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J. Differential Equations 190 (2003) 466–503

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**Journal of  
Differential  
Equations**

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<http://www.elsevier.com/locate/jde>

# Secular growth estimates for hyperbolic systems

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Received January 15, 2002; revised October 4, 2002

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## Abstract

We consider a model problem for the secular growth, which covers all the cases likely to happen in multi-scales BKW expansions for nonlinear hyperbolic systems. This model problem consists in studying the growth in time of the solution of a nonhomogeneous hyperbolic system whose source term is a product of various functions which solve homogeneous hyperbolic systems. The secular growth is due to resonances, that we try to control. When this is not possible, other tools such as decay properties or Strichartz estimates must be used.

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*MSC:* 35B05; 35L40; 35B40; 35B45; 35B34

*Keywords:* Secular growth; Hyperbolic systems; BKW expansions; Interaction of waves; Strichartz estimates

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## 1. Introduction

Secular growth is often the most tricky difficulty to handle in the justification of multiscales BKW expansions. It consists in the growth in the “fast” variables of the profiles used in the expansion. Such a growth has two main drawbacks: it may break the coherence of the ansatz, and it worsens the final error estimates. For instance, the ansatz

$$u^\varepsilon(t, x) = \mathcal{U}_0(\varepsilon t, t, x) + \varepsilon \mathcal{U}_1(\varepsilon t, t, x) + \varepsilon^2 \mathcal{U}_2(\varepsilon t, t, x)$$

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is not coherent for times  $\tau \sim 1$  (i.e.  $t \sim 1/\varepsilon$ ) if  $\mathcal{U}_1(\tau, t, x)$  grows linearly in  $t$  since  $\varepsilon\mathcal{U}_1$  is then of size  $O(1)$ , and therefore ceases to be a corrector. In this particular case, the secular growth must therefore be sublinear to ensure the coherence of the ansatz.

Similarly, suppose that the residual obtained by plugging this ansatz into the system of equations considered reads

$$r^\varepsilon(t, x) = \varepsilon\mathcal{R}_1(\varepsilon t, t, x) + \varepsilon^2\mathcal{R}_2(\varepsilon t, t, x) + \dots;$$

this residual is of size  $O(1)$  and hence not necessarily small if  $\mathcal{R}_1$  grows linearly in  $t$  for times  $t \sim 1/\varepsilon$ . More generally, the faster the secular growth in  $\mathcal{R}_1$ , the larger the residual, and hence the worse the error estimate one obtains.

As far as the system of equations under consideration is of hyperbolic kind (though vaguely as for the Euler equations for the water wave problem), the secular growth can be controlled thanks to the peculiar way in which the profiles depend on the secular variables: they always solve a nonhomogeneous hyperbolic system whose source term is a product of functions solving another homogeneous hyperbolic system (which reduces very often to a transport operator). More precisely, all the cases of secular growth we know are covered by the following model problem, which consists in studying the solution  $u(t, x)$  defined on  $\mathbb{R}^{1+d}$  and with values in  $\mathbb{R}^D$  of

$$\begin{cases} M(\partial)u = g(v^1, \dots, v^J), \\ u|_{t=0} = u_0, \end{cases} \quad \text{with} \quad \begin{cases} N^j(\partial)v^j = 0, \\ v^j|_{t=0} = v_0^j, \quad j = 1, \dots, J, \end{cases} \quad (1)$$

where  $g$  and the operators  $M(\partial)$  and  $N^j(\partial)$ ,  $j = 1, \dots, J$ , satisfy:

**Assumption 1.1.** (i)  $J \in \mathbb{N}^*$ ,  $g$  is a  $J$ -linear mapping defined on  $(\mathbb{R}^D)^J$  and with values in  $\mathbb{R}^D$ .

(ii) The operators  $M(\partial)$  and  $N^j(\partial)$  are of the form

$$M(\partial) = M_0\partial_t + \sum_{k=1}^d M_k\partial_k \quad \text{and} \quad N^j(\partial) = N_0^j\partial_t + \sum_{k=1}^d N_k^j\partial_k, \quad j = 1 \dots J,$$

where  $M_k$  and  $N_k^j$  are real  $D \times D$  symmetric matrices for all  $0 \leq k \leq d$  and  $1 \leq j \leq J$ . Moreover,  $M_0$  and  $N_0^j$ ,  $j = 1 \dots J$ , are positive definite.

**Remark 1.1.** Making in the first system the change of dependent variable to  $M_0^{-1/2}u$  and multiplying the resulting equations by  $M_0^{-1/2}$  preserves the assumptions and brings us back to the case  $M_0 = \text{Id}$ . We therefore assume from now on that  $M_0 = \text{Id}$ , and similarly that  $N_0^j = \text{Id}$  for all  $j = 1, \dots, J$ .

In all the studies where it appeared, the problem of handling the secular growth has been treated in quite different ways. For instance, sublinear secular growth is needed to justify the Schrödinger approximation in diffractive optics [13,17], the

envelope equations for oscillations of large amplitude [7,9], the long wave approximation in ferromagnetic media [8]; sub-square root growth is needed in the long wave approximation for general classes of quasilinear hyperbolic systems [2,3] and for the water wave problem [19,20]; more precise estimates are needed in the short wave approximation [22]. The results, such as the error estimates, given in these papers are very rarely optimal. The aim of this paper is to study precisely the phenomenon of secular growth in order to give sharp results for all the situations likely to arise in the justification of BKW expansions, thus allowing accurate error estimates. As a simple consequence of the results presented here, one can for instance improve the  $o(1)$  error estimates of [13,17] to  $O(\sqrt{\varepsilon})$  or even  $O(\varepsilon)$ , which can be quite useful in practice, since the physical values of  $\varepsilon$  are not always extremely small.

When  $J = 1$  in system (1), it is well known that  $u$  has a sublinear secular growth, provided that the characteristic varieties of the two hyperbolic operators involved have no common sheet. Our first result (Proposition 2.3) is based on the control of the set of resonances and shows that if the initial condition  $v_0^1$  has its Fourier transform in  $L^p(\mathbb{R}^d)$  with  $p$  big enough, then the growth rate can be made more precise. We give examples proving that this rate is sharp. When the Fourier transform of the initial condition  $v_0^1$  vanishes on the set of resonances, Proposition 2.4 shows that no secular growth is possible.

When  $J \geq 2$  and in space dimension  $d = 1$ , Proposition 3.3 shows that the secular growth of  $u$  is at most  $O(\sqrt{t})$  (or  $o(\sqrt{t})$  in a particular case). This rate can be improved (Propositions 3.3 and 3.4) when the initial conditions  $v_0^1, \dots, v_0^J$  have their Fourier transform in  $L^p$  with  $p$  big enough. Here again, we prove that these results are sharp. In Proposition 3.5, we consider the case when the initial conditions have certain quantified decay properties.

When  $J \geq 2$  and in space dimension  $d \geq 2$ , the results of the one-dimensional case can be generalized if all the operators considered are transport operators; otherwise, we use Strichartz estimates to give in Proposition 3.7 a secular growth rate which depends on  $d$  and on  $J$ . When  $J = 2$  and  $d = 3$  for instance, we find that the growth rate is  $O(\sqrt{t})$ , as in the one-dimensional case. When  $(d - 1)(J - 1) \geq 4$ , no secular growth is possible.

The plan of the paper is as follows: the next two subsections introduce the basic geometrical and functional tools we need in this paper; in Section 2, we give sharp estimates in the particular case  $J = 1$  of system (1). What we abusively call the “nonlinear” case  $J \geq 2$  is addressed in Section 3: after treating the one dimensional case (Section 3.2), where all the operators  $M(\partial)$  and  $N^j(\partial)$  can be reduced to transport operators, we address (Section 3.3) the general multi-dimensional case which requires completely different tools based on Strichartz estimates. Two technical proofs have been postponed to two appendices.

**Notation.** (i) Throughout this paper, constants are uniformly denoted by Cst.

(ii) Given any distribution  $f \in \mathcal{S}'(\mathbb{R}^d)'$ , we denote indifferently by  $\hat{f}$  or  $\mathcal{F}f$  the Fourier transform of  $f$ .

(iii) To any real number  $1 \leq p \leq \infty$ , we denote by  $p'$  its conjugate,  $1/p + 1/p' = 1$ .

(iv) Given any real number  $s$ ,  $[s]$  denotes its integer part, i.e. the biggest integer smaller than  $s$ .

1.1. A few properties of the characteristic varieties

First of all, let us recall the definition of the characteristic varieties.

**Definition 1.1.** The characteristic varieties  $\mathcal{C}_M$  and  $\mathcal{C}_{N^j}$  associated with the operators  $M(\partial)$  and  $N^j(\partial)$ ,  $j = 1, \dots, J$ , are defined as

$$\mathcal{C}_M = \left\{ (\tau, \xi) \in \mathbb{R}^{1+d}, \det \left( \tau I + \sum_{k=1}^d M_k \xi_k \right) = 0 \right\},$$

$$\mathcal{C}_{N^j} = \left\{ (\tau, \xi) \in \mathbb{R}^{1+d}, \det \left( \tau I + \sum_{k=1}^d N_k^j \xi_k \right) = 0 \right\},$$

for all  $j = 1, \dots, J$ .

**Remark 1.2.** Since we have supposed in Assumption 1.1 that all the matrices  $M_k$  and  $N_k^j$  are real, it is easy to see that  $(0, 0)$  is a center of symmetry for  $\mathcal{C}_M$  and  $\mathcal{C}_{N^j}$ .

Assuming moreover that  $M(\partial)$  and  $N^j(\partial)$  are strongly hyperbolic, we can parameterize  $\mathcal{C}_M$  and  $\mathcal{C}_{N^j}$  in a simple way. We recall the definition of a strongly hyperbolic operator:

**Definition 1.2** (Strong hyperbolicity). An operator  $L(\partial)$  satisfying the same assumptions as  $M(\partial)$  and  $N^j(\partial)$  is *strongly* hyperbolic (in the direction  $(O\omega)$ ) if the different sheets of  $\mathcal{C}_L$  do not intersect outside the axis  $(O\omega)$ .

**Notation.** If  $M(\partial)$  and  $N^j(\partial)$ ,  $j = 1 \dots J$ , are strongly hyperbolic, then there exists  $\mathbf{m}$  (resp.  $\mathbf{n}^j$ ,  $j = 1 \dots J$ ) functions  $\mu_1 \dots \mu_{\mathbf{m}}$  (resp.  $v_1^j \dots v_{\mathbf{n}^j}^j$ ) continuous on  $\mathbb{R}^d$  and smooth on  $\mathbb{R}^d \setminus \{0\}$  such that

$$\mathcal{C}_M = \bigcup_{m=1}^{\mathbf{m}} \{(\mu_m(\xi), \xi), \xi \in \mathbb{R}^d\} \quad \text{and} \quad \mathcal{C}_{N^j} = \bigcup_{n=1}^{\mathbf{n}^j} \{(v_n^j(\xi), \xi), \xi \in \mathbb{R}^d\}.$$

**Remark 1.3.** The functions  $\mu_m$  and  $v_n^j$  are homogeneous of degree one, while the eigenprojectors  $\pi_m(\xi)$  and  $\pi_n^j(\xi)$  onto  $\ker(\mu_m(\xi) + \sum_k M_k \xi_k)$  and  $\ker(v_n^j(\xi) + \sum_k N_k^j \xi_k)$  are homogeneous of degree zero.

When we have to control the resonances or to use the decay rate in  $L^\infty$  norm of the solutions of any strongly hyperbolic operator  $L(\partial)$ , it is convenient to suppose

that the hypersurfaces  $\{\xi \in \mathbb{R}^d, \lambda(\xi) = 1\}$ , where  $\lambda$  parameterizes any sheet of  $\mathcal{C}_L$ , have everywhere positive curvature. This condition appears very often in the literature. Following Strichartz [21], we call it the *well roundedness* condition.

**Definition 1.3** (Well roundedness). A strongly hyperbolic operator  $L(\partial)$  is called *well rounded* if

$$\forall \xi \neq 0, \quad \text{rank}(\text{Hess } \lambda(\xi)) = d - 1,$$

for all the  $\lambda$  which parameterize a sheet of  $\mathcal{C}_L$ .

A first application of this notion is given in the following lemma which provides information on the relative behavior of the different sheets of the characteristic varieties of two strongly hyperbolic operators  $L_1(\partial)$  and  $L_2(\partial)$  in two particular cases.

**Lemma 1.1.** *Let  $L_1(\partial) \neq L_2(\partial)$  be two strongly hyperbolic operators.*

(i) *If  $\mathcal{C}_{L_1} \cap \mathcal{C}_{L_2} = \{0\}$  then there exists a constant  $c_{L_1, L_2} > 0$  such that*

$$\forall \xi \in \mathbb{R}^d, \quad |\lambda_1(\xi) - \lambda_2(\xi)| \geq c_{L_1, L_2} |\xi|,$$

where  $(\lambda_1, \lambda_2)$  parameterizes any pair of sheets of  $\mathcal{C}_{L_1} \times \mathcal{C}_{L_2}$ .

(ii) *If  $L_1(\partial) = \partial_t + c \cdot \partial_x$  and  $L_2(\partial)$  is well rounded, and if the hyperplane  $\tau + c \cdot \xi = 0$  is tangent to a sheet  $\{(\lambda_2(\xi), \xi), \xi \in \mathbb{R}^d\}$  of  $\mathcal{C}_{L_2}$  then there exists a unique  $\sigma_0 \in \mathbb{S}^{d-1}$  such that  $\nabla_{\sigma_0} \lambda_2 = c$ , and a constant  $\gamma > 0$  such that*

$$\forall \sigma \in \mathbb{S}^{d-1}, \quad |c \cdot \sigma - \lambda_2(\sigma)| \geq \gamma |\sigma_{\perp}|^2,$$

where  $\sigma_{\perp}$  denotes the orthogonal projection of  $\sigma$  on  $\sigma_0^{\perp} = \{\xi \in \mathbb{R}^d, \xi \cdot \sigma_0 = 0\}$ , and  $\mathbb{S}^{d-1}$  the unit sphere of  $\mathbb{R}^d$ .

**Proof.** (i) Due to the fact that  $\lambda_1$  and  $\lambda_2$  are homogeneous of degree one, one has

$$\inf_{\xi \in \mathbb{R}^d} \frac{|\lambda_1(\xi) - \lambda_2(\xi)|}{|\xi|} = \inf_{\sigma \in \mathbb{S}^{d-1}} (|\lambda_1(\sigma) - \lambda_2(\sigma)|).$$

Since  $(\lambda_1 - \lambda_2)$  is continuous and does not vanish on the compact set  $\mathbb{S}^{d-1}$ , we obtain the existence of  $c_{L_1, L_2} > 0$ .

(ii) Since  $L_2(\partial)$  is well rounded,  $\nabla \lambda_2$  is a diffeomorphism from  $\mathbb{S}^{d-1}$  onto  $\nabla \lambda_2(\mathbb{S}^{d-1})$  [21]. Since moreover  $\mathcal{C}_{L_1}$  and  $\mathcal{C}_{L_2}$  are tangent, the existence and uniqueness of  $\sigma_0$  are proved.

