Asymptotic Analysis 18 (1998) 111–146
IOS Press

Dispersive effects for nonlinear geometrical optics with rectification

David Lannes
MAB, Université Bordeaux I, 33405 Talence, France
E-mail: lannes@math.u-bordeaux.fr

Abstract. Oscillating approximate solutions to nonlinear hyperbolic dispersive systems are studied. Ansatz of three scales are used in order to deal with diffractive effects. The scaling of the approximate solutions is chosen so that diffractive, dispersive effects and rectification are present in the leading term.

The propagation along the rays of geometrical optics of the oscillating Fourier coefficients of the leading terms is corrected by a Schrödinger dispersion which appears for long times only. The propagation of the nonoscillating Fourier coefficient depends on the properties of a symmetric hyperbolic system, whose characteristic variety is the tangent cone at 0 to the characteristic variety of the initial operator.

Equations determining the leading term require a sublinear growth condition for the corrector and the inroduction of the analytical "average operators" which convey this sublinear growth condition in a simple way and sort the nonlinearities out.

In the last part, detailed physical examples are given.

1. Introducing the problem and the first profile equations

1.1. Setting up the problem

The aim of this paper is to construct approximate solutions to the nonlinear symmetric hyperbolic system of equations

$$L^{\varepsilon}(v, \partial_x)v + F(v) = 0, \tag{1}$$

where the principal part is the quasilinear operator

$$L^{\varepsilon}(., \partial_x) := L_1(., \partial_x) + L_0/\varepsilon := \sum_{\mu=0}^{d} A_{\mu}(.)\partial_{\mu} + L_0/\varepsilon.$$

We want to study the behavior of high frequency solutions to this dispersive problem for time scales at which diffractive, dispersive and nonlinear effects (as rectification) are present in the leading term of approximate solutions.

The term L_0 induces dispersion of the different frequencies; that is why the effects here described differ qualitatively from those described in [7] where one has $L_0 = 0$.

In the last section, we give physical examples where these phenomena occur.

As in [5] and [7], we seek ansatz with three scales (such a structure being typical of diffractive geometric optics)

$$u^{\varepsilon}(x) = \varepsilon^{p} u(\varepsilon, \varepsilon x, x, x \cdot \beta/\varepsilon), \tag{2}$$

where

$$u(\varepsilon, X, x, \theta) = a(X, x, \theta) + \varepsilon b(X, x, \theta) + \varepsilon^2 c(X, x, \theta). \tag{3}$$

Here $x = (t, y) \in \mathbb{R}^{1+d}$, $\beta = (\tau, \eta) \in \mathbb{R}^{1+d}$ and the profiles $a(X, x, \theta)$, $b(X, x, \theta)$ and $c(X, x, \theta)$ are smooth with respect to X, x, θ and periodic in θ.

The exponent p is a critical exponent for which the time scales for diffractive and nonlinear effects are the same.

Since the above expansion is used for times $t \sim 1/\varepsilon$ one must actually control the growth of the profile in t. In order for the corrector b to be effectively smaller than the leading term a for such times, we want it to satisfy the *sublinear growth condition* introduced in [5]:

$$\lim_{t\to\infty} \frac{1}{t} \|\partial^{\gamma}_{X,x,\theta} b(T, Y, t, y, \theta)\|_{L^2([0,\underline{T}]\times\mathbb{R}^d_Y\times\mathbb{R}^d_y\times\mathbb{T})} = 0, \tag{4}$$

for all $\gamma \in \mathbb{N}^{2d+3}$.

The following assumption expresses the symmetric hyperbolicity of our problem (1).

Assumption 1 (symmetric hyperbolicity). *The coefficients A_μ are smooth real hermitian symmetric valued functions of u on a neighborhood of $0 \in \mathbb{C}^N$, and, for each u, $A_0(u)$ is positive-definite.*

We also assume that the system is conservative in the sense that $L_0^* = -L_0$.

Remark. Thanks to the assumption made on $A_0(u)$, we can make the change of dependant variable to $A_0(0)^{-1/2}u$. Multiplying the resulting equations by $A_0(0)^{-1/2}$ preserves the hypotheses and reduces to the case $A_0(0) = I$.

From now on, we will take $A_0(0) = I$.

