives (one essentially has to implement the same approach to (94)). In the second equation of (82) we use the first equation to replace $\partial_x^2\partial_t q$ by $-\partial_t\partial_x(1+\varepsilon c'(\theta)\partial_t\theta$, obtaining that

$$\partial_x\theta + \kappa^2\partial_t\partial_x\big((1+\varepsilon c'(\theta))\partial_t\theta\big) = -\partial_t q + 2\varepsilon q(1+\varepsilon c'(\theta))\partial_t\theta.$$

We reorganize this equation using that

$$\partial_x\big[(1+\varepsilon c'(\theta))\partial_t\theta\big] = (1+\varepsilon c'(\theta)\partial_t Y + \varepsilon Y\partial_t c'(\theta),$$

with $Y = \partial_x\theta$ so that

$$\partial_t\partial_x\big[(1+\varepsilon c'(\theta))\partial_t\theta\big] = (1+\varepsilon c'(\theta)\partial_t^2 Y + 2\varepsilon\partial_t c'(\theta)\partial_t Y + \varepsilon\partial_t^2 c'(\theta)Y,$$

and Y appears as a solution of the equation

$$a_0\kappa^2\partial_t^2 Y + \varepsilon\kappa^2 a_1\partial_t Y + (1+\varepsilon\kappa a_2)Y = \chi + \varepsilon\psi \tag{109}$$

where

$$a_0 = 1+\varepsilon c', \quad a_1 = 2\partial_t c', \quad a_2 = \kappa\partial_t^2 c', \quad c' = c'(\theta), \tag{110}$$

and

$$\chi = -\partial_t q, \qquad \psi = 2q(1+\varepsilon c'(\theta))\partial_t\theta.$$

For our purposes, we are looking at solutions which remain uniformly bounded (with respect to $\kappa = \sqrt{\mu/3}$ and ε) for times of order $O(\varepsilon^{-1})$. Since the ODE is singular because of the coefficient κ^2 in front of the higher order term, a direct application of Duhamel's formula requires that the right-hand-side is of order $O(\varepsilon\kappa)$ in order for the solution to be uniformly bounded on the $O(\varepsilon^{-1})$ time scale. A refined study, that takes advantage of the oscillating nature of the solutions of the homogeneous equation, is required to handle the contribution of $O(\varepsilon)$ and even $O(1)$ source terms and provide the control we need on $Y = \partial_x\theta$ (see [13] for details).

Step 4. End of the proof. Gathering these elements, it is possible to show that the blow up conditions in Theorem 8 do not occur over a time scale of order $O(1/(\varepsilon+\mu))$, as stated in Theorem 9. This is done using a quite technical iterative scheme that will not be reproduced here; we refer to [13] for the details.

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