Remark now that  $c \cdot \sigma_0 = \lambda_2(\sigma_0)$  since, by definition,  $c = \nabla_{\sigma_0} \lambda_2$  and because  $\lambda_2$  is homogeneous of degree one. Moreover,  $\sigma_0$  is the only element of  $\mathbb{S}^{d-1}$  satisfying this property:  $\forall \sigma \in \mathbb{S}^{d-1}, \sigma \neq \sigma_0$ , one has  $c \cdot \sigma - \lambda_2(\sigma) \neq 0$ . In order to prove this claim, we use the fact that the bounded connected component of  $\mathbb{R}^d \setminus (\nabla \lambda_2(\mathbb{S}^{d-1}))$  is strictly

convex and that  $\lambda_2$  is its support function, as a consequence of the well roundedness of  $L_2(\partial)$  [11]. It follows that if  $\nabla_{\sigma_0} \lambda_2 \cdot \sigma = \lambda_2(\sigma)$  then necessarily  $\nabla_{\sigma_0} \lambda_2 = \nabla_{\sigma} \lambda_2$ , and hence  $\sigma = \sigma_0$ .

Consequently, the inequality stated in the lemma is proved for  $\sigma$  away from a neighborhood of  $\sigma_0$ , provided that  $\gamma$  is taken small enough. For  $\sigma$  near  $\sigma_0$ , write

$$\lambda_2(\sigma) = \lambda_2(\sigma_0) + \nabla_{\sigma_0} \lambda_2 \cdot (\sigma - \sigma_0) + \frac{1}{2} d_{\sigma_0}^2 \lambda_2(\sigma - \sigma_0, \sigma - \sigma_0) + O(|\sigma - \sigma_0|^3).$$

Since  $\lambda_2(\sigma_0) = \nabla_{\sigma_0} \lambda_2 \cdot \sigma_0$  with  $\nabla_{\sigma_0} \lambda_2 = c$ , and since  $d_{\sigma_0}^2 \lambda_2(\sigma_0, \cdot) = 0$ , we then have  $\lambda_2(\sigma) - c \cdot \sigma = \frac{1}{2} d_{\sigma_0}^2 \lambda_2(\sigma_{\perp}, \sigma_{\perp}) + O(|\sigma - \sigma_0|^3)$ , and the estimation of the lemma for  $\sigma$  near  $\sigma_0$  follows from the fact that  $d_{\sigma_0}^2$  is positive definite on  $\sigma_0^{\perp}$ , as a consequence of the well roundedness of  $L_2(\partial)$ .  $\square$

### 1.2. The spaces $E_p^s$

Thanks to the preceding section, we are able to take advantage of the geometric properties of the characteristic varieties to control the secular growth phenomena. The behavior (in space or frequency variables) of the initial conditions can also be exploited. The spaces  $E_p^s$  are introduced to take advantage of the regularity of the Fourier transform of the initial conditions (in as much as elements of  $L^p$ ,  $p \geq 2$ , are more regular than elements of  $L^2$ ).

**Definition 1.4.** Let  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ . The space  $E_p^s(\mathbb{R}^d)$  is defined as

$$E_p^s(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d)', |f|_{E_p^s} := |(1 + |\xi|^2)^{s/2} \hat{f}|_p < \infty\},$$

where  $|\cdot|_p$  denotes the usual norm of  $L^p(\mathbb{R}^d)$ .

**Remark 1.4.** When  $p = 2$ , we have the usual Sobolev space  $E_2^s(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ . Note also that these spaces belong to the class of spaces  $B_{p,k}$  studied by Hörmander [12].

We first give a useful embedding property of these spaces.

**Proposition 1.1.** Let  $s, s' \in \mathbb{R}$ , and  $1 \leq p, p' \leq \infty$ .

If  $p \geq p'$  and  $s > s' + d(1/p' - 1/p)$  then  $E_p^s(\mathbb{R}^d) \subset E_{p'}^{s'}(\mathbb{R}^d)$ , and the injection is continuous.

**Proof.** Suppose that  $1 < p, p' < \infty$ ; usual modifications yield the limiting cases. One has

$$|u|_{E_{p'}^{s'}} = \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^{\frac{p's'}{2}} |\hat{u}(\xi)|^{p'} d\xi \right)^{1/p'}$$

$$\begin{aligned}
 &= \left( \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi|^2)^{\frac{p'(s-s')}{2}}} (1 + |\xi|^2)^{\frac{p's}{2}} |\hat{u}(\xi)|^{p'} d\xi \right)^{1/p'} \\
 &\leq \left| \frac{1}{(1 + |\cdot|^2)^{\frac{p'(s-s')}{2}}} \right|_{\frac{p}{p-p'}}^{1/p'} \left( \int_{\mathbb{R}^d} (1 + |\xi|^2)^{\frac{ps}{2}} |\hat{u}(\xi)|^p d\xi \right)^{1/p} \\
 &\leq \text{Cst } |u|_{E_p^s},
 \end{aligned}$$

thanks to the assumptions made on  $s, s', p$  and  $p'$ , which proves the result.  $\square$

It is quite obvious that the  $E_p^s$  are Banach spaces. The following theorem gives their properties as an algebra. Its proof is given in Appendix A.

**Theorem 1.1.** *Let  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ . Then*

(i) *If  $u, v \in E_1^0 \cap E_p^s$ , then  $uv \in E_1^0 \cap E_p^s$  and*

$$|uv|_{E_p^s} \leq \text{Cst} (|u|_{E_1^0} |v|_{E_p^s} + |u|_{E_p^s} |v|_{E_1^0});$$

(ii) *If  $u, v \in E_p^s$  and if  $s > \frac{d}{p}(p - 1)$ , then  $uv \in E_p^s$  and*

$$|uv|_{E_p^s} \leq \text{Cst} |u|_{E_p^s} |v|_{E_p^s}.$$

**Remark 1.5.** Thanks to these properties of algebra, the spaces  $E_p^s$  are well adapted to the study of semilinear PDE. One can for instance check that the classical cubic Schrödinger equation is well posed in  $E_p^s$  if  $s > \frac{d}{p}(p - 1)$ . Even for some quasilinear equations, as KdV or KP equations, these spaces are of interest [18].

**2. Estimates of the secular growth in the linear case**

We consider here the “linear” version of (1), i.e. the case  $J = 1$  which is written as

$$\begin{cases} M(\partial)u = g(v), \\ u|_{t=0} = u_0, \end{cases} \quad \text{with} \quad \begin{cases} N(\partial)v = 0, \\ v|_{t=0} = v_0, \end{cases} \tag{2}$$

where  $g \in \mathcal{L}(\mathbb{R}^n)$  and the operators  $M(\partial)$  and  $N(\partial)$  are as in Assumption 1.1.

When  $M(\partial)$  and  $N(\partial)$  are strongly hyperbolic, we denote by  $\mu_1, \dots, \mu_m$  the  $\mathbf{m}$  sheets of  $\mathcal{C}_M$  and by  $\nu_1, \dots, \nu_n$  the  $\mathbf{n}$  sheets of  $\mathcal{C}_N$ .

The worst secular growth that one can expect for  $u$  is linear in time. Under simple geometrical assumptions on the characteristic varieties, it is easy to obtain a *sublinear* secular growth, but making this result more precise is more difficult.

2.1. The general case: linear growth

If we denote by  $S_M(\cdot)$  the semigroup associated with  $M(\partial)$ , and similarly  $S_N(\cdot)$  the semigroup associated with  $N(\partial)$ , it is easy to see that the solution  $u$  of (2) is written, for any initial condition  $v_0$  in  $E_p^s(\mathbb{R}^d)$ ,

$$\begin{aligned} u(t) &= S_M(t)u_0 + \int_0^t S_M(t-t')g(v(t')) dt' \\ &= S_M(t)u_0 + \int_0^t S_M(t-t')g(S_N(t')v_0) dt'. \end{aligned} \tag{3}$$

Since both  $S_M(t)$  and  $S_N(t)$  act unitarily on all the  $E_p^s(\mathbb{R}^d)$ , one can deduce from Eq. (3) the following proposition.

**Proposition 2.1.** *Let  $s \in \mathbb{R}$  and  $1 \leq p \leq \infty$ . If  $u_0$  and  $v_0$  are in  $E_p^s(\mathbb{R}^d)$ , then the unique solution  $u \in C(\mathbb{R}^+, E_p^s(\mathbb{R}^d))$  of (2) satisfies*

$$|u(t)|_{E_p^s} \leq |u_0|_{E_p^s} + \text{Cst } t|v_0|_{E_p^s}.$$

**Optimality.** The particular case of (2) given by  $\partial_t u = v$  and  $\partial_t v = 0$  can be explicitly solved:  $u(t) = u_0 + tv_0$ . This expression shows that the linear growth stated in Proposition 2.1 can be obtained.

2.2. Sublinear secular growth

2.2.1. Nonspecific sublinear growth

In the example given above to prove the optimality of Proposition 2.1, one can notice that the characteristic varieties  $\mathcal{C}_M$  and  $\mathcal{C}_N$  are the same. If we do not allow such a situation, i.e. if  $\mathcal{C}_M$  and  $\mathcal{C}_N$  have no common sheet, then the secular growth of  $u$  is sublinear.

**Proposition 2.2.** *Let  $s \in \mathbb{R}$ ,  $1 \leq p < \infty$  and  $u_0, v_0 \in E_p^s(\mathbb{R}^d)$ .*

*If  $\mathcal{C}_M$  and  $\mathcal{C}_N$  have no common sheet (i.e. if  $\mu_m(\xi) \neq \nu_n(\xi)$  for almost all  $\xi \neq 0$ ,  $m = 1, \dots, \mathbf{m}$  and  $n = 1, \dots, \mathbf{n}$ ), then the unique solution  $u \in C(\mathbb{R}^+, E_p^s(\mathbb{R}^d))$  of (2) satisfies*

$$\lim_{t \rightarrow \infty} \frac{1}{t} |u(t)|_{E_p^s} = 0.$$



**Proof.** If  $u_0$  and  $v_0$  are in  $H^s$ , that is, if  $p = 2$ , this proposition can be found in [13,17]. Though the generalization to the case  $1 < p < \infty$  does not raise any difficulty, we give it here (for strongly hyperbolic operators) for the sake of completeness. If  $M(\partial)$  and  $N(\partial)$  are strongly hyperbolic, one has for all  $f \in \mathcal{S}'(\mathbb{R}^d)$ ,

$$\widehat{S_M(t)}f(\xi) = \sum_{m=1}^m e^{i\mu_m(\xi)} \pi_m(\xi) \widehat{f}(\xi) \quad \text{and} \quad \widehat{S_N(t)}f(\xi) = \sum_{n=1}^n e^{i\nu_n(\xi)} \pi_n(\xi) \widehat{f}(\xi),$$

so that one obtains with Eq. (3),

$$\begin{aligned} |u(t)|_{E_p^s} &\leq |u_0|_{E_p^s} + \text{Cst} \sum_{m=1}^m \sum_{n=1}^n \left( \int_{\mathbb{R}^d} \left| \int_0^t e^{-it'(\mu_m(\xi) - \nu_n(\xi))} dt' \right|^p \right. \\ &\quad \left. \times (1 + |\xi|^2)^{ps/2} |\widehat{v_0}(\xi)|^p d\xi \right)^{1/p}. \end{aligned}$$

Thanks to the assumptions made in the statement of the proposition, it is easy to see that the time dependent integral above is  $o(t)$  for almost all  $\xi$ ; a dominated convergence theorem concludes the proof.  $\square$

**Remark 2.1.** (i) The strong hyperbolicity assumption is not necessary here; the important point in the proof is the fact that the set of *resonances* (which is here the projection on the  $\mathbb{R}_\xi^d$  plane of  $\mathcal{C}_M \cap \mathcal{C}_N$ ) has zero Lebesgue measure.

(ii) The case  $p = \infty$  is excluded here; this is because the proof relies on an argument of dominated convergence which cannot be used when  $p = \infty$ .

**Optimality.** We show here that, when  $p = \infty$  and even though  $\mathcal{C}_M$  and  $\mathcal{C}_N$  do not have any common sheet, one cannot expect a sublinear secular growth.

In order to do so, we consider the particular case of system (2) given by  $(\partial_t + \partial_x)u = v$  and  $\partial_t v = 0$ . If  $u_0 = 0$  then

$$\hat{u}(t, \xi) = t e^{-it\xi/2} \text{sinc}(t\xi/2) \hat{v}(\xi),$$

where  $\text{sinc}(x) = \sin(x)/x$ , extended by 1 at 0.

Taking  $v = v_0 = \delta \in E_\infty^0$  yields  $\hat{v}(\cdot) \equiv 1$ , and hence  $|\hat{u}(\xi)|_\infty = t$ , or equivalently,  $|u|_{E_\infty^0} = t$ , so that the secular growth is linear.

### 2.2.2. Specific sublinear growth

At this stage, we have seen that the general linear secular growth is in fact only sublinear when  $\mathcal{C}_M$  and  $\mathcal{C}_N$  have no common sheet. In the proof of Proposition 2.2, one can see that the secular growth is due to *resonances*, i.e. to all those  $\xi$  satisfying  $\phi(\xi) := \mu_m(\xi) - \nu_n(\xi) = 0$ . This set is a real algebraic submanifold of dimension at most  $d - 1$ .

When no more can be said about it, we can only use the fact that it is of zero Lebesgue measure to obtain the result of Proposition 2.2. But in some particular cases, it is possible to know this set more precisely, as well as the order of

cancellation of  $\phi$  on it. The cases we present in the following proposition cover almost all the configurations one can meet in physical examples. The most simple is when  $\mathcal{C}_M \cap \mathcal{C}_N = \{0\}$  since the set of resonances then reduces to  $\{0\}$ . When the set of resonances is not trivial, transversality of the operators is used to control the order of cancellation of the resonance function  $\phi$ .

**Definition 2.1.** Let  $L_1(\partial)$  and  $L_2(\partial) \neq L_1(\partial)$  be two strongly hyperbolic operators such that  $\mathcal{C}_{L_1} \cap \mathcal{C}_{L_2} \neq \{0\}$ .

(i) We say that  $L_1(\partial)$  is transverse to  $L_2(\partial)$  if whenever one sheet of  $\mathcal{C}_{L_1}$  intersect nontrivially a sheet of  $\mathcal{C}_{L_2}$ , these two sheets are transverse in the ordinary sense of transversality of manifolds.

(ii) We say that  $L_1(\partial)$  is tangent to  $L_2(\partial)$  if whenever one sheet of  $\mathcal{C}_{L_1}$  intersect nontrivially a sheet of  $\mathcal{C}_{L_2}$ , these two sheets are tangent in the ordinary sense of tangency of manifolds.

**Proposition 2.3.** Suppose that  $M(\partial) \neq N(\partial)$  are strongly hyperbolic. Let  $s, \alpha \in \mathbb{R}$ ,  $1 \leq p, r < \infty$ . Then the following holds:

(i) When  $\mathcal{C}_M \cap \mathcal{C}_N = \{0\}$ : the unique solution  $u \in C(\mathbb{R}^+, E_p^s(\mathbb{R}^d))$  of (2) satisfies

$$|u(t)|_{E_p^s} \leq \begin{cases} |u_0|_{E_p^s} + \text{Cst } t^{1-\frac{d}{pr}} |v_0|_{E_{pr}^s} & \text{if } pr > d \text{ (and } \alpha = 0), \\ |u_0|_{E_p^s} + \text{Cst}(1 + \ln t) |v_0|_{E_{pr}^{s+2\alpha}} & \text{if } pr = d \text{ and } \alpha > 0, \\ |u_0|_{E_p^s} + \text{Cst} |v_0|_{E_{pr}^{s+2\alpha}} & \text{if } pr < d \text{ and } 2\alpha + 1 > d/pr, \end{cases}$$

for all  $t \geq 1$  and  $u_0 \in E_p^s(\mathbb{R}^d)$  and  $v_0 \in E_{pr}^{s+2\alpha}(\mathbb{R}^d)$ .