By the way, we also make the following assumption on the order of the nonlinearities:

Assumption 2 (order of nonlinearities). *The quasilinear terms are of order $2 \leqslant K \in \mathbb{N}$ which means*

$$|\alpha| \leqslant K - 2 \quad \Rightarrow \quad \partial^{\alpha}_{\mathrm{Re}\, u,\ \mathrm{Im}\, u} |A_\mu(u) - A_\mu(0)|_{u=0} = 0.$$

Thus, $A_\mu(u) - A_\mu(0) = \mathrm{O}(|u|^{K-1})$ and the quasilinear term, $(A_\mu(u) - A_\mu(0))\partial_\mu u$, is of order K.

As for the semilinear term F, we assume it is smooth on a neighborhood of $0 \in \mathbb{C}^N$, and of order $J \geqslant 2$ which means

$$|\gamma| \leqslant J - 1 \quad \Rightarrow \quad \partial^{\gamma}_{\mathrm{Re}\, u,\ \mathrm{Im}\, u} F(0) = 0.$$

Then, $F(u) = \mathrm{O}(|u|^J)$.

As usual we have to chose the scaling of the problem, i.e., the size of the solutions we are looking for. We must therefore chose the exponent p. The standard normalization, whose goal is having a time of nonlinear interaction of order ε^{-1}, is the following *standard normalization*:

$$p = \max\left\{\frac{2}{K-1}, \frac{1}{J-1}\right\}.$$

In the approximate solution we are looking for, only the leading order nonlinear terms will be of importance; we now define them:

Definition 1 (leading order nonlinear terms). If $p = 2/(K-1)$, the degree $K-1$ Taylor polynomial of $A_\mu(u) - A_\mu(0)$ at $u = 0$ is denoted $\Gamma_\mu(u)$.

If $p < 2/(K-1)$, set $\Gamma_\mu := 0$ for all μ.

If $p = 1/(J-1)$, let $\Phi(u)$ denote the degree J Taylor polynomial of F at $u = 0$. If $p < (J-1)$, set $\Phi := 0$.

1.2. Equations for the profiles

Plugging (2) into Eq. (1) and applying the chain rules leads to

$$L^\varepsilon(u^\varepsilon, \partial_x)u^\varepsilon + F(u^\varepsilon) = \left[L\left(\varepsilon^p u, \varepsilon\partial_X + \partial_x + \frac{\beta}{\varepsilon}\partial_\theta\right)\varepsilon^p u + F(\varepsilon^p u)\right]_{X=\varepsilon x,\ \theta = x\cdot\beta/\varepsilon}.$$

Now plugging expansion (3) of u into this equation and expanding the function of (X, x, θ) in brackets in powers of ε gives the following result:

Lemma 1. *Suppose that a, b, c are smooth functions on* $[0, T] \times \mathbb{R}^d \times \mathbb{R}^{1+d} \times \mathbb{T}$ *and that u is given by* (3). *Then*

$$\begin{aligned} &L^\varepsilon\left(\varepsilon^p u, \varepsilon\partial_X + \partial_x + \frac{\beta}{\varepsilon}\partial_\theta\right)\varepsilon^p u + F(\varepsilon^p u) \\ &\quad = \varepsilon^{p-1} r_0(X, x, \theta) + \varepsilon r_1(X, x, \theta) + \varepsilon^2 r_2(X, x, \theta) + \varepsilon^2 l(\varepsilon, X, x, \theta), \end{aligned} \tag{5}$$

where the $r_j(X, x, \theta)$ are given by

$$\begin{aligned} r_0 &= \left(L_0 + L_1(0, \beta)\partial_\theta\right)a, \\ r_1 &= \left(L_0 + L_1(0, \beta)\partial_\theta\right)b + L_1(0, \partial_x)a, \\ r_2 &= \left(L_0 + L_1(0, \beta)\partial_\theta\right)c + L_1(0, \partial_x)b + L_1(0, \partial_X)a + \Phi(a) + \sum_{\mu=0}^{d} \beta_\mu \Gamma_\mu(a)\partial_\theta a. \end{aligned}$$

The function $l(\varepsilon, X, x, \theta)$ depends on the functions a, b, c, is periodic in θ and each of its derivatives with respect to X, x, θ is continuous on $[-1, 1] \times \mathbb{R}^{1+d} \times \mathbb{R}^{1+d} \times \mathbb{T}$.