(ii) When  $\mathcal{C}_M \cap \mathcal{C}_N \neq \{0\}$ , and  $M(\partial)$  is transverse to  $N(\partial)$ : for all  $u_0 \in E_p^s(\mathbb{R}^d)$  and  $v_0 \in E_{pr}^{s+2\alpha}(\mathbb{R}^d)$ , one has

$$|u(t)|_{E_p^s} \leq |u_0|_{E_p^s} + \text{Cst } t^{1-\frac{1}{pr}} |v_0|_{E_{pr}^{s+2\alpha}}, \quad \text{if } pr > 1 \text{ and } 2\alpha pr > d - 1;$$

If moreover both  $M(\partial)$  and  $N(\partial)$  are transport operators, then

$$|u(t)|_{E_p^s} \leq |u_0|_{E_p^s} + \text{Cst}(1 + \ln t) |v_0|_{E_{pr}^{s+2\alpha}} \quad \text{if } pr = 1 \text{ and } 2\alpha > d - 1.$$

(iii) When one of the operators  $M(\partial)$  and  $N(\partial)$  is a transport operator while the other is well rounded, and when these two operators are tangent: if  $4\alpha pr > d + 1$  and  $2pr > d - 1$ , then

$$|u(t)|_{E_p^s} \leq |u_0|_{E_p^s} + \text{Cst } t^{1-\frac{d-1}{2pr}} |v_0|_{E_{pr}^{s+2\alpha}},$$

for all  $u_0 \in E_p^s(\mathbb{R}^d)$  and  $v_0 \in E_{pr}^{s+2\alpha}(\mathbb{R}^d)$ .

The proof of the proposition relies strongly on the following two lemma.

**Lemma 2.1.** *Let  $1 \leq q < \infty$  and  $\varphi$  be a smooth, bounded function defined on  $\mathbb{R}$  and such that  $\varphi(0) = 0$ . For all  $\xi \in \mathbb{R}^d \setminus \{0\}$ , denoting  $\Psi_t(\xi) := \frac{\varphi(t|\xi|)}{|\xi|}$  and  $w_\alpha(\xi) := (1 + |\xi|^2)^{\alpha/2}$ , one has*

$$\begin{aligned} |\psi_t|_q &\leq \text{Cst } t^{1-\frac{d}{q}} \quad \text{if } q > d, \\ |\psi_t w_{-2\alpha}|_q &\leq \text{Cst}(1 + \ln t) \quad \text{if } q = d \text{ and } \alpha > 0, \\ |\psi_t w_{-2\alpha}|_q &\leq \text{Cst} \quad \text{if } q < d \text{ and } 2\alpha + 1 > d/q, \end{aligned}$$

for all  $t \geq 1$ .

**Proof.** First consider the case  $q > d$ . One then has

$$\begin{aligned} |\psi_t|_q &= \left( \int_{\mathbb{R}^d} \frac{|\varphi(t|\xi|)|^q}{|\xi|^q} d\xi \right)^{1/q} \\ &= t^{1-\frac{d}{q}} \left( \int_{\mathbb{R}^d} \frac{|\varphi(|\xi|)|^q}{|\xi|^q} d\xi \right)^{1/q} \\ &\leq \text{Cst } t^{1-\frac{d}{q}}, \end{aligned}$$

since the condition  $q > d$  implies the convergence of the above integral.

For  $q \leq d$ , one has

$$\begin{aligned} |\psi_t w_{-2\alpha}|_q &= \left( \int_{\mathbb{R}^d} \frac{|\varphi(t|\xi|)|^q}{|\xi|^q} \frac{1}{(1 + |\xi|^2)^{\alpha q}} d\xi \right)^{1/q} \\ &\leq \text{Cst} + \left( \int_{|\xi| \leq 1} \frac{|\varphi(t|\xi|)|^q}{|\xi|^q} d\xi \right)^{1/q}, \end{aligned}$$

provided that  $2\alpha + 1 > d/q$ . But

$$\begin{aligned} \left( \int_{|\xi| \leq 1} \frac{|\varphi(t|\xi|)|^q}{|\xi|^q} d\xi \right)^{1/q} &\leq t^{1-\frac{d}{q}} \left( \int_{|\xi| \leq t} \frac{|\varphi(|\xi|)|^q}{|\xi|^q} d\xi \right)^{1/q} \\ &\leq \text{Cst } t^{1-\frac{d}{q}} \left( 1 + \left( \int_1^t \rho^{d-1-q} d\rho \right)^{1/q} \right), \end{aligned}$$

which is bounded by a constant if  $d > q$  and by  $\text{Cst}(1 + \ln t)$  if  $d = q$ .  $\square$

**Lemma 2.2.** *Let  $1 \leq q < \infty$ . For all  $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ , define  $\psi_t(\xi) := \frac{e^{-it\xi_1} - 1}{\xi_1}$  (or  $\psi_t(\xi) := -it$  if  $\xi_1 = 0$ ) and  $w_\alpha(\xi) := (1 + |\xi|^2)^{\alpha/2}$ . Then*

$$|\psi_t w_{-2\alpha}|_q \leq \text{Cst } t^{1-\frac{d}{q}} \quad \text{if } 2\alpha q > d - 1,$$

$$|\psi_t w_{-2\alpha}|_1 \leq \text{Cst}(1 + \ln t) \quad \text{if } 2\alpha > d - 1,$$

for all  $t \geq 1$ .

**Proof.** We only prove the case  $q = 1$ , slight modifications yielding the case  $q > 1$ . Denoting  $\xi_{II} := (\xi_2, \dots, \xi_d)$ , one has

$$\begin{aligned} |\psi_t w_{-2\alpha}|_1 &\leq \int_{\mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} \left| \frac{e^{-it\xi_1} - 1}{\xi_1} \right| \frac{1}{(1 + |\xi|^2)^\alpha} d\xi_1 \right) d\xi_{II} \\ &\leq \int_{\mathbb{R}^{d-1}} \left( \frac{1}{(1 + |\xi_{II}|^2)^\alpha} \int_{|\xi_1| \leq 1} \left| \frac{e^{-it\xi_1} - 1}{\xi_1} \right| d\xi_1 \right. \\ &\quad \left. + 2 \int_{|\xi_1| \geq 1} \frac{1}{|\xi_1|(1 + |\xi|^2)^\alpha} d\xi_1 \right) d\xi_{II}. \end{aligned}$$

Following the same line as in the case  $d = q$  of Lemma 2.1, one obtains

$$|\psi_t w_{-2\alpha}|_1 \leq \text{Cst}(1 + \ln t) \int_{\mathbb{R}^{d-1}} \frac{1}{(1 + |\xi_{II}|^2)^\alpha} d\xi_{II} + \int_{\xi \in \mathbb{R}^d, |\xi_1| \geq 1} \frac{1}{|\xi_1|(1 + |\xi|^2)^\alpha} d\xi.$$

Since these two integrals converge when  $2\alpha > d - 1$ , the lemma is proved.  $\square$

**Proof.** As in the proof of Proposition 2.2, one can prove that the solution  $u$  of (2) satisfies, for all  $1 \leq p < \infty$ ,

$$\begin{aligned} |u(t)|_{E_p^s} &\leq |u_0|_{E_p^s} + \text{Cst} \sum_{m,n} \left[ \int_{\mathbb{R}^d} \left| \int_0^t e^{-it'(\mu_m(\xi) - \nu_n(\xi))} dt' \right|^p (1 + |\xi|^2)^{ps/2} |\widehat{v}_0(\xi)|^p d\xi \right]^{1/p} \\ &:= |u_0|_{E_p^s} + \text{Cst} \sum_{m,n} A_{m,n}^{1/p}. \end{aligned} \tag{4}$$

Now, by Hölder’s inequality and for all  $1 \leq r < \infty$ ,

$$\begin{aligned} A_{m,n} &\leq \text{Cst} \left[ \int_{\mathbb{R}^d} \left| \int_0^t e^{-it'(\mu_m(\xi) - \nu_n(\xi))} dt' \right|^{pr} \frac{1}{(1 + |\xi|^2)^{\alpha pr}} d\xi \right]^{1/r} |v_0|_{E_{pr}^{s+2\alpha}}^p \\ &:= B_{m,n}^{1/r} |v_0|_{E_{pr}^{s+2\alpha}}^p. \end{aligned} \tag{5}$$

In order to give an upper bound for  $B_{m,n}$ , remark that

$$\begin{aligned} \left| \int_0^t e^{-it'(\mu_m(\xi) - v_n(\xi))} dt' \right| &= t \left| \frac{2 \sin\left(\frac{t}{2}(\mu_m(\xi) - v_n(\xi))\right)}{t(\mu_m(\xi) - v_n(\xi))} \right| \\ &\leq t \frac{\varphi(t|\mu_m(\xi) - v_n(\xi)|)}{t|\mu_m(\xi) - v_n(\xi)|}, \end{aligned} \tag{6}$$

where  $\varphi$  is a real-valued positive function defined on  $\mathbb{R}^+$  as  $\varphi(r) = 2 \sin(r/2)$  if  $r \in [0, \pi]$  and  $\varphi(r) = 2$  if  $r > \pi$ . We now distinguish the three cases of the proposition.

*Case i:* Since  $r \mapsto \varphi(r)/r$  is decreasing and  $\mathcal{C}_M \cap \mathcal{C}_N = \{0\}$ , one can deduce from Eq. (6) and Lemma 1.1(i) that

$$\left| \int_0^t e^{-it'(\mu_m(\xi) - v_n(\xi))} dt' \right| \leq \frac{\varphi(tc_{M,N}|\xi|)}{c_{M,N}|\xi|},$$

where the positive constant  $c_{M,N}$  is as in Lemma 1.1. Applying Lemma 2.1 with  $\varphi$  defined as in (6) and  $q = pr$ , and using estimates (4) and (5) yields the result.

*Case ii:* We suppose that  $\mu_m$  and  $v_n$  parameterize a sheet of  $\mathcal{C}_N$  and  $\mathcal{C}_M$ , respectively, which intersect nontrivially. Otherwise, the (better) estimates of (i) hold. From Eqs. (5), (6), and using the fact that  $\mu_m$  and  $v_n$  are homogeneous of degree one, one obtains

$$B_{m,n} = B_n \leq t^{pr} \int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{\varphi(t\rho|\mu_m(\sigma) - v_n(\sigma)|)^{pr}}{t\rho|\mu_m(\sigma) - v_n(\sigma)|^{pr}} \frac{\rho^{d-1}}{(1 + \rho^2)^{qpr}} \Sigma(d\sigma) d\rho, \tag{7}$$

where  $\mathbb{S}^{d-1}$  denotes the unit sphere of  $\mathbb{R}^d$  and  $\Sigma$  its Lebesgue measure.

Since  $M(\partial)$  is transverse to  $N(\partial)$ , there exists  $\beta > 0$  such that  $|\nabla\mu_m(\sigma) - \nabla v_n(\sigma)| \geq \beta$  for all  $\sigma \in \mathbb{S}^{d-1}$ . There exists therefore a finite (because  $\mathbb{S}^{d-1}$  is a compact set) recovering of  $\mathbb{S}^{d-1}$  by open subsets  $\Omega_1, \dots, \Omega_L$  such that on each  $\Omega_l$  there exists a diffeomorphism

$$\begin{aligned} f^l : \Omega_l &\rightarrow f_l(\Omega_l), \\ \sigma &\mapsto f^l(\sigma) = (f_1^l(\sigma) := v_n(\sigma) - \mu_m(\sigma), f_2^l(\sigma), \dots, f_d^l(\sigma)), \end{aligned}$$

where all the  $f_2^l, \dots, f_d^l$  are real valued. Let  $(\chi_l)_l$  be a partition of unity subordinated to  $(\Omega_l)_l$ , and decompose  $B_n$  into  $B_n = B_n^1 + \dots + B_n^L$ , with obvious notations. Since for all  $l \in [1, L]$ ,  $f^l(\text{Supp } \chi_l)$  is a compact set, the changes of variables  $\sigma' = f^l(\sigma)$  and

then  $\sigma''_1 = t\rho\sigma'_1$  yield

$$\begin{aligned} B_n^I &\leq t^{pr} \int_0^\infty \int_{\Omega_t} \left| \frac{\varphi(t\rho|f_1^I(\sigma)|)}{t\rho|f_1^I(\sigma)|} \right|^{pr} \frac{\rho^{d-1}}{(1+\rho^2)^{\alpha pr}} \Sigma(d\sigma) d\rho \\ &\leq \text{Cst } t^{pr} \int_0^\infty \int_{\mathbb{R}} \left| \frac{\varphi(t\rho|\sigma'_1|)}{t\rho\sigma'_1} \right|^{pr} \frac{\rho^{d-1}}{(1+\rho^2)^{\alpha pr}} d\sigma'_1 d\rho \\ &\leq \text{Cst } t^{pr-1} \int_0^\infty \frac{\rho^{d-2}}{(1+\rho^2)^{\alpha pr}} d\rho \times \int_{\mathbb{R}} \left| \frac{\varphi(|\sigma''_1|)}{\sigma''_1} \right|^{pr} d\sigma''_1. \end{aligned}$$

Therefore,  $B_n^I \leq \text{Cst } t^{pr-1}$ , provided that  $2\alpha pr - (d - 2) > 1$  and  $pr > 1$ , and hence  $B_n \leq \text{Cst } t^{pr-1}$ . The end of the proof follows as in the first case.

For the particular case  $pr = 1$  and both  $M(\partial)$  and  $N(\partial)$  are transport operators, note that up to a linear change of variables, the left-hand side of (6) is equal to  $\psi_t(\xi)$  as defined in Lemma 2.2. Therefore, the result is a simple consequence of this lemma and of (4) and (5).

Case iii: We suppose here that  $M(\partial) = \partial_t + c \cdot \partial_x$  and  $N(\partial)$  is well rounded. The case obtained by a permutation of  $M(\partial)$  and  $N(\partial)$  can be treated exactly in the same way.