Our stategy is to choose a, b and c so that one has $r_j = 0$ for $j = 0, 1, 2$. Thanks to the above lemma, this is equivalent to the following equations:

$$\left(L_0 + L_1(0, \beta)\partial_\theta\right)a = 0, \tag{6}$$

$$\left(L_0 + L_1(0, \beta)\partial_\theta\right)b + L_1(0, \partial_x)a = 0, \tag{7}$$

$$\left(L_0 + L_1(0, \beta)\partial_\theta\right)c + L_1(0, \partial_x)b + L_1(0, \partial_X)a + \Phi(a) + \sum_{\mu=0}^{d} \beta_\mu \Gamma_\mu(a)\partial_\theta a = 0. \tag{8}$$

Notations: From now on, when no mistake is possible, we will write $L^\varepsilon(\partial_x)$ and $L_1(\partial_x)$ instead of $L^\varepsilon(0, \partial_x)$ and $L_1(0, \partial_x)$.

The nonlinearity due to quasilinearity will be noted

$$\Psi(a) := \sum_{\mu=0}^{d} \beta_\mu \Gamma_\mu(a)\partial_\theta(a).$$

1.3. *A brief commentary on these equations*

As usual in geometric optics, (6) will impose resolubility conditions on the Fourier coefficients a_n of a. Indeed, if we want a_n to be a non-trivial solution to (6), it is well known that $n\beta$ must belong to the characteristic variety Char L of the symbol $L(\tau, \eta) := \tau I + \sum \eta_\mu A_\mu(0) + L_0/\mathrm{i}$. The projector $\pi(n\beta)$ on ker $L(n\beta)$ is in that case non-zero, and one must have $\pi(n\beta)a_n = a_n$. This is an algebraic *resolubility condition* which will be useful to derive necessary conditions on a from Eqs. (6)–(8).

The projectors $\pi(n\beta)$ are also used to exploit Eq. (7), which is often referred to as the "transport equation". It is well known indeed that $T_n(\partial_x) := \pi(n\beta)L_1(\partial_x)\pi(n\beta)$ is a scalar transport operator when $n\beta$ is a smooth point of Char L, and that a_n annihilates T_n.

The case of the non-oscillating coefficient a_0 is singular since 0 is often a crosspoint of Char L. In [7], the non-dispersive $L_0 = 0$ case has been considered: one then has $\pi(0) = \mathrm{Id}$. In that case the operator $T_0 := \pi(0)L_1(\partial_x)\pi(0)$ annihilated by a_0 coincides with the original operator L_1. In this paper we have to deal with the dispersive case where $L_0 \neq 0$ and, therefore, $\pi(0) \neq \mathrm{Id}$ and $T_0 \neq L_1$. We thus have to study this new operator T_0. We will prove that its characteristic variety is the tangent variety at 0 to Char L, in a sense to be precised later. We can also decompose a_0 into simpler waves, using the spectral decomposition of the symbol of T_0. Indeed, $T_0(\partial_x)a_0 = 0$ implies the following decomposition (cf. Section 2.5):

$$a_0(X, t, y) = \sum_\alpha \mathrm{e}^{\mathrm{i}t\tau_\alpha^0(D_y)} E_\alpha(D_y) f(X, y),$$

where the functions τ_α^0 parametrize Char T_0 and $E_\alpha(D_y)$ are the pseudo-differential orthogonal projectors associated to the spectral decomposition of T_0. We will denote $\mathcal{A}_\mathrm{f}$ the set of all α such the τ_α^0 parametrizes a flat part of Char T_0, and $\mathcal{A}_\mathrm{c}$ its complementary.

In order to determine the profile a, we still lack a few necessary conditions; they may be obtained using the sublinear growth condition (4). We work indeed with times $\mathrm{O}(1/\varepsilon)$ for which diffractive and dispersive effects, as well as rectification are not negligible. The sublinear growth condition (4) is therefore a non-trivial constraint. We recall that $\pi(\beta)$ has been introduced to derive resolubility conditions from (6). In this paper we introduce a new kind of projectors called "average projectors". They are of analytical nature and are used to derive *new resolubility conditions* from the sublinear growth condition (4). These resolubility conditions take the form of a Schrödinger-like equation with respect to the slow time scale T, which involves symmetric second order operators denoted R_n and nonlinearities involving the a_n and $a_{0,f}$ transported at the same speed, denoted $g_n(a_j, a_{0,f})$.

It will then be possible to determine the profile a.

Theorem 1. *Let $h(Y, y, \theta)$ be a trigonometric polynomial in θ with Fourier coefficients $h_n = \pi(n\beta)h_n$.*

There exists $T_* > 0$ and a unique $a(T, Y, t, y, \theta)$ such that $\pi(n\beta)a_n = a_n$ for all n, satisfying the initial condition $a(0, Y, 0, y, \theta) = h(Y, y, \theta)$ at $T = t = 0$ together with the evolution equations

$$T_m(\partial_x)a_m = 0 \quad \textit{for all } m,$$

and

$$\begin{aligned}
&T_n(\partial_X)\pi(n\beta)a_n + \mathrm{i}R_n(\partial_y)\pi(n\beta)a_n + \pi(n\beta)g_n(a_j, a_{0,f}) = 0,\\
&\{\partial_T + v_f \cdot \partial_Y + \mathrm{i}E_f(D_y)R_0(\partial_y)\}E_f(D_y)a_{0,f} + E_f(D_y)g_f(a_j, a_{0,f}) = 0,\\
&\{\partial_T + \tau_c^0(D_Y) + \mathrm{i}E_c(D_y)R_0(\partial_y)\}E_c(D_y)a_{0,c} = 0,
\end{aligned}$$

for $n \neq 0$, $f \in \mathcal{A}_\mathrm{f}$ and $c \in \mathcal{A}_\mathrm{c}$.

Remark. The nonlinearities couple the evolution equations for the a_n and the $a_{0,f}$, if they are transported at the same speed. If, for instance, all the a_n are transported at different speeds, and if there is no flat part in Char T_0, then all the evolution equations above are linear, and in particular rectification does not occur.

It is then easy to find the correctors b and c. Once proved the existence of the approximate solution $u = a + \varepsilon b + \varepsilon^2 c$, we will prove that its remainder is small and an essential stability theorem: the approximate solution u just defined remains close to an exact solution to problem (1), for times $O(1/\varepsilon)$.

Theorem 2. *Let v^ε be the exact solution to the initial value problem*

$$L^\varepsilon(v^\varepsilon, \partial_x)v^\varepsilon + F(v^\varepsilon) = 0, \quad v^\varepsilon|_{t=0} = u^\varepsilon|_{t=0}.$$

Then, there exists $T_- > 0$ so that

(i) *For each $T \in]0, T_-[$ and for ε small enough, v^ε exists and is smooth on $[0, T/\varepsilon] \times \mathbb{R}^d$.*
(ii) *We can write $v^\varepsilon = \varepsilon^p \mathcal{V}^\varepsilon(t, y, y \cdot \eta/\varepsilon)$ and $u^\varepsilon = \varepsilon^p \mathcal{U}^\varepsilon(t, y, y \cdot \eta/\varepsilon)$, with $\mathcal{U}^\varepsilon, \mathcal{V}^\varepsilon \in C^\infty([0, T/\varepsilon]; H^\infty(\mathbb{R}^d \times \mathbb{T}))$, and one has, for all s,*

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant t \leqslant T/\varepsilon} \left\| \mathcal{U}^\varepsilon(t) - \mathcal{V}^\varepsilon(t) \right\|_{H^s(\mathbb{R}^d \times \mathbb{T})} = 0.$$

2. Algebraic analysis of Eqs (6)–(8)

2.1. Algebraic resolubility conditions

Definition 2. For $\beta = (\tau, \eta) \in \mathbb{R}^{1+d}$, $\pi(\beta)$ denotes the linear projection on the kernel of $L(\beta) := \tau I + \sum_{\mu=1}^d \eta_\mu A_\mu(0) + L_0/\mathrm{i}$ along its range, and $Q(\beta)$ is the partial inverse defined by

$$Q(\beta)\pi(\beta) = 0 \quad \text{and} \quad Q(\beta)L(\beta) = I - \pi(\beta).$$

Thanks to symmetric hyperbolicity, one observes that $\pi(\beta)$ and $Q(\beta)$ are hermitian symmetric and, in particular, that $\pi(\beta)$ is an orthogonal projector.