The proof follows the proof of the second case until Eq. (7). At this point, we use Lemma 1.1(ii) and the fact that  $r \mapsto \varphi(r)/r$  is decreasing to obtain

$$\begin{aligned} B_n &\leq t^{pr} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left| \frac{\varphi(t\rho\gamma|\sigma_\perp|^2)}{t\rho\gamma|\sigma_\perp|^2} \right|^{pr} \frac{\rho^{d-1}}{(1+\rho^2)^{\alpha pr}} \Sigma(d\sigma) d\rho \\ &\leq \text{Cst } t^{pr} \int_0^\infty \int_{B_{d-1}} \left| \frac{\varphi(t\rho\gamma|\sigma_\perp|^2)}{t\rho\gamma|\sigma_\perp|^2} \right|^{pr} \frac{\rho^{d-1}}{(1+\rho^2)^{\alpha pr}} d\sigma_\perp d\rho, \end{aligned}$$

where  $B_{d-1}$  denotes the unit ball of  $\mathbb{R}^{d-1}$ . The change of variables  $\sigma'_\perp = \sqrt{t\rho}\sigma_\perp$  then yields

$$\begin{aligned} B_n &\leq \text{Cst } t^{pr-\frac{d-1}{2}} \int_0^\infty \int_{\mathbb{R}^{d-1}} \left| \frac{\varphi(\gamma|\sigma'_\perp|^2)}{\gamma|\sigma'_\perp|^2} \right|^{pr} \frac{\rho^{d-1-\frac{d-1}{2}}}{(1+\rho^2)^{\alpha pr}} d\sigma'_\perp d\rho \\ &= \text{Cst } t^{pr-\frac{d-1}{2}} \int_0^\infty \frac{\rho^{\frac{d-1}{2}}}{(1+\rho^2)^{\alpha pr}} d\rho \times \int_{\mathbb{R}^{d-1}} \left| \frac{\varphi(\gamma|\sigma'_\perp|^2)}{\gamma|\sigma'_\perp|^2} \right|^{pr} d\sigma'_\perp, \end{aligned}$$

and hence  $B_n \leq \text{Cst } t^{pr-\frac{d-1}{2}}$ , provided that  $2\alpha pr - \frac{d-1}{2} > 1$  and  $2pr > d - 1$ .  $\square$

**Remark 2.2.** (i) The secular growth is lower in point (i) of the proposition than in point (iii), and lower (when  $d \geq 3$ ) in this latter than in the second one. This had to be expected since the cause of the secular growth is the presence of resonances. In point (i) the set of resonances is just a point, while in (ii) it is a submanifold of dimension  $d - 1$ , and in (iii) of dimension 1. Note that when  $d = 2$ , the submanifold of

resonances has the same dimension in the last two cases, but that the secular growth given by the proposition is not the same for both cases. This is because the *order of cancellation* of the function of resonances  $\mu_m(\xi) - \nu_n(\xi)$  on the set of resonances is also important. The higher it is, the bigger the secular growth.

(ii) In the first case of point (i) of the proposition, we have taken  $v_0 \in E_{pr}^{s+2\alpha}$  with  $\alpha = 0$  for the sake of simplicity; in fact,  $\alpha$  can take small negative values (more precisely, we can allow  $2\alpha > d - pr$ ), enhancing a slight regularizing effect.

**Optimality.** (i) We show here that the exponent of  $t$  given in Proposition 2.3(i) is the best that one can expect under the assumptions made in that proposition.

In order to do so, we take  $v_0$  defined as

$$\widehat{v}_0(\xi) = \frac{1}{|\xi|^a} \chi_B(\xi),$$

where  $0 < a < 1/2$  and  $\chi_B$  is the indicatrix function of the closed unit ball  $B$  of  $\mathbb{R}$ . One has  $v_0 \in E_{1/a-\varepsilon}^\infty$  for any  $\varepsilon > 0$  (here  $d = 1$ ).

For the particular case of system (2) given by  $(\partial_t + \partial_x)u = v$  and  $\partial_t v = 0$ , and for  $u_0 = 0$  the solution  $u$  is given by

$$\widehat{u}(t, \xi) = e^{-it\xi} \frac{e^{it\xi} - 1}{i\xi} \widehat{v}_0(\xi),$$

and therefore,  $|u(t)|_{E_2^0} = |u(t)|_{L^2}$  is given by

$$\begin{aligned} |u(t)|_{E_2^0} &= \left( \int_{|\xi| \leq 1} \left| \frac{e^{it\xi} - 1}{\xi} \right|^2 \frac{1}{|\xi|^{2a}} d\xi \right)^{1/2} \\ &= t^{1+a} \left( \int_{|\xi| \leq 1} \left| \frac{e^{it\xi} - 1}{t\xi} \right|^2 \frac{1}{|t\xi|^{2a}} d\xi \right)^{1/2} \\ &= t^{1+a-\frac{1}{2}} \left( \int_{|\xi| \leq t} \left| \frac{e^{i\xi} - 1}{\xi} \right|^2 \frac{1}{|\xi|^{2a}} d\xi \right)^{1/2}. \end{aligned}$$

Since for all  $0 < a < 1/2$ , we have  $\int_{\mathbb{R}} \left| \frac{e^{i\xi} - 1}{\xi} \right|^2 \frac{1}{|\xi|^{2a}} d\xi < \infty$ , we can deduce from the above inequalities that

$$|u(t)|_{E_2^0} \sim \text{Cst } t^{1+a-\frac{1}{2}} \quad \text{for } t \sim \infty. \tag{8}$$

Using the results of Proposition 2.3(i), we would have obtained  $|u(t)|_{E_2^0} = O(t^{1-\frac{1}{2}+\frac{a}{1-aw}})$ . Since the limit exponent obtained by letting  $\varepsilon \rightarrow 0$  is the exponent given in (8), one cannot expect, under the assumptions of Proposition 2.3(i), a general result giving a secular growth with a better rate.

(ii) Under the assumptions of Proposition 2.3(iii), the best growth rate one can expect for the  $H^s$  norm of  $u(t)$  is  $t^{1/2}$  when  $d = 3$ . The example of large corrector for diffractive optics given in [13, Section 7.5] proves that this rate can be obtained.

If, as we have just seen, it is not possible to improve the growth rate given in Proposition 2.3 without any additional assumption, one can however improve this result in the following way.

**Corollary 2.1.** *With the assumptions giving an algebraic secular growth in Proposition 2.3, if  $u_0 \in E_p^s$  and  $v_0 \in E_{pr}^{s+2\alpha}$  (with  $\alpha = 0$  in case (i)), then one has*

$$\lim_{t \rightarrow \infty} \frac{1}{t^\beta} |u(t)|_{E_p^s} = 0,$$

where  $\beta = 1 - \frac{d}{pr}$  in case (i),  $\beta = 1 - \frac{1}{pr}$  in (ii) and  $\beta = 1 - \frac{d-1}{2pr}$  in (iii).

**Proof.** The proof follows the same lines in the three cases. We detail it for (ii) for instance.

Denote by  $\mathcal{R}$  the set of resonances  $\mathcal{R} := \{\xi \in \mathbb{R}^d, \exists m, \exists n, \mu_m(\xi) - \nu_n(\xi) = 0\}$ . We first prove the following claim: if  $\widehat{v}_0 \in \mathcal{D}(\mathbb{R}^d \setminus \mathcal{R})$ , i.e. if  $\widehat{v}_0$  is smooth and compactly supported in  $\mathbb{R}^d \setminus \mathcal{R}$ , then  $|u(t)|_{E_p^s}$  is bounded in time. Indeed, instead of integrating over the full space  $\mathbb{R}^d$  in the expression defining  $B_{m,n}$  in Eq. (5), one can integrate only on the support of  $\widehat{v}_0$ . Therefore, the r.h.s. of (6) can be majorized by  $\frac{2}{\inf_{\xi \in \text{Supp } v_0} |\mu_m(\xi) - \nu_n(\xi)|}$  which is finite thanks to the assumption made on  $v_0$ . The claim then follows from Eqs. (4) and (5).

Remark now that  $\mathbb{A} := \{f \in \mathcal{S}'(\mathbb{R}^d), \widehat{f} \in \mathcal{D}(\mathbb{R}^d \setminus \mathcal{R})\}$  is dense in  $E_{pr}^{s+2\alpha}$ , so that one can find a sequence  $(v_{0,n})_{n \in \mathbb{N}} \in \mathbb{A}^{\mathbb{N}}$  such that  $v_{0,n} \rightarrow v_0$  in  $E_{pr}^{s+2\alpha}$  when  $n \rightarrow \infty$ . We want to prove that for any  $\varepsilon > 0$ , there exists  $t_0$  such that for any  $t > t_0$ , one has  $t^{-\beta} |u(t)|_{E_p^s} < \varepsilon$ . We first write

$$|u(t)|_{E_p^s} \leq |u(t) - u_n(t)|_{E_p^s} + |u_n(t)|_{E_p^s},$$

where  $u_n$  is the solution of (2) with data  $v_{0,n}$  instead of  $v_0$ . Thanks to Proposition 2.3, one has

$$|u(t)|_{E_p^s} \leq \text{Cst}(1 + t^\beta) |v_0 - v_{0,n}|_{E_{pr}^{s+2\alpha}} + |u_n(t)|_{E_p^s}.$$

Let  $n_0$  be big enough to have  $\text{Cst} |v_0 - v_{0,n}|_{E_{pr}^{s+2\alpha}} < \varepsilon/3$ . Now, thanks to the claim proved above, there exists  $t_0$  such that  $t^{-\beta} |u_{n_0}(t)|_{E_p^s} < \varepsilon/2$  when  $t > t_0$ . Possibly taking a bigger  $t_0$ , one can assume that for such times,  $(1 + t^\beta)t^{-\beta} < 3/2$ , and hence that

$$\frac{1}{t^\beta} |u(t)|_{E_p^s} < \varepsilon/2 + \varepsilon/2,$$

which concludes the proof of the corollary.  $\square$



2.3. *Absence of secular growth*

If Proposition 2.3(i) shows that in some cases, no secular growth occurs, in most situations, it does, and we gave sharp results of secular growth. However, one can notice that the example used to prove the optimality of these results relies on the fact that  $\widehat{v}_0$  takes important values near the origin. We now show that if we exclude such a behavior, then no secular growth is possible for  $u$ . Before stating the proposition, we need the following definition.

**Definition 2.2.** Let  $P$  be the real-valued function defined on  $\mathbb{R}$  as  $P(r) = |r|$  if  $|r| \leq 1$  and  $P(r) = 1$  for  $|r| > 1$ . The Fourier multiplier  $\mathbf{P}$  is then defined as  $\mathbf{P}f := \mathcal{F}^{-1} \left( P(|\cdot|) \widehat{f}(\cdot) \right)$ , for any  $f \in \mathcal{S}'(\mathbb{R}^d)$ .

**Proposition 2.4.** *Suppose that  $M(\partial)$  and  $N(\partial)$  are strongly hyperbolic, and assume that  $\mathcal{C}_M \cap \mathcal{C}_N = \{0\}$ . Let  $s \in \mathbb{R}$  and  $1 \leq p, r \leq \infty$ .*

*If  $1 > d/pr$  then the unique solution  $u \in C(\mathbb{R}^+, E_p^s(\mathbb{R}^d))$  of (2) satisfies*

$$|u(t)|_{E_p^s} \leq |u_0|_{E_p^s} + \text{Cst } |v_0|_{E_{pr}^s},$$

for any  $u_0 \in E_p^s$  and  $v_0 = \mathbf{P}v_0$  with  $v_0 \in E_{pr}^s$ .

**Proof.** With the notations used in the proof of Proposition 2.3, and under the new assumptions made here, one can estimate the terms  $A_{m,n}$  which appear in Eq. (4) as follows:

$$\begin{aligned} A_{m,n} &\leq \int_{\mathbb{R}^d} \left| \frac{2 \sin(t/2(\mu_m(\xi) - v_n(\xi)))}{\mu_m(\xi) - v_n(\xi)} \right|^p (1 + |\xi|^2)^{ps/2} |P(|\xi|)|^p |\widehat{v}_0|^p d\xi \\ &\leq \left| \left( \frac{2 \sin(t/2(\mu_m(\xi) - v_n(\xi)))}{\mu_m(\xi) - v_n(\xi)} \right)^p |P(|\xi|)|^p \right|_r |v_0|_{E_{pr}^s}^p, \end{aligned} \tag{9}$$

for  $1 \leq p < \infty$  and  $1 \leq r \leq \infty$  and thanks to Hölder’s inequality.

From Lemma 2.1 we then have

$$A_{m,n} \leq 2^p \left| \frac{P(|\xi|)^p}{(c_{M,N}|\xi|)^p} \right|_r |v_0|_{E_{pr}^s}^p,$$

and hence  $A_{m,n} \leq \text{Cst } |v_0|_{E_{pr}^s}^p$ , provided that  $1 > d/pr$ . The proposition is thus proved for  $1 \leq p < \infty$  and  $1 \leq r \leq \infty$ .

In order to obtain the result when  $p = \infty$ , one must replace Eq. (5) by

$$|u(t)|_{E_\infty^s} \leq |u_0|_{E_\infty^s} + \text{Cst } \sum_{m,n} A'_{m,n},$$

where

$$\begin{aligned}
 A'_{m,n} &= \left| \frac{2 \sin(t/2(\mu_m(\xi) - v_n(\xi)))}{\mu_m(\xi) - v_n(\xi)} (1 + |\xi|^2)^{s/2} P(|\xi|) \widehat{v_0} \right|_{\infty} \\
 &\leq \left| \frac{2 \sin(t/2(\mu_m(\xi) - v_n(\xi)))}{\mu_m(\xi) - v_n(\xi)} P(|\xi|) \right|_{\infty} |v_0|_{E^s_{\infty}}.
 \end{aligned}$$

Here again, the first term of the r.h.s. is bounded independently of  $t$ .  $\square$

**Remark 2.3.** We used in the proof of Corollary 2.1(i) the fact that when  $\mathcal{C}_M \cap \mathcal{C}_N = \{0\}$ ,  $u$  is not secular if  $\widehat{v_0} \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ . This is also a direct consequence of Proposition 2.4.  $\square$

We also have the following useful corollary in dimension  $d = 1$ .

**Corollary 2.2.** *Suppose that  $d = 1$  and that the assumptions of Proposition 2.4 are satisfied.*

*If  $u_0 \in E^s_p$  and  $v_0 = \partial_x \widetilde{v_0}$  with  $\widetilde{v_0} \in E^{s+1}_{pr}$ , then*

$$|u(t)|_{E^s_p} \leq |u_0|_{E^s_p} + \text{Cst} |\widetilde{v_0}|_{E^{s+1}_{pr}}.$$

**Proof.** If  $v_0 = \partial_x \widetilde{v_0}$  with  $\widetilde{v_0} \in E^{s+1}_{pr}$ , then one has  $v_0 = \mathbf{P} \underline{v_0}$ , with  $\underline{v_0} = \mathbf{Q} \widetilde{v_0}$  and  $\mathbf{Q}$  is the Fourier multiplier associated with the function  $Q(\xi) = i \text{sgn}(\xi)$  if  $|\xi| \leq 1$  and  $Q(\xi) = i\xi$  otherwise. It follows that  $\underline{v_0} \in E^s_{pr}$ , and that

$$|\underline{v_0}|_{E^s_{pr}} \leq |\widetilde{v_0}|_{E^{s+1}_{pr}},$$

so that the end of the proof is now a consequence of Proposition 2.4.  $\square$

### 3. Estimates of the secular growth in the nonlinear case

We consider here the “nonlinear” version of (1), i.e. the case  $J \geq 2$ ,

$$\begin{cases} M(\partial)u = g(v^1, \dots, v^J), \\ u|_{t=0} = u_0 \end{cases} \quad \text{with} \quad \begin{cases} N^j(\partial)v^j = 0, \\ v^j|_{t=0} = v^j_0, \quad j = 1, \dots, J, \end{cases}$$

where the operators  $M(\partial)$  and  $N^j(\partial)$  are as in Assumption 1.1. When these operators are strongly hyperbolic, we denote by  $\mu_1 \dots \mu_m$  and  $v^j_1 \dots v^j_{n_j}$  the different sheets of  $\mathcal{C}_M$  and  $\mathcal{C}_{N^j}$ , respectively.

As a general rule, the secular growth of  $u$  is weaker in the nonlinear case ( $J \geq 2$ ) than in the linear case ( $J = 1$ ), and the decay properties in space variables of the  $v_0^j$  are more important.

### 3.1. Nonspecific sublinear growth

A first important difference between the cases  $J = 1$  and  $J \geq 2$  is that in the nonlinear case, the secular growth of  $u$  is always at most sublinear in time.

**Proposition 3.1.** *Let  $1 \leq p < \infty$ . If  $u_0, v_0^1, \dots, v_0^J \in E_p^s$ , and if there is not any plane sheet (i.e. hyperplane) common to  $\mathcal{C}_M, \mathcal{C}_{N^1}, \dots, \mathcal{C}_{N^J}$ , then the secular growth of  $u$  is sublinear,*

$$\lim_{t \rightarrow \infty} \frac{1}{t} |u(t)|_{E_p^s} = 0.$$

**Proof.** This proposition is a consequence of Proposition 4.1 of [13] or Lemma 6 and Proposition 5 of [17] when  $p = 2$ , i.e. when  $E_p^s = H^s$ . The case  $1 \leq p < \infty$  can be proved with the same methods.  $\square$

**Remark 3.1.** If a plane sheet is common to  $\mathcal{C}_M, \mathcal{C}_{N^1}, \dots, \mathcal{C}_{N^J}$ , then we can extract a subsystem from (1) of the form  $(\partial_t + c \cdot \partial_x)\tilde{u} = \tilde{g}(\tilde{v}_1, \dots, \tilde{v}_J)$  with  $(\partial_t + c \cdot \partial_x)\tilde{v}_j = 0$  for  $j = 1, \dots, J$ , and the secular growth of  $\tilde{u}$  is therefore linear.

Since one then has  $(\partial_t + c \cdot \partial_x)\tilde{g}(\tilde{v}_1, \dots, \tilde{v}_J) = 0$ , this subsystem belongs to the ‘‘linear’’ class  $J = 1$  studied in Section 2. This is the reason why we have claimed that in the (nondegenerated) nonlinear case  $J \geq 2$ , the secular growth is at most sublinear.

### 3.2. Specific sublinear growth in space dimension $d = 1$

The one-dimensional case deserves special attention because the operators  $M(\partial)$  and  $N^j(\partial)$  can be reduced to a combination of transport operators whose properties are completely different from those observed for general hyperbolic operators in higher dimensions. When  $d = 1$ , the properties of the general system (1) can be easily deduced from the behavior of the solution of

$$\begin{cases} (\partial_t + c\partial_x)u = g(v^1, \dots, v^J), \\ u|_{t=0} = u_0 \end{cases} \quad \text{with} \quad \begin{cases} (\partial_t + c_j\partial_x)v^j = 0, \\ v^j|_{t=0} = v_0^j, \quad j = 1, \dots, J, \end{cases} \quad (10)$$

and where, for all  $i \neq j$ ,  $c_i \neq c_j$ .

3.2.1. General estimates

It is expected that the source term  $g(v^1, \dots, v^J)$  in (10) has a less important contribution than in the linear case  $J = 1$  because the waves  $v^1, \dots, v^J$  overlap for only a finite period of time. This is the object of the following proposition.

**Proposition 3.2.** *Let  $s \geq 0$  and  $v_0^1, \dots, v_0^J \in H^s(\mathbb{R})$ ;*

(i) *if  $J = 2$  then for all  $t \geq 0$ , the solution  $u$  of (10) satisfies*

$$|u(t)|_{H^s(\mathbb{R})} \leq |u_0|_{H^s(\mathbb{R})} + \text{Cst} \frac{\sqrt{t}}{\sqrt{|c_1 - c_2|}} |v_0^1|_{H^s(\mathbb{R})} |v_0^2|_{H^s(\mathbb{R})};$$

(ii) *if  $J = 2$  and  $c = c_1$  then for all  $t \geq 0$ , the solution  $u$  of (10) satisfies*

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} |u(t)|_{H^s(\mathbb{R})} = 0;$$

(iii) *if  $J \geq 3$  then for all  $t \geq 0$ , the solution  $u$  of (10) satisfies*

$$|u(t)|_{H^s(\mathbb{R})} \leq |u_0|_{H^s(\mathbb{R})} + \text{Cst} C(c, c_1, \dots, c_J) \prod_{j=1}^J |v_0^j|_{H^s(\mathbb{R})},$$

where Cst does not depend on  $c, c_1, \dots, c_J$  and

$$C(c, c_1, \dots, c_J) = \min_{i \neq j} \left( \frac{1}{\sqrt{|c_i - c_j|}} \max_{k \neq i, j} \frac{1}{\sqrt{|c - c_k|}} \right) \text{ if } c \neq c_j \ \forall j \in [1, J],$$

and

$$C(c, c_1, \dots, c_J) = \min_j \left( \frac{1}{\sqrt{|c - c_j|}} \max_{k \neq j} \frac{1}{\sqrt{|c - c_k|}} \right) \text{ if } c = c_1.$$

The proof of this proposition relies strongly on the  $L_t^2 L_x^2$  estimate given in the following lemma.

**Lemma 3.1.** *With the usual notations, if  $v_0^1, v_0^2 \in L^2(\mathbb{R})$  and  $B : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{R}^D$  is bilinear, then  $h := B(v^1, v^2)$  is in  $L^2(\mathbb{R}_{t,x}^{1+1})$  and*

$$\|h\|_{L^2(\mathbb{R}_{t,x}^{1+1})} \leq \text{Cst} \frac{1}{\sqrt{|c_1 - c_2|}} |v_0^1|_2 |v_0^2|_2,$$

where the constant depends only on  $B$ .

**Proof.** If  $v_0^1, v_0^2 \in \mathcal{D}(\mathbb{R})$  then

$$\begin{aligned} \|h\|_{L^2(\mathbb{R}_{t,x}^{1+1})} &= \left( \int_{\mathbb{R}^2} |B(v_0^1(x - c_1 t'), v_0^2(x - c_2 t'))|^2 dx dt' \right)^{1/2} \\ &= \text{Cst} \left( \frac{1}{|c_1 - c_2|} \int_{\mathbb{R}^2} |v_0^1(P_1)|^2 |v_0^2(P_2)|^2 dP_1 dP_2 \right)^{1/2}, \end{aligned}$$

the last step being a consequence of the change of variables  $(t', x) \mapsto (P_1(x - c_1 t'), P_2(x - c_2 t'))$ .

Fubini’s rule and the density of  $\mathcal{D}(\mathbb{R})$  in  $L^2(\mathbb{R})$  then allow the end of the proof.  $\square$

**Proof of Proposition 3.2.**

(i) From the equation satisfied by  $u$ , one obtains easily

$$\begin{aligned} |u(t)|_2^2 &= |u_0|_2^2 + 2 \int_{\mathbb{R}} \int_0^t g(v^1(t', x), v^2(t', x)) \cdot u(t', x) dt' dx \\ &\leq |u_0|_2^2 + 2 \left( \int_0^t |u(t')|_2^2 dt' \right)^{1/2} \left( \int_{\mathbb{R}} \int_0^t |g(v^1, v^2)|^2 dt' dx \right)^{1/2}, \end{aligned}$$

the last inequality being a consequence of Cauchy–Schwarz inequality. Lemma 3.1 therefore yields

$$|u(t)|_2^2 \leq |u_0|_2^2 + \text{Cst} \frac{1}{\sqrt{|c_1 - c_2|}} \left( \int_0^t |u(t')|_2^2 dt' \right)^{1/2} |v_0^1|_2 |v_0^2|_2,$$

from which the first point of the lemma follows for  $s = 0$ . The case  $s \in \mathbb{N}$  is obtained in the same way after differentiating the equation satisfied by  $u$ . The case  $s \in \mathbb{R}$  then follows by (complex) multilinear interpolation [4,23].

- (ii) This result can be found in [2].
- (iii) The solution  $u$  of (10) can be written

$$u(t, x) = u_0(x - ct) + \int_0^t g(v_0^1(x - c_1 t' - c(t - t')), \dots, v_0^J(x - c_J t' - c(t - t'))) dt',$$

and hence, for all  $s \in \mathbb{N}$ ,

$$\begin{aligned} |\partial_x^s u(t, x)| &\leq |D^s u_0(x - ct)| \\ &+ \text{Cst} \sum_{\alpha_j=s} c_{\alpha_1, \dots, \alpha_J} \int_0^t \prod_{j=1}^J |D^{\alpha_j} v_0^j(x - c_j t' - c(t - t'))| dt'. \quad (11) \end{aligned}$$

We first consider the case  $c \neq c_j, \forall j \in [1, J]$ . We bound the integrals which appear in (11) as follows

$$\begin{aligned} & \int_0^t \prod_{j=1}^J |D^{\alpha_j} v_0^j(x - c_j t' - c(t - t'))| dt' \\ & \leq \left( \int_0^t |\widetilde{D^{\alpha_1} v_0^1} \widetilde{D^{\alpha_2} v_0^2}|^2 dt' \right)^{1/2} |\widetilde{D^{\alpha_3} v_0^3} \dots \widetilde{D^{\alpha_J} v_0^J}|_{L^2_{t'}}, \end{aligned} \tag{12}$$

where for all  $j \in [0, J]$  and all  $\alpha_j \in \mathbb{N}$ ,  $\widetilde{D^{\alpha_j} v_0^j}(t') := D^{\alpha_j} v_0^j(x - ct - (c_j - c)t')$ . Supposing for instance that  $\alpha_3 = \max_{j=3, \dots, J} \alpha_j$ , we also have

$$\begin{aligned} |\widetilde{D^{\alpha_3} v_0^3} \dots \widetilde{D^{\alpha_J} v_0^J}|_{L^2_{t'}} & \leq |\widetilde{D^{\alpha_3} v_0^3}|_{L^2_{t'}} \prod_{j=4}^J |D^{\alpha_j} v_0^j|_{\infty} \\ & \leq \frac{1}{\sqrt{|c - c_3|}} |D^{\alpha_3} v_0^3|_{L^2_x} \prod_{j=4}^J |D^{\alpha_j} v_0^j|_{\infty} \\ & \leq \frac{1}{\sqrt{|c - c_3|}} \prod_{j=3}^J |v_0^j|_{H^s}, \end{aligned} \tag{13}$$

the last inequality being a consequence of the fact that for all  $j \geq 4$  (if there is such),  $|D^{\alpha_j} v_0^j|_{\infty} \leq |v_0^j|_{H^s}$  because  $s - \alpha_j \geq 1 > 1/2 = d/2$ . Eqs. (11) and (12) thus yield

$$|\partial_x^s u(t)|_2 \leq |D^s u_0|_2 + \frac{\text{Cst}}{\sqrt{|c - c_3|}} \prod_{j=3}^J |v_0^j|_{H^s} \sum_{\alpha_1 + \dots + \alpha_J = s} c_{\alpha_1, \dots, \alpha_J} \|\widetilde{D^{\alpha_1} v_0^1} \widetilde{D^{\alpha_2} v_0^2}\|_{L^2(\mathbb{R}^2_{t',x})}. \tag{14}$$

Since Lemma 3.1 yields

$$\begin{aligned} \|\widetilde{D^{\alpha_1} v_0^1} \widetilde{D^{\alpha_2} v_0^2}\|_{L^2(\mathbb{R}^2_{t',x})} & \leq \frac{\text{Cst}}{\sqrt{|c_1 - c_2|}} |D^{\alpha_1} v_0^1|_2 |D^{\alpha_2} v_0^2|_2 \\ & \leq \frac{\text{Cst}}{\sqrt{|(c_1 - c_2)|}} |v_0^1|_{H^s} |v_0^2|_{H^s}, \end{aligned} \tag{15}$$

the lemma follows from Eq. (14) when  $s \in \mathbb{N}$ . The case  $s \in \mathbb{R}^+$  is then obtained again by multilinear interpolation.

We now consider the case where one of the  $c_j$  is equal to  $c$ . Assume for instance that  $c = c_1$ . While Eq. (14) remains true, one cannot use Lemma 3.1 anymore to obtain Eq. (15). But since  $c = c_1, \widetilde{v_0^1}$  does not depend on  $t'$ , and a direct computation

yields

$$\begin{aligned} \|\widetilde{D^{\alpha_1} v_0^1} \widetilde{D^{\alpha_2} v_0^2}\|_{L^2(\mathbb{R}_{r,\alpha}^2)} &= \frac{1}{\sqrt{|c - c_2|}} |D^{\alpha_1} v_0^1|_2 |D^{\alpha_2} v_0^2|_2 \\ &\leq \frac{1}{\sqrt{|c - c_2|}} |v_0^1|_{H^s} |v_0^2|_{H^s}. \end{aligned} \tag{16}$$

The result of the proposition follows from Eqs. (14) and (16) when  $s \in \mathbb{N}$ . The case  $s \in \mathbb{R}^+$  is then obtained again by multilinear interpolation.  $\square$

**Optimality.** The case  $J \geq 3$  is optimal in the sense that there is no secular growth. When  $J = 2$ , we can prove that the secular growth in  $t^{1/2}$ , as stated in Proposition 3.2 is the best one can expect. This will be done after Proposition 3.3 in the next section.

*3.2.2. Estimates using  $L^r$  regularity in frequency*

In the only secular case of Proposition 3.2, i.e. when  $J = 2$ , the growth rate  $t^{1/2}$  can be improved assuming that  $v_0^1$  and  $v_0^2$  are in  $\mathcal{F}L^r(\mathbb{R})$  with  $r$  big enough. This is done in the following two propositions; the proof of the first one is given in Appendix B.

**Proposition 3.3.** *Let  $J = 2$ ,  $s \in \mathbb{R}$ ,  $1 \leq p, r < \infty$  and suppose  $c \neq c_1$  and  $c \neq c_2$ .*

*Consider also  $v_0^1, v_0^2 \in E_p^s \cap E_r^{2\alpha}$ , with  $\alpha$  given below. Then the unique solution  $u \in C(\mathbb{R}^+, E_p^s(\mathbb{R}))$  of (10) satisfies the estimates*

$$|u(t)|_{E_p^s} \leq \begin{cases} \text{Cst } C(c, c_1, c_2) t^{1-1/r} |v_0^1|_{E_p^s \cap E_r^0} |v_0^2|_{E_p^s \cap E_r^0} & \text{if } r > 1 \text{ (and } \alpha = 0), \\ \text{Cst } C(c, c_1, c_2) (1 + \ln t) |v_0^1|_{E_p^s \cap E_\infty^{2\alpha}} |v_0^2|_{E_p^s \cap E_\infty^{2\alpha}} & \text{if } r = 1 \text{ and } \alpha > 0, \end{cases}$$

for all  $t \geq 1$  and where Cst does not depend on  $c, c_1, c_2$  and  $C(c, c_1, c_2)$  is given by

$$C(c, c_1, c_2) := \frac{1}{|c_2 - c_1|^{1/pr}} \max \left( \frac{1}{|c - c_1|^{1/pr}}, \frac{1}{|c - c_2|^{1/pr}} \right).$$

If moreover  $s > 1/r - 1/p$  and  $p \geq r$  then

$$|u(t)|_{E_p^s} \leq \text{Cst } C(c, c_1, c_2) t^{1-1/r} |v_0^1|_{E_p^s} |v_0^2|_{E_p^s}.$$

The next proposition deals with the case  $c = c_1$  which is excluded in Proposition 3.3. This case is particular because the explicit solution can be written as a convolution product.