We also introduce the following operators which act on trigonometric polynomials:

Definition 3. If $b(X,x,\theta)$ is a trigonometric polynomial given by $b(X,x,\theta) = \sum_n b_n(X,x)\mathrm{e}^{\mathrm{i}n\theta}$, then we define $\mathbf{\Pi}(\beta)$ by

$$\mathbf{\Pi}(\beta)b(X,x,\theta) := \sum_n \pi(n\beta)b_n(X,x)\mathrm{e}^{\mathrm{i}n\theta}.$$

Similarly, we define the operators $\mathbf{Q}(\beta)$ and $\mathbf{L}(\beta)$.

We can now formulate the straightforward following lemma which expresses the resolubility condition of an equation of the type $L(\beta)x = y$.

Lemma 2. *For $x, y \in \mathbb{R}^N$, one has the equivalence*

$$L(\beta)x = y \quad \Rightarrow \quad \pi(\beta)y = 0 \quad \textit{and} \quad x = \pi(\beta)x + Q(\beta)y.$$

The same property also holds for the operators $\mathbf{\Pi}$*,* $\mathbf{Q}$ *and* $\mathbf{L}$*.*

2.2. Consequences for the profile equations

We can now apply the above lemma to our Eqs (6)–(8).

Equation $r_0 = 0$: On the Fourier coefficients of a, (6) reads $\mathrm{i}L(n\beta)a_n = \mathrm{i}(L_1(n\beta) + L_0/\mathrm{i})a_n = 0$.

We make here the following assumption:

Assumption 3. *If $L(\beta)$ is non-invertible, $L(n\beta)$ is non-invertible for only a finite number of $n \in \mathbb{Z}$. Let us denote $\mathcal{N}$ the set of such non-zero values of n.*

Remark. In order to satisfy this assumption it is sufficient to have $L_1(\beta)$ invertible, which is the case in all the physical examples we have met.

In order to have non-trivial solutions to the above equation, we take β such that $\det L(\beta) = 0$ (we will introduce in the next section the characteristic variety of L which is the set of all such β). The assumption thus forces a_n to be equal to 0 for almost all values of n. Therefore, a is a trigonometric polynomial of degree at most $\sup \mathcal{N}$.

Equation (6) then reads $\mathbf{L}(\beta)a = 0$, and Lemma 2 says this is equivalent to

$$a = \mathbf{\Pi}(\beta)a. \tag{9}$$

Equation $r_1 = 0$: Since we know that a is a trigonometric polynomial, it is easy to see with the same reasoning as above that b is also a trigonometric polynomial. Equation (7) then reads $\mathrm{i}\mathbf{L}(\beta)b = -L_1(\partial_x)a$.

Thanks to Lemma 2, this is equivalent to

$$\mathbf{\Pi}(\beta)L_1(\partial_x)a = 0 \quad \text{and} \quad b = \mathbf{\Pi}(\beta)b + \mathrm{i}\mathbf{Q}(\beta)L_1(\partial_x)a.$$

Thanks to (9), this reads

$$\mathbf{\Pi}(\beta)L_1(\partial_x)\mathbf{\Pi}(\beta)a = 0 \quad \text{and} \quad \big(I - \mathbf{\Pi}(\beta)\big)b = \mathrm{i}\mathbf{Q}(\beta)L_1(\partial_x)\mathbf{\Pi}(\beta)a. \tag{10}$$

Equation $r_2 = 0$: Since a and b are trigonometric polynomials, it is quite easy to see that so are $\Phi(a)$, $\Psi(a)$ and c. Equation (8) thus reads $\mathrm{i}\mathbf{L}(\beta)c = -L_1(\partial_x)b - L_1(\partial_X)a - \Phi(a) - \Psi(a)$.

Thanks to Lemma 2, this is equivalent to

$$\mathbf{\Pi}(\beta)L_1(\partial_x)b + \mathbf{\Pi}(\beta)L_1(\partial_X)a + \mathbf{\Pi}(\beta)\big(\Phi(a) + \Psi(a)\big) = 0$$

and

$$\big(I - \mathbf{\Pi}(\beta)\big)c = \mathrm{i}\mathbf{Q}(\beta)L_1(\partial_x)b + \mathrm{i}\mathbf{Q}(\beta)L_1(\partial_X)a + \mathrm{i}\mathbf{Q}(\beta)\big(\Phi(a) + \Psi(a)\big). \tag{11}$$