**Proposition 3.4.** *Let  $J = 2$ ,  $s \in \mathbb{R}$ ,  $1 \leq p, r < \infty$  and suppose  $c = c_1$  (and  $c \neq c_2$ ).*

Consider also  $v_0^1 \in E_p^s$  and  $v_0^2 \in E_t^{s+2\alpha}$ , with  $\alpha$  given below. Then the unique solution  $u \in C(\mathbb{R}^+, E_p^s(\mathbb{R}))$  of (10) satisfies the estimates

$$|u(t)|_{E_p^s} \leq \begin{cases} |u_0|_{E_p^s} + \frac{\text{Cst}}{|c_1 - c_2|^{1/r}} t^{1-1/r} |v_0^1|_{E_p^s} |v_0^2|_{E_t^s} & \text{if } r > 1 \text{ (and } \alpha = 0), \\ |u_0|_{E_p^s} + \frac{\text{Cst}}{|c_1 - c_2|^{1/r}} (1 + \ln t) |v_0^1|_{E_p^s} |v_0^2|_{E_\infty^{s+2\alpha}} & \text{if } r = 1 \text{ and } \alpha > 0, \end{cases}$$

for all  $t \geq 1$  and where Cst does not depend on  $c_1$  nor  $c_2$ .

**Proof.** The solution of (10) can be written in this particular case  $c = c_1$  as a convolution product,

$$\hat{u}(t, \xi) = e^{-ict\xi} \hat{u}_0(\xi) + e^{-ict\xi} \hat{v}_0^1 *_g (\psi_t \hat{v}_0^2),$$

where  $*_g$  denotes the convolution product associated to the bilinear mapping  $g$ , and  $\psi_t(\xi) := \frac{e^{it(c_1 - c_2)\xi} - 1}{i(c_1 - c_2)\xi}$ . It is then easy to deduce the result from Hölder’s and Lemma 2.1.  $\square$

**Optimality.** We prove here that in Proposition 3.3, one cannot expect a better growth rate than  $t^{1/2}$  when  $p = r = 2$ . The example we give below can be adapted for other values of  $p$  and  $r$ .

Let  $u$  solve  $\partial_t u + c\partial_x u = v_0^1(x - c_1 t)v_0^2(x - c_2 t)$  with  $c_1 \neq c_2$  and  $u_0 = 0$ . With the notations already used to prove optimality after Proposition 2.1, take  $v_0^1$  and  $v_0^2$  defined as

$$\hat{v}_0^1(\xi) = \hat{v}_0^2(\xi) = \frac{1}{|\xi|^a} \chi_B(\xi), \quad \forall \xi \in \mathbb{R} \setminus \{0\},$$

with  $0 < a < 1/2$ . One then has  $v_0^1, v_0^2 \in H^s(\mathbb{R})$  for all  $s \geq 0$ , and  $u$  is given by

$$\hat{u}(t, \xi) = e^{-ict\xi} \int_{\mathbb{R}} \frac{e^{it[(c-c_1)\xi + (c_1-c_2)\xi']} - 1}{i[(c-c_1)\xi + (c_1-c_2)\xi']} \frac{\chi_B(\xi - \xi')}{|\xi - \xi'|^a} \frac{\chi_B(\xi')}{|\xi'|^a} d\xi',$$



and therefore

$$\begin{aligned}
 |u(t)|_2 &= |\hat{u}(t)|_2 \\
 &\geq |\chi_B \hat{u}(t, \cdot)|_2 \\
 &= \left( \int_B \left| \int_{\mathbb{R}} \frac{e^{i[(c-c_1)\xi + (c_1-c_2)\xi']} - 1}{i[(c-c_1)\xi + (c_1-c_2)\xi']} \frac{\chi_B(\xi - \xi') \chi_B(\xi')}{|\xi - \xi'|^a |\xi'|^a} d\xi' \right|^2 d\xi \right)^{1/2} \\
 &= t^{2a-\frac{1}{2}} \left( \int_{tB} \left| \int_{\mathbb{R}} \frac{e^{i[(c-c_1)\xi + (c_1-c_2)\xi']} - 1}{i[(c-c_1)\xi + (c_1-c_2)\xi']} \frac{\chi_{tB}(\xi - \xi') \chi_{tB}(\xi')}{|\xi - \xi'|^a |\xi'|^a} d\xi' \right|^2 d\xi \right)^{1/2},
 \end{aligned}$$

the last equality being a consequence of the change of variables  $\xi \rightsquigarrow \xi/t$  and  $\xi' \rightsquigarrow \xi'/t$ . For  $t \geq 1$ , one has therefore

$$|u(t)|_2 \geq t^{2a-\frac{1}{2}} \left( \int_B \left| \int_{\mathbb{R}} \frac{e^{i[(c-c_1)\xi + (c_1-c_2)\xi']} - 1}{i[(c-c_1)\xi + (c_1-c_2)\xi']} \frac{\chi_{tB}(\xi - \xi') \chi_{tB}(\xi')}{|\xi - \xi'|^a |\xi'|^a} d\xi' \right|^2 d\xi \right)^{1/2}.$$

Fatou’s Lemma then yields

$$\begin{aligned}
 &\liminf_{t \rightarrow \infty} \frac{1}{t^{2a-\frac{1}{2}}} |u(t)|_2 \\
 &\geq \left( \int_B \left| \int_{\mathbb{R}} \frac{e^{i[(c-c_1)\xi + (c_1-c_2)\xi']} - 1}{i[(c-c_1)\xi + (c_1-c_2)\xi']} \frac{1}{|\xi - \xi'|^a} \frac{1}{|\xi'|^a} d\xi' \right|^2 d\xi \right)^{1/2} > 0.
 \end{aligned}$$

Such a result being true for all  $0 < a < 1/2$ , this yields that for any  $0 < \alpha < 1/2$ , it is possible to find  $v_0^1$  and  $v_0^2$  such that  $u$  has a secular growth rate bigger than  $t^\alpha$ .

*3.2.3. Estimates using the decay properties in space variables*

It is sometimes possible to solve the profile equations obtained by the BKW method in functional spaces satisfying some decay property. The secular growth is then lower than in the general case (we only consider the case  $J = 2$  since no secular growth is possible when  $J \geq 3$ ). The following proposition is a slight generalization of a result of [19,20] used to control the secular growth arising in the derivation of the KdV approximation from Boussinesq or Euler equations.

**Proposition 3.5.** *Let  $J = 2$ ,  $s > 1/2$ , and  $v_0^1, v_0^2 \in H^s(\mathbb{R})$ .*

*Assume also that for some  $\alpha > 0$ ,  $\underline{v}_0^1 := (1 + x^2)^\alpha v_0^1$  and  $\underline{v}_0^2 := (1 + x^2)^\alpha v_0^2$  are in  $H^s(\mathbb{R})$ .*

Then the unique solution  $u$  of (10) satisfies

$$|u(t)|_{H^s} \leq \begin{cases} |u_0|_{H^s} + \text{Cst } t^{1-2\alpha} |\underline{v}_0^1|_{H^s} |\underline{v}_0^1|_{H^s} & \text{if } 2\alpha < 1, \\ |u_0|_{H^s} + \text{Cst } \ln(1+t) |\underline{v}_0^1|_{H^s} |\underline{v}_0^1|_{H^s} & \text{if } 2\alpha = 1, \\ |u_0|_{H^s} + \text{Cst } |\underline{v}_0^1|_{H^s} |\underline{v}_0^1|_{H^s} & \text{if } 2\alpha > 1. \end{cases}$$

**Proof.** Classically, one has

$$\begin{aligned} |u(t)|_{H^s} &\leq |u_0|_{H^s} + \int_0^t |g(v_0^1(x - c_1 t'), v_0^2(x - c_2 t'))|_{H^s} dt' \\ &\leq |u_0|_{H^s} + \int_0^t \sup_{x \in \mathbb{R}} \left| \frac{1}{(1 + (x - c_1 t')^2)^\alpha} \frac{1}{(1 + (x - c_2 t')^2)^\alpha} \right| \\ &\quad \times \left| g(\underline{v}_0^1(x - c_1 t'), \underline{v}_0^2(x - c_2 t')) \right|_{H^s} dt'. \end{aligned}$$

Since we also have

$$\begin{aligned} &\sup_{x \in \mathbb{R}} \left| \frac{1}{(1 + (x - c_1 t')^2)^\alpha} \frac{1}{(1 + (x - c_2 t')^2)^\alpha} \right| \\ &= \begin{cases} \frac{1}{|c_1 - c_2|^{2\alpha} t'^{2\alpha}} & \text{if } t' \geq \frac{2}{|c_1 - c_2|}, \\ \frac{1}{(1 + \frac{|c_1 - c_2|^2 t'^2}{4})^{2\alpha}} & \text{otherwise,} \end{cases} \end{aligned}$$

we obtain the following estimate:

$$|u(t)|_{H^s} = |u_0|_{H^s} + \frac{2}{|c_1 - c_2|} \left( \int_0^1 \frac{1}{(1 + t'^2)^{2\alpha}} dt' + \int_1^{t|c_1 - c_2|/2} \frac{2^{2\alpha}}{t'^{2\alpha}} dt' \right) |\underline{v}_0^1|_{H^s} |\underline{v}_0^1|_{H^s},$$

from which the results of the proposition follow easily.  $\square$

### 3.3. Specific sublinear growth in space dimension $d \geq 2$

As far as only transport operators are considered, the arguments used to control the secular growth in the multi-dimensional case  $d \geq 2$  are quite similar to those used in dimension one. In order to generalize the results of Lemma 3.1 and Proposition 3.2, we use ideas introduced in [5,6] for the Boltzmann equations.

**Definition 3.1.** Let  $p \in [1, d]$  and denote by  $\mathbb{R}_p^d$  the set of all the subspaces of  $\mathbb{R}^d$  of dimension  $p$ . For any function  $f$  define

$$\delta_p(f) = \sup_{\pi \in \mathbb{R}_p^d} \sup_{z \in \pi^*} \text{ess} \left( \int_{\pi} |f(z+x)|^2 d^p x \right)^{1/2},$$

where for all  $\pi \in \mathbb{R}_p^d$ ,  $\pi^*$  denotes any supplementary subspace of  $\pi$ .

The link between the  $\delta_p$  and the  $H^s$  norms is made in the following lemma.

**Lemma 3.2.** For any  $f \in \mathcal{S}'(\mathbb{R}^d)$ , one has:

- (i)  $\delta_d(f) = |f|_2$ ;
- (ii) For any  $p \in [1, d]$ ,  $\delta_p(f) \leq 2^{\frac{d-p}{2}} |f|_{H^{d-p}(\mathbb{R}^d)}$ .

**Proof.** Condition (i) is straightforward; for (ii), remark that the case  $d = 2$  is classical (see [9, Lemma 11] for instance). The result for  $d \geq 2$  is obtained by induction.  $\square$

Lemma 3.1 can be generalized in dimension  $d \geq 1$  as follows.

**Lemma 3.3.** With the usual notations, if  $v_0^1 \in L^2(\mathbb{R}^d)$ ,  $\delta_1(v_0^2) < \infty$  and  $B : \mathbb{R}^D \times \mathbb{R}^D \rightarrow \mathbb{R}^D$  is bilinear, then  $h := B(v^1, v^2)$  is in  $L^2(\mathbb{R}_{t,x}^{1+d})$  and

$$\|h\|_{L^2(\mathbb{R}_{t,x}^{1+d})} \leq \text{Cst} \frac{1}{\sqrt{|c_1 - c_2|}} |v_0^1|_2 \delta_1(v_0^2),$$

where the constant depends only on  $B$ .

**Proof.** One has

$$\begin{aligned} \|h\|_{L^2(\mathbb{R}_{t,x}^{1+d})} &= \left( \int_{\mathbb{R}^{d+1}} |B(v_0^1(x - c_1 t), v_0^2(x - c_2 t))|^2 dx dt \right)^{1/2} \\ &= \left( \int_{\mathbb{R}^{d+1}} |B(v_0^1(x), v_0^2(x + (c_1 - c_2)t))|^2 dx dt \right)^{1/2} \\ &\leq \text{Cst} \left( \int_{\mathbb{R}^d} |v_0^1(x)|^2 \int_{\mathbb{R}} |v_0^2(x + (c_1 - c_2)t)|^2 dt dx \right)^{1/2}, \end{aligned}$$

and the integral in  $t$  of the last inequality can be majorized by  $\frac{1}{|c_1 - c_2|} \delta_1(v_0^2)^2$ , which yields the lemma.  $\square$

When considering the particular case of system (1) where all the operators are transport operators, i.e. systems of the form

$$\begin{cases} (\partial_t + c \cdot \partial_x)u = g(v^1, \dots, v^J), \\ u|_{t=0} = u_0, \end{cases} \quad \text{with} \quad \begin{cases} (\partial_t + c_j \cdot \partial_x)v^j = 0, \\ v^j|_{t=0} = v_0^j, \quad j = 1 \dots J, \end{cases} \quad (17)$$

and where for all  $i \neq j$ ,  $c_i \neq c_j$ , Proposition 3.2 can be generalized as follows:

**Proposition 3.6.** *Let  $d \geq 2$ ,  $s \geq 0$  and  $v_0^1, \dots, v_0^J \in H^{s+\mu}(\mathbb{R}^d)$  for a certain  $\mu \geq 0$ .*

(i) *If  $J = 2$  and  $\mu \geq d - [\frac{s+1}{2}] - 1$ , then for all  $t \geq 0$ , the solution  $u$  of (17) satisfies*

$$|u(t)|_{H^s(\mathbb{R}^d)} \leq |u_0|_{H^s(\mathbb{R}^d)} + \text{Cst} \frac{\sqrt{t}}{\sqrt{|c_1 - c_2|}} |v_0^1|_{H^{s+\mu}(\mathbb{R}^d)} |v_0^2|_{H^{s+\mu}(\mathbb{R}^d)};$$

(ii) *If  $J = 2$  and  $c = c_1$ , and with the same  $\mu$  as in (i), then the solution  $u$  of (10) satisfies for all  $t \geq 0$ ,*

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} |u(t)|_{H^s(\mathbb{R}^d)} = 0;$$

(iii) *If  $J \geq 3$ ,  $\mu \geq d - 1$  and if  $\mu > \frac{d}{2} - [\frac{s+1}{2}]$  when  $J \geq 4$ , then for all  $t \geq 0$ , the solution  $u$  of (10) satisfies*

$$|u(t)|_{H^s(\mathbb{R}^d)} \leq |u_0|_{H^s(\mathbb{R}^d)} + \text{Cst} C(c, c_1, \dots, c_J) \prod_{j=1}^J |v_0^j|_{H^{s+\mu}(\mathbb{R}^d)},$$

where Cst does not depend on  $c, c_1, \dots, c_J$  and  $C(c, c_1, \dots, c_J)$  is as in Proposition 3.2.

**Proof.** Since the proof is very similar to the proof of Proposition 3.2, we only give the main lines.

(i) The proof is exactly the same, provided that one uses Lemma 3.3 instead of Lemma 3.1. One also has to use Lemma 3.2 to bound the  $\delta_1(v_0^j)$  by Sobolev norms:  $\delta_1(\partial^\alpha v_0^j) \leq |v_0^j|_{H^{|\alpha|+d-1}}$  for any multiindex  $\alpha \in \mathbb{N}^2$ .

(ii) If  $\widehat{v_0^2} \in \mathcal{D}(\mathbb{R}^d \setminus \{(c - c_2) \cdot \xi = 0\})$ , i.e. is smooth and compactly supported outside the hyperplane  $\{(c - c_2) \cdot \xi = 0\}$  (which is the set of resonances in this particular case), then it is easy to see on the explicit expression of  $\widehat{u}(t)$  that  $|u(t)|_{H^s}$  is bounded in time, and hence that  $\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} |u(t)|_{H^s} = 0$ .