The first of these two equations with the help of (10) gives

$$\mathbf{\Pi}(\beta)L_1(\partial_x)\mathbf{\Pi}(\beta)b = -\mathrm{i}\mathbf{\Pi}L_1(\partial_x)\mathbf{Q}L_1(\partial_x)\mathbf{\Pi}a - \mathbf{\Pi}L_1(\partial_X)\mathbf{\Pi}a - \mathbf{\Pi}(\beta)\big(\Phi(a) + \Psi(a)\big). \tag{12}$$

2.3. About the operator $\mathbf{\Pi}(\beta)L_1(\partial_x)\mathbf{\Pi}(\beta)$

We have met in the previous section conditions of the type $L(\beta)x = 0$; in order to have non-trivial solutions to this equation, we need to have $\det L(\beta) = 0$. That is why we introduce the characteristic variety:

Definition 4 (characteristic variety). The characteristic variety of the operator L is defined by

$$\operatorname{Char} L = \left\{\beta = (\tau, \eta) \in \mathbb{R}^{1+d},\ \det\left(\tau I + \sum_{\mu=1}^{d} \eta_\mu A_\mu(0) + L_0/\mathrm{i}\right) = 0\right\}.$$

Remark. If L_0 were not here, that is, if the problem were not dispersive, this variety would be homogeneous; it is no longer the case here, that is why Assumption 3 makes sense.

Thanks to symmetric hyperbolicity, the polynomial equation in τ defining $\operatorname{Char} L$, $\det L(\tau, \eta) = 0$, has real roots for all η and, therefore, one can parametrize $\operatorname{Char} L$ by η on $\mathbb{R}^d$ with a finite number of functions $\tau_i(\eta)$:

$$\det\big(\tau_i(\eta), \eta\big) = 0, \quad \forall \eta \in \mathbb{R}^d.$$

Such a variety may have singular points; these are the points where the multiplicity of the roots of the polynomial in τ, $\det(L(\beta))$ changes, i.e., the points where the graphs of the functions τ_i intersect. The origin $0 \in \mathbb{R}^{1+d}$ is often singular, and we will make the following assumption:

Assumption 4. *The singular points of* $\operatorname{Char} L$ *are isolated and located on* $\mathbb{R}_\tau \times \{0\}_{\mathbb{R}^d}$.

It follows that for each $\eta \neq 0$ the numbers of distinct $\tau_i(\eta)$ (i.e., the number of distinct sheets of $\mathrm{Char}\,L$) is the same.

It also follows that every τ_i belong to $C^\infty(\mathbb{R}^d\backslash\{0\})$.

In Eqs (10), (12) derived in the previous section, operators involving $\mathbf{\Pi}(\beta)$, $\mathbf{Q}(\beta)$ and $L_1(\partial_x)$, $L_1(\partial_X)$ appeared. When β corresponds to a smooth point of the characteristic variety of L, these operators admit simple scalar expressions. It is no more the case when β is singular.

Proposition 1. *If $\beta = (\tau_i(\eta), \eta)$ is a smooth point of the characteristic variety of L, then*

$$\pi(\beta)L_1(\partial_x)\pi(\beta) = \pi(\beta)\big(\partial_t - \tau_i'(\eta)\cdot\partial_y\big)$$

and

$$\pi(\beta)L_1(\partial_x)Q(\beta)L_1(\partial_x)\pi(\beta) = \frac{1}{2}\pi(\beta)\tau_i''(\eta)(\partial_y, \partial_y).$$

Proof. See [5], for instance. □

The first part of this proposition implies that the characteristic variety of $\pi(\beta)L_1(\partial_x)\pi(\beta)$ is the tangent plane to $\mathrm{Char}\,L$ in β. This property, as we show now, remains true if $\beta = 0$ is singular, though we do not have the exact expression of $\pi(0)L_1(\partial_x)\pi(0)$.

Proposition 2. *Suppose that $\beta = 0$ is an isolated singular point of* $\mathrm{Char}\,L$. *Then,* $\mathrm{Char}\,\pi(0)L_1(\partial_x)\pi(0)$ *is the tangent cone of* $\mathrm{Char}\,L$ *at* 0.