Since the hyperplane  $\{(c - c_2) \cdot \xi = 0\}$  has zero Lebesgue measure,  $\mathbb{A} := \{f \in \mathcal{S}'(\mathbb{R}^d), \widehat{f} \in \mathcal{D}(\mathbb{R}^d \setminus \{(c - c_2) \cdot \xi = 0\})\}$  is dense in  $H^{s+\mu}(\mathbb{R}^d)$ , and the result follows from a density argument, as in the proof of Corollary 2.1.

(iii) Here again the proof follows the proof of Proposition 3.2. Eq. (13) can be replaced by the estimate

$$|\widetilde{D^{\alpha_3} v_0^3} \dots \widetilde{D^{\alpha_J} v_0^J}|_{L^2_\tau} \leq \frac{1}{\sqrt{|c - c_3|}} \prod_{j=3}^J |v_0^j|_{H^{s+\mu'}},$$

provided that  $\mu' \geq d - 1$  and, if  $J \geq 4$ , that  $\mu' > \frac{d}{2} + [\frac{s}{2}] - s$ .

If  $c \neq c_j$  for all  $j \in [1, J]$ , and using Lemma 3.3, one can replace Eq. (15) by

$$\begin{aligned} \|\widetilde{D^{\alpha_1} v_0^1} \widetilde{D^{\alpha_2} v_0^2}\|_{L^2(\mathbb{R}^{1+d}_{t,x})} &\leq \frac{\text{Cst}}{\sqrt{|c_1 - c_2|}} |D^{\alpha_1} v_0^1|_2 \delta_1(D^{\alpha_2} v_0^2) \\ &\leq \frac{\text{Cst}}{\sqrt{|c_1 - c_2|}} |v_0^1|_{H^s} |v_0^2|_{H^{s+\mu''}}, \end{aligned}$$

the last inequality being a consequence of Lemma 3.2 provided that  $\mu'' \geq d - 1$ .

If  $c = c_1$ , the same estimate holds by a direct computation.

The same methods as for Proposition 3.2 allow the end of the proof.  $\square$

**Remark 3.2.** As in space dimension  $d = 1$ , the estimates of Proposition 3.6 can be improved using the spaces  $E_p^s$ . Using Lemma 2.2 and performing slight modifications in the proofs of Propositions 3.3 and 3.4, one obtains that the estimates of these propositions remain valid when  $d \geq 1$ . Only the condition on  $\alpha$  has to be changed to  $2\alpha r > d - 1$  ( $1 \leq r < \infty$ ) in both propositions.

We finally consider the general case where the operators considered in (1) are not transport operators. The tools we must use to control the secular growth in this case differ radically from what has been used above. One must use the decay properties of the free solutions of such operators, and Strichartz estimates play a central role. The general secular growth that one can expect for the solution  $u$  of (1) when  $d \geq 2$  and  $J \geq 2$  is given in the following proposition.

**Proposition 3.7.** *Suppose that  $d, J \geq 2$  and that for all  $j \in [1, J]$ , the operators  $N^j(\partial)$  are strongly hyperbolic and well rounded. Let also  $s \geq 0$ ,  $\mu \geq 0$ , and  $v_0^1 \dots v_0^J \in H^{s+\mu}(\mathbb{R}^d)$ . Then the solution  $u$  of (1) satisfies*

$$|u(t)|_{H^s(\mathbb{R}^d)} \leq |u_0|_{H^s(\mathbb{R}^d)} + \text{Cst } t^{1 - \frac{(d-1)(J-1)}{4}} \prod_{j=1}^J |v_0^j|_{H^{s+\mu}(\mathbb{R}^d)},$$

if  $(d - 1)(J - 1) < 4$  and  $\mu = \frac{(d+1)(J-1)}{4J}$ , and

$$|u(t)|_{H^s(\mathbb{R}^d)} \leq |u_0|_{H^s(\mathbb{R}^d)} + \text{Cst} \prod_{j=1}^J |v_0^j|_{H^{s+\mu}(\mathbb{R}^d)},$$

if  $(d - 1)(J - 1) \geq 4$  and  $\mu = \frac{d(J-1)-2}{2J}$ .

**Proof.** Let  $S_M(t)$  be the unitary group associated with  $M(\partial)$ . Then  $u$  can be written

$$u(t, x) = S_M(t)u_0(x) + \int_0^t S_M(t - t')g(v^1(t', x), \dots, v^J(t', x)) dt',$$

and therefore, for all  $n \in \mathbb{N}$  and  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$  such that  $|\alpha| := \alpha_1 + \dots + \alpha_d = n$ , and denoting  $\partial^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$ , one then has

$$\partial^\alpha u(t) = S_M(t)\partial^\alpha u_0 + \int_0^t S_M(t - t')\partial^\alpha g(v^1, \dots, v^J) dt'. \tag{18}$$

Expanding the derivative of  $g$  yields for the last term of the r.h.s.

$$\begin{aligned} & \int_0^t S_M(t - t')\partial^\alpha g(v^1, \dots, v^J) dt' \\ &= \sum_{\beta \in \mathbb{N}^{Jd}, \beta_1 + \dots + \beta_J = \alpha} C(\beta) \int_0^t S_M(t - t')g(\partial^{\beta_1} v^1, \dots, \partial^{\beta_J} v^J) dt' \\ &:= \sum_{\beta \in \mathbb{N}^{Jd}, \beta_1 + \dots + \beta_J = \alpha} C(\beta)A_\beta(t). \end{aligned} \tag{19}$$

For all  $\beta = (\beta_1, \dots, \beta_J) \in \mathbb{N}^{Jd}$  such that  $\beta_1 + \dots + \beta_J = \alpha$ , the norm of  $A_\beta$  can be majorized as follows

$$\begin{aligned} |A_\beta(t)|_2 &\leq \int_0^t |g(\partial^{\beta_1} v^1, \dots, \partial^{\beta_J} v^J)|_2 dt' \\ &\leq t^{1/p} \|g(\partial^{\beta_1} v^1, \dots, \partial^{\beta_J} v^J)\|_{L^p(\mathbb{R}_t; L^2(\mathbb{R}_x^d))}, \end{aligned} \tag{20}$$

where  $1 \geq p \geq \infty$ .

We now prove the following claim: for any  $J$ -uplet of functions  $(f_1, \dots, f_J)$ , we have the following estimate:

$$\|g(f_1, \dots, f_J)\|_{L^p(\mathbb{R}_t; L^2(\mathbb{R}_x^d))} \leq \text{Cst} \prod_{j=1}^J \|f_j\|_{L^{2^j}(\mathbb{R}_t; L^{2^j}(\mathbb{R}_x^d))}.$$

Indeed, one has

$$\begin{aligned} \|g(f_1, \dots, f_J)\|_{L^p(\mathbb{R}_t; L^2(\mathbb{R}_x^d))} &\leq \text{Cst} \left\| \sum_{j=1}^J |f_j|^J \right\|_{L^p(\mathbb{R}_t; L^2(\mathbb{R}_x^d))} \\ &\leq \text{Cst} \sum_{j=1}^J \|f_j\|_{L^{2^j}(\mathbb{R}_t; L^{2^j}(\mathbb{R}_x^d))}^J. \end{aligned}$$

From the above inequality, one can deduce that for any  $\gamma_1, \dots, \gamma_{J-1} > 0$ , one has

$$\begin{aligned} \|g(f_1, \dots, f_J)\|_{L^p(\mathbb{R}_t; L^2(\mathbb{R}_x^d))} &\leq \text{Cst} \sum_{j=1}^{J-1} \gamma_j^J \|f_j\|_{L^{p_j}(\mathbb{R}_t; L^{2j}(\mathbb{R}_x^d))}^J \\ &\quad + \frac{1}{(\gamma_1 \dots \gamma_{J-1})^J} \|f_J\|_{L^{p_J}(\mathbb{R}_t; L^{2J}(\mathbb{R}_x^d))}^J. \end{aligned}$$

Taking  $\gamma_j^J := \frac{1}{\|f_j\|_{L^{p_j}(\mathbb{R}_t; L^{2j}(\mathbb{R}_x^d))}^{J-1}} \prod_{j' \neq j} \|f_{j'}\|_{L^{p_{j'}}(\mathbb{R}_t; L^{2j'}(\mathbb{R}_x^d))}$  then gives the result of the claim.

Thanks to this claim, Eq. (20) yields

$$|A_\beta(t)|_2 \leq \text{Cst} t^{1/p} \prod_{j=1}^J \|\partial^{\beta_j} v^j\|_{L^{p_j}(\mathbb{R}_t; L^{2j}(\mathbb{R}_x^d))}. \tag{21}$$

The  $L_t^{p_j} L_x^{2j}$  estimate of the r.h.s. of (21) can be controlled thanks to the Strichartz estimates given in the lemma below. We use here the notation  $\dot{H}^\mu(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^2), (\sqrt{-\Delta})^{-\mu} f \in L^2(\mathbb{R}^d)\}$ .

**Lemma 3.4** (Strichartz estimates). *Let  $d \geq 2$  and  $L(\partial)$  be a well rounded strongly hyperbolic operator.*

*Let also  $2 \leq a, b < \infty$  be such that*

$$0 \leq \frac{2}{a} \leq \min\left((d-1)\left(\frac{1}{2} - \frac{1}{b}\right), 1\right) \quad \text{and} \quad \left(\frac{2}{a}, (d-1)\left(\frac{1}{2} - \frac{1}{b}\right)\right) \neq (1, 1).$$

*For any  $f_0 \in \dot{H}^\mu(\mathbb{R}^d)$ , with  $\mu := d(\frac{1}{2} - \frac{1}{b}) - \frac{1}{a}$ , the solution  $f$  of  $L(\partial)f = 0$ ,  $f|_{t=0} = f_0$  satisfies*

$$\|f\|_{L^a(\mathbb{R}_t; L^b(\mathbb{R}_x^d))} \leq \text{Cst} \|f_0\|_{\dot{H}^\mu(\mathbb{R}^d)}.$$

**Proof.** The lemma reduces to the well-known Strichartz estimates when  $L(\partial)$  is the wave operator. This classical result can be easily extended to the present case. The usual proof ([10,14] for instance) uses an abstract duality argument (the  $TT^*$  method) and interpolation between the energy and the decay estimates. Only the last two points depend on the equation considered.

Since we have trivially an energy estimate in the problem we are concerned with here, and since the same decay estimate as for the wave equation also holds if  $L(\partial)$  is well rounded [11], the usual Strichartz estimates remain true.  $\square$

We now use these Strichartz estimates in Eq. (21). In order to minimize the secular growth, we want the exponent  $1/p$  which appears in this equation to be as small as possible. This means that we want  $p$  and hence  $pJ$  to be as small as possible. Applying Lemma 3.4 with  $a = pJ$  and  $b = 2J$  yields the following optimal values for

$p$  (which must of course satisfy  $1 \leq p \leq \infty$ ):  $p = 1$  if  $(-1)(J - 1) \geq 4$  and  $p = \frac{4}{(d-1)(J-1)}$  otherwise.

With this value of  $p$ , we deduce from Eq. (21) and Lemma 3.4 that

$$|A_\beta(t)|_2 \leq \text{Cst } t^{1/p} \prod_{j=1}^J |v_0^j|_{H^{s+\mu}(\mathbb{R}^d)},$$

with  $\mu = \frac{d(J-1)}{2J} - \frac{1}{pJ}$ .

This estimate, together with Eqs. (18) and (19) achieves the proof of the proposition in the case  $s \in \mathbb{N}$ . The general case  $s \geq 0$  follows by multilinear interpolation.  $\square$

**Remark 3.3.** It is known that when the initial conditions are in certain classes of functions, the solution of the homogeneous wave equation decays uniformly as  $O(t^{-(d-1)/2})$  (see [15,16,24] for instance). These results can sometimes be extended to the general hyperbolic systems considered here (we have used such a property in the proof of Lemma 3.4). When such a decay rate is satisfied by the  $v_0^j$ , the results of Proposition 3.4 can be considerably improved. If for instance  $J = 2$  and  $|v_0^1|_{W^{s,\infty}} \leq \frac{\text{Cst}}{(1+t)^{(d-1)/2}}$ , then

$$|u(t)|_{H^s} \leq |u_0|_{H^s} + \text{Cst} \int_0^t \frac{1}{(1+t')^{(d-1)/2}} dt' |v_0^2|_{H^s},$$

so that the secular growth is at most in  $O(\sqrt{t})$  when  $d = 2$ , compared to the  $O(t^{3/4})$  given by Proposition 3.7.  $\square$

### Acknowledgments

The author is very grateful to T. Colin and J.-L. Joly for their fruitful suggestions about this work. The suggestions of R. Carles also proved very useful.

### Appendix A. Proof of Theorem 1.1

The proof is a generalization of the proof of the algebra properties of  $H^s(\mathbb{R}^d)$  for  $s > d/2$ , as one can find it in [1] for instance.

Let  $u \in \mathcal{S}'(\mathbb{R}^d)$ . In order to make the Littlewood–Paley decomposition of  $u$ , introduce  $\psi \in C_0^\infty(\mathbb{R})$  such that  $\psi(r) = 1$  for  $|r| \leq 1/2$  and  $\psi(r) = 0$  for  $|r| \geq 1$ ,  $\psi$  is even and decreasing on  $\mathbb{R}^+$ . We define  $\Psi$  on  $\mathbb{R}^d$  as  $\Psi(\xi) = \psi(|\xi|)$  for all  $\xi \in \mathbb{R}^d$ . Finally, let  $\varphi(\xi) := \Psi(\xi/2) - \Psi(\xi)$ . It is easy to see that  $\varphi$  is always positive;



one also has

$$1 = \Psi(\xi) + \sum_{n \geq 0} \varphi(2^{-n}\xi), \tag{A.1}$$

and we define the Littlewood–Paley decomposition of  $u$  as

$$u = u_{-1} + \sum_{n \geq 0} u_n,$$

with  $u_{-1} = \Psi(D)u$  and  $u_n = \varphi(2^{-n}D)u$ , for  $n \geq 0$ .

Since for  $\xi$  fixed, only two terms of the r.h.s. of Eq. (A.1) can be nonzero, and thanks to the inequality  $a^p + b^p \leq (a + b)^p \leq 2^{p-1}(a^p + b^p)$ , we obtain the following double inequality, which generalizes the double inequality of almost orthogonality,

$$\frac{1}{2^{p-1}} \leq \Psi^p(\xi) + \sum_{n \geq 0} \varphi^p(2^{-n}\xi) \leq 1.$$

Multiplying this double inequality by  $|\hat{u}(\xi)|^p$  and integrating with respect to  $\xi$  yields

$$\sum_{n \geq -1} |\hat{u}_n|_p^p \leq |\hat{u}|_p^p \leq 2^{p-1} \sum_{n \geq -1} |\hat{u}_n|_p^p. \tag{A.2}$$

Consider now the norm of  $u_n$  in  $E_p^s$ ; one has

$$\begin{aligned} |u_n|_{E_p^s}^p &= |(1 + |\cdot|^2)^{s/2} \hat{u}_n|_p^p \\ &= \int_{\mathbb{R}^d} (1 + |\xi|^2)^{ps/2} \varphi(2^{-n}\xi)^p |\hat{u}(\xi)|^p d\xi. \end{aligned} \tag{A.3}$$

Since  $\text{Supp } \varphi \subset \{\xi, 1/2 \leq |\xi| \leq 2\}$ , we have, for any  $\xi \in \text{Supp } \varphi(2^{-n}\cdot)$ ,

$$2^{pns} \frac{1}{C^p} \leq (1 + |\xi|^2)^{ps/2} \leq (1 + 2^{2(n+1)})^{ps/2} \leq C^p 2^{pns}, \tag{A.4}$$

where  $C$  is a constant which depends on  $s$  but not on  $n$  nor on  $p$ .

From Eqs. (A.3) and (A.4), we then deduce

$$|\hat{u}_n|_p \frac{2^{ns}}{C} \leq |u_n|_{E_p^s} \leq C 2^{ns} |\hat{u}_n|_p. \tag{A.5}$$

It is then easy to obtain a dyadic characterization of the spaces  $E_p^s$  which generalizes characterization (A.2) of the spaces  $E_p^0$ , as well as the classical ([1] for instance) characterization of the Sobolev spaces  $H^s$ . This is summarized in the following lemma, whose proof is a direct consequence of Eqs. (A.2) and (A.5).

**Lemma A.1.** (i) *If  $u \in E_p^s$  then  $\forall n \geq -1, |\hat{u}_n|_p \leq C|u|_{E_p^s} c_n 2^{-ns}$ , where  $c_n = c_n(u)$  satisfies  $\sum_{n \geq -1} c_n^p \leq 1$ .*

(ii) Reciprocally, if  $\forall n \geq -1, |\widehat{u}_n|_p \leq Cc_n 2^{-ns}$ , with  $\sum_{n \geq -1} c_n^p \leq 1$ , then  $u \in E_p^s$  and  $|u|_{E_p^s} \leq \text{Cst } C$ .

Consider now two functions  $u, v \in E_p^s \cap E_1^0$ . Multiplying their Littlewood–Paley decomposition yields

$$\begin{aligned} uv &= \sum_{m, n \geq -1} u_m v_n \\ &= \sum_{n \geq 0} (S_n u) v_n + \sum_{m \geq -1} u_m (S_{m+1} v) := \Sigma_1 + \Sigma_2, \end{aligned} \tag{A.6}$$

where for all  $j_0 \geq 0$  and  $w \in \mathcal{S}'$ ,  $S_{j_0} w := \sum_{j=-1}^{j_0-1} w_j$ .

First remark that  $|(\widehat{S_n u} v_n)_p$  can be controlled by a convolution inequality,

$$|(\widehat{S_n u} v_n)_p \leq |\widehat{S_n u}|_1 |\widehat{v_n}|_p,$$

and then that  $|\widehat{S_n u}|_1 \leq |\widehat{u}|_1$ , to deduce from Lemma A.1 that

$$|(\widehat{S_n u} v_n)_p \leq C |\widehat{u}|_1 |v|_{E_p^s} c_n 2^{-ns}, \tag{A.7}$$

with  $\sum c_n^p \leq 1$ .

Since moreover  $\text{Supp } (\widehat{S_n u} v_n) \subset \{|\xi| \leq 2^{n+2}\}$ , one has  $(\Sigma_1)_n = \sum_{k \geq n-3} ((S_k u) v_k)_n$ , and hence

$$\begin{aligned} |(\Sigma_1)_n|_p &\leq \sum_{k \geq n-3} |((S_k u) v_k)_n|_p \\ &\leq \sum_{k \geq n-3} |(S_k u) v_k|_p \\ &\leq C |\widehat{u}|_1 |v|_{E_p^s} \sum_{k \geq n-3} c_k 2^{-ks}. \end{aligned} \tag{A.8}$$

Now, write  $\sum_{k \geq n-3} c_k 2^{-ks} = \sum_{k \geq n-3} c_k 2^{-(k-n)s/p} 2^{-ks/q - ns/p}$ , with  $1/p + 1/q = 1$ , and use Hölder’s inequality to find

$$\begin{aligned} \sum_{k \geq n-3} c_k 2^{-ks} &\leq \left( \sum_{k \geq n-3} 2^{-ks - ns(q-1)} \right)^{1/q} \left( \sum_{k \geq n-3} c_k^p 2^{-(k-n)s} \right)^{1/p} \\ &\leq 2^{-ns} \text{Cst} \left( \sum_{k \geq n-3} c_k^p 2^{-(k-n)s} \right)^{1/p} := \text{Cst } 2^{-ns} \widetilde{c}_n. \end{aligned} \tag{A.9}$$

Since  $\sum_n \widetilde{c}_n^p = \sum_n \sum_{k \geq n-3} c_k^p 2^{-(k-n)s} = \text{Cst} < \infty$ , we can use point (ii) of Lemma A.1 to deduce from Eqs. (A.8) and (A.9) that  $\Sigma_1 \in E_p^s$  and that  $|\Sigma_1|_{E_p^s} \leq \text{Cst } |\widehat{u}|_1 |v|_{E_p^s}$ .

Similarly, one can show that  $\Sigma_2 \in E_p^s$  and that  $|\Sigma_2|_{E_p^s} \leq \text{Cst} |\hat{v}|_1 |u|_{E_p^s}$ , which achieves the proof of the first point of the theorem.

The second point follows from the first one and from the embedding properties of Proposition 1.1.

### Appendix B. Proof of Proposition 3.3

The first step in the proof of Proposition 3.3 is the following key lemma.

**Lemma B.1.** *Let  $1 \leq p < \infty$  and  $1 \leq r < \infty$ . Consider  $f_1 \in L^p(\mathbb{R})$  and  $w_{2\alpha} f_2 \in L^r(\mathbb{R})$ , with  $\alpha$  given below and  $w_\alpha(\xi) = (1 + |\xi|^2)^{\alpha/2}$ . Then the function  $A$  defined as*

$$A(\xi) := \int_{\mathbb{R}} \frac{e^{it[(c-c_2)\xi + (c_2-c_1)\xi']}}{i[(c-c_2)\xi + (c_2-c_1)\xi']} g(f_1(\xi'), f_2(\xi - \xi')) d\xi',$$

satisfies  $A \in L^p(\mathbb{R})$  and

$$|A|_p \leq \begin{cases} \frac{\text{Cst}}{|c-c_2|^{1/pr} |c_2-c_1|^{1/pr}} t^{1-1/r} |f_1|_p |f_2|_r & \text{if } r > 1 \text{ (and } \alpha = 0), \\ \frac{\text{Cst}}{|c-c_2|^{1/pr} |c_2-c_1|^{1/pr}} (1 + \ln t) |f_1|_p |w_{2\alpha} f_2|_\infty & \text{if } r = 1 \text{ and } \alpha > 0, \end{cases}$$

where the constant does not depend on  $c, c_1, c_2$ .

**Proof.** We first prove the lemma in the case  $p = 1$ . Introduce, for all  $\xi, \xi' \in \mathbb{R}^2$ ,

$$q_t(\xi, \xi') := \frac{e^{it[(c-c_2)\xi + (c_2-c_1)\xi']} - 1}{i[(c-c_2)\xi + (c_2-c_1)\xi']} \quad \text{and} \quad F_t(\xi, \xi') := q_t(\xi, \xi') g(f_1(\xi'), f_2(\xi - \xi')).$$

For almost all  $\xi' \in \mathbb{R}$ , one has

$$\begin{aligned} \int |F_t(\xi, \xi')| d\xi &= \text{Cst} |f_1(\xi')| \int_{\mathbb{R}} |q_t(\xi, \xi')| |f_2(\xi - \xi')| d\xi \\ &\leq \text{Cst} |f_1(\xi')| \left( \int_{\mathbb{R}} |q_t(\xi, \xi')|^r d\xi \right)^{1/r} |f_2|_r, \end{aligned} \tag{B.1}$$

where  $1 \leq r < \infty$ . We only consider the case  $r > 1$ ; the modifications to treat the case  $r = 1$  are straightforward. One can compute

$$\begin{aligned} \left( \int_{\mathbb{R}} |q_t(\xi, \xi')|^r d\xi \right)^{1/r} &= \frac{1}{|c-c_2|^{1/r}} \left( \int_{\mathbb{R}} \left| \frac{e^{it\xi} - 1}{i\xi} \right|^r d\xi \right)^{1/r} \\ &\leq \frac{1}{|c-c_2|^{1/r}} \text{Cst} t^{1-1/r}, \end{aligned} \tag{B.2}$$

the last inequality being obtained as in Lemma 2.1; note also that the constant which appears above does not depend on  $c, c_1, c_2$ . From Eqs. (B.1) and (B.2), one can deduce

$$\int_{\mathbb{R}} |F_t(\xi, \xi')| d\xi \leq \frac{\text{Cst}}{|c - c_2|^{1/r}} t^{1-1/r} |f_1(\xi')| |f_2|_r.$$

Integrating this inequality with respect to  $\xi'$  then yields the result of the lemma when  $p = 1$ .

Suppose now that  $p > 1$  and that  $f_1 \in L^p(\mathbb{R})$ . Then for almost all  $\xi \in \mathbb{R}$ , the mapping  $\xi' \mapsto |q_t(\xi, \xi')| |f_1(\xi')|^p |f_2(\xi - \xi')|$  is integrable as a consequence of the result proved for  $p = 1$ . In other words, the mapping  $\xi' \mapsto |q_t(\xi, \xi')|^{1/p} |f_1(\xi')| |f_2(\xi - \xi')|^{1/p}$  is in  $L^p(\mathbb{R})$ . Moreover  $\xi' \mapsto |q_t(\xi, \xi')|^{1/p} |f_2(\xi - \xi')|^{1/p}$  is in  $L^p(\mathbb{R})$ . Indeed, one has

$$\int_{\mathbb{R}} |q_t(\xi, \xi')| |f_2(\xi - \xi')| d\xi' \leq \frac{\text{Cst}}{|c_2 - c_1|^{1/r}} t^{1-1/r} |f_2|_r,$$

the above majoration being established as in Eqs. (B.1) and (B.2).

It follows from these points that for almost all  $\xi \in \mathbb{R}$ , the mapping  $\xi' \mapsto F_t(\xi, \xi')$  is in  $L^1(\mathbb{R})$ , and that

$$\begin{aligned} \int_{\mathbb{R}} |F_t(\xi, \xi')| d\xi' &\leq \left( \int_{\mathbb{R}} |q_t(\xi, \xi')| |f_1(\xi')|^p |f_2(\xi - \xi')| d\xi' \right)^{1/p} \\ &\quad \times \frac{\text{Cst}}{|c_2 - c_1|^{1/pr}} (t^{1-1/r})^{1/p} |f_2|_r^{1/p}. \end{aligned}$$

Therefore,  $|A|_p = (\int_{\mathbb{R}} (\int_{\mathbb{R}} |F_t(\xi, \xi')| d\xi')^p d\xi)^{1/p}$  is bounded by

$$\begin{aligned} |A|_p &\leq \left( \int_{\mathbb{R}^2} |q_t(\xi, \xi')| |f_1(\xi_1)|^p |f_2(\xi - \xi')| d\xi d\xi' \right)^{1/p} \\ &\quad \times \frac{\text{Cst}}{|c_2 - c_1|^{1/pr}} (t^{1-1/r})^{1/p} |f_2|_r^{1/p}. \end{aligned}$$

Using the result proved when  $p = 1$ , we have therefore

$$\begin{aligned} |A|_p &\leq \frac{\text{Cst}}{|c - c_2|^{1/pr}} (t^{1-1/r})^{1/p} |f_1|_p |f_2|_r^{1/p} \times \frac{\text{Cst}}{|c_2 - c_1|^{1/pr}} (t^{1-1/r})^{1/p} |f_2|_r^{1/p} \\ &\leq \text{Cst} \frac{1}{|c - c_2|^{1/pr}} \frac{1}{|c_2 - c_1|^{1/pr}} t^{1-1/r} |f_1|_p |f_2|_r, \end{aligned}$$

which is the result stated in the lemma.  $\square$

**Proof of the Proposition 3.3.** Here again, we only consider the case  $r > 1$ , the case  $r = 1$  being obtained in the same way up to easy modifications.

The solution of (10) with  $J = 2$  can be written,

$$\hat{u}(t, \xi) = e^{-ict\xi} \hat{u}_0(\xi) + e^{-ict\xi} \int_{\mathbb{R}} q_t(\xi, \xi') g(\widehat{v}_0^1(\xi'), \widehat{v}_0^2(\xi - \xi')) d\xi'.$$

Denoting by  $(v_{0,n}^1)_{n \geq -1}$  and  $(v_{0,n}^2)_{n \geq -1}$  the Littlewood–Paley decomposition of  $v_0^1$  and  $v_0^2$ , one can write, with the same notations as in the proof of Theorem 1.1,

$$\begin{aligned} \hat{u}(t, \xi) &= e^{-ict\xi} \hat{u}_0(\xi) + e^{-ict\xi} \sum_{n \geq 0} \int_{\mathbb{R}} q_t(\xi, \xi') g((\widehat{S}_n v_0^1)(\xi'), \widehat{v}_{0,n}^2(\xi - \xi')) d\xi' \\ &\quad + e^{-ict\xi} \sum_{m \geq -1} \int_{\mathbb{R}} q_t(\xi, \xi') g(\widehat{v}_{0,m}^1(\xi'), (\widehat{S}_{m+1} v_0^2)(\xi - \xi')) d\xi' \\ &:= e^{-ict\xi} \hat{u}_0(\xi) + e^{-ict\xi} \left( \sum_{n \geq 0} A_n(t, \xi) + \sum_{m \geq -1} B_m(t, \xi) \right). \end{aligned} \tag{B.3}$$

Thanks to Lemma B.1, we have

$$|A_n(t, \cdot)|_p \leq \frac{\text{Cst}}{|c - c_1|^{1/pr} |c_2 - c_1|^{1/pr}} t^{1-1/r} |\widehat{v}_{0,n}^1|_p |(\widehat{S}_n v_0^1)|_r$$

and

$$|B_m(t, \cdot)|_p \leq \frac{\text{Cst}}{|c - c_2|^{1/pr} |c_2 - c_1|^{1/pr}} t^{1-1/r} |\widehat{v}_{0,m}^1|_p |(\widehat{S}_{m+1} v_0^2)|_r.$$

Since for  $j = 1, 2$  and  $n \geq -1$  one has  $|\widehat{S}_n v_0^j|_r \leq |v_0^j|_r$ , and thanks to Lemma A.1 which asserts that  $|v_{0,n}^j|_p \leq C |v_0^j|_{E_p^s} c_n 2^{-ns}$ , with  $\sum_{n \geq -1} c_n^p \leq 1$ , we obtain

$$|A_n(t, \cdot)|_p \leq \frac{\text{Cst}}{|c - c_1|^{1/pr} |c_2 - c_1|^{1/pr}} t^{1-1/r} C |v_0^2|_{E_p^s} |v_0^1|_{E_r^0} c_n 2^{-ns}, \tag{B.4}$$

and a similar expression for  $|B_m(t, \cdot)|_p$ .

Replacing Eq. (A.7) in the proof of Theorem 1.1 by Eq. (B.4) above, one can then mimic the end of the proof of Theorem 1.1 to conclude the proof of Proposition 3.3.

The last point of the proposition is a consequence of Proposition 1.1.  $\square$

**Remark B.1.** If  $r = p$  then one can check that the constant  $C(c, c_1, c_2)$  of the proposition can be replaced by  $\tilde{C}(c, c_1, c_2) = \frac{1}{|c_2 - c_1|^{1/p^2}} \min(\frac{1}{|c - c_1|^{1/pp}}, \frac{1}{|c - c_2|^{1/pp}})$ , and therefore, when  $c \sim c_1$ ,  $\tilde{C}(c, c_1) \sim \frac{1}{|c_1 - c_2|^{1/p^2}}$ , which is in accordance with the constants given in Proposition 3.2.

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