Remark. By tangent cone at a singular point we mean the contingent cone, as in [3]. Let f_i, $i = 1, \dots, s$, be s functions of $\mathcal{D}(\mathbb{R}^d)$ such that $f_1(0) = \cdots = f_s(0) = 0$. The contingent cone at $\eta = 0$ of the set $K := \{(f_i(\eta), \eta),\ \eta \in \mathbb{R}^d,\ i = 1, \dots, s\}$ is given by

$$T_K(0) := \big\{(u, v) \in \mathbb{R}^{1+d},\ \exists i,\ u = \mathrm{d}f_i(0, v)\big\},$$

where $\mathrm{d}f_i(0, v)$ denotes the directional derivative of f_i at the point 0 and in the direction v. It is not necessary for the f_i to be in $\mathcal{D}(\mathbb{R}^d)$, but these directional derivatives must exist.

Proof. In this proof we will denote by $\tau_1(\eta)$ for $i = 1, \dots, s$ the functions parametrizing $\mathrm{Char}\,L$ and whose value is 0 in 0, and $\pi_i(\eta) := \pi(\tau_i(\eta), \eta)$ the orthogonal projectors on $\ker(\tau_i(\eta)I + \sum \eta_\mu A_\mu + L_0/\mathrm{i})$, for $\eta \neq 0$. Introduce also the global projector $P(\eta) := \pi_1(\eta) + \cdots + \pi_s(\eta)$, for $\eta \neq 0$. For $\eta = 0$, one takes of course $P(0) := \pi(0)$.

The result we want to prove is thus the following:

$$\mathrm{Char}\,\pi(0)L_1\pi(0) = \big\{(u, v) \in \mathbb{R}^{1+d},\ \exists i,\ u = \mathrm{d}\tau_i(0, v)\big\}. \tag{13}$$

(1) The projector P is C^∞ in a neighborhood of 0; this is a consequence of the following expression of P:

$$P(\eta) := \frac{1}{2\mathrm{i}\pi}\int_{\mathcal{C}(0,r)} \Big(zI - \mathrm{i}\sum \eta_\mu A_\mu - L_0\Big)^{-1}\mathrm{d}z,$$

where r is small enough, so that there are no other roots than the $\tau_i,\ i = 1,\dots,s$, in $D(0,r)$.

(2) We now have to prove that the directional derivatives $\mathrm{d}\tau_i(0,v)$ make sense, which is not obvious since τ_i is not differentiable at 0. We have to prove that the directional derivative $\mathrm{d}\tau_{i_0}(v/n,v)$ admits a limit $\mathrm{d}\tau_{i_0}(0,v)$ when $n \to \infty$.

It is sufficient to prove that the function $t \in \mathbb{R} \to \mathrm{d}\tau_{i_0}(tv,v)$ defined on $]0,1]$ with value in $\mathbb{R}$ can be extended by continuity in 0.

Since this function is semi-algebraic continuous on $]0,r]$, it is sufficient to prove that it is bounded (see [2, Proposition 2.5.3]).

We introduce here the function τP defined by $\tau P(\eta) := \tau_1(\eta)\pi_1(\eta) + \cdots + \tau_s(\eta)\pi_s(\eta)$, for $\eta \in \mathbb{R}^d \backslash \{0\}$ and $\tau P(0) := 0$.

The function τP is C^∞ in a neighborhood of 0 since it has the following expression:

$$\tau P(\eta) = \frac{1}{2\mathrm{i}\pi}\int_{C(0,r)} z\Bigl(zI - \mathrm{i}\sum \eta_\mu A_\mu - L_0\Bigr)^{-1}\,\mathrm{d}z.$$

By the way, one has, for $\eta \neq 0$,

$$(\tau P)'(\eta) = \sum_{i=1}^{s}\bigl(\tau_i'(\eta)\pi_i(\eta) + \tau_i(\eta)\pi_i'(\eta)\bigr). \tag{14}$$

Since $\pi_i\pi_j = \delta_{i,j}\pi_i$, one has $\pi_i(\eta)\pi_j'(\eta)\pi_i(\eta) = 0$, for all i and j.

Multiplying (14) on both right and left sides by $\pi_{i_0}(\eta)$ thus yields

$$\pi_{i_0}(\eta)(\tau P)'(\eta)\pi_{i_0}(\eta) = \tau_{i_0}'(\eta)\pi_{i_0}(\eta).$$

Taking the norm of both sides of this equality yields

$$\bigl\|\tau_{i_0}'(\eta)\bigr\| \leqslant \bigl\|(\tau P)'(\eta)\bigr\|.$$

Since $\tau P(\eta)$ is C^∞ on a neighborhood of 0, it is bounded; therefore, $\tau_{i_0}'(\eta)$ is bounded and, a fortiori, $t \to \mathrm{d}\tau(tv,v)$ is bounded.

As said above, we can therefore extend the directional derivatives in 0 and we denote $\mathrm{d}\tau_{i_0}(0,v)$ its value.

(3) We now want to prove the direct inclusion in (13).

Let $(u,v) \in \mathbb{R}^{1+d}$ and $x = \pi(0)x \neq 0$ so that $\pi(0)L_1(u,v)\pi(0)x = 0$. This relation says that (u,v) is in Char $\pi(0)L_1\pi(0)$.

The first step is the following property: there exists an i_0 such that

$$\bigl(u - \mathrm{d}\tau_{i_0}(v/n,v)\bigr)\pi_{i_0}(v/n)x \to 0 \quad \text{when } n \to 0. \tag{15}$$

Since $P(0) = \pi(0)$ and P is continuous, one has

$$P(v/n)L_1(u,v)P(v/n)x \to 0 \quad \text{when } n \to \infty.$$

Since $v/n \neq 0$ corresponds to a smooth point of Char L, replacing P by its expression $P = \pi_1 + \cdots + \pi_s$ and using Proposition (1) yields

$$\sum_{i=1}^{s} \bigl(u - \mathrm{d}\tau_i(v/n, v)\bigr)\pi_i(v/n)x + \sum_{k\neq l} \pi_k(v/n) \sum_\mu v_\mu A_\mu \pi_l(v/n)x \to 0. \tag{16}$$

But, thanks to a continuity argument,

$$\pi_1(v/n)x + \cdots + \pi_s(v/n)x = P(v/n)x \to x \neq 0.$$

Thus, there exist i_0 such that $\pi_{i_0}(v/n)x \nrightarrow 0$.

We then multiply (16) on the left by $\pi_{i_0}(v/n)$. Since $\pi_i\pi_j = 0$ if $i \neq j$ and $\pi_i^2 = \pi_i$, this yields

$$\bigl(u - \mathrm{d}\tau_{i_0}(v/n, v)\bigr)\pi_{i_0}(v/n)x - \pi_{i_0}(v/n) \sum_\mu v_\mu A_\mu \sum_{k\neq i_0} \pi_k(v/n)x \to 0. \tag{17}$$

By definition of the π_i, one has, for all k,

$$\pi_{i_0}(v/n)\Bigl(\tau_{i_0}(v/n)I + \sum_\mu \frac{v_\mu}{n} A_\mu + L_0/\mathrm{i}\Bigr)\pi_k(v/n) = 0.$$

Thus, it follows that

$$\pi_{i_0} \sum_\mu v_\mu A_\mu \sum_{k\neq i_0} \pi_k(v/n)x = -\frac{n}{\mathrm{i}}\pi_{i_0}(v/n)L_0 \sum_{k\neq i_0} \pi_k(v/n)x. \tag{18}$$

Introduce the matrix M_0 such that $M_0 = M_0^*$, $M_0^2 = L_0/\mathrm{i}$ and $\ker M_0 = \ker L_0$. We then have

$$\bigl\|\pi_{i_0}(v/n)L_0\pi_k(v/n)x\bigr\| \leqslant \bigl\|P(v/n)M_0\bigr\|^2\|x\|. \tag{19}$$

The mapping $\phi : y \to P(y)M_0$ is C^∞ and satisfies $\phi(0) = 0$. From the Mean Value Theorem, we therefore deduce the existence of a real positive constant M such that

$$\bigl\|P(v/n)M_0\bigr\| \leqslant M/n. \tag{20}$$

From (18), (19) and (20) we deduce that

$$\pi_{i_0}(v/n) \sum_\mu v_\mu A_\mu \sum_{k\neq i_0} \pi_k(v/n)x \to 0.$$

This relation together with relation (17) yields $(u - \mathrm{d}\tau_{i_0}(v/n, v))\pi_{i_0}(v/n)x \to 0$.

Thanks to the second point of this proof, Eq. (15) reads $(u - \mathrm{d}\tau_{i_0}(0, v))\pi_{i_0}(v/n)x \to 0$.

But since we have chosen i_0 so that $\pi_{i_0}(v/n)x \nrightarrow 0$, we have shown

$$u = \mathrm{d}\tau_{i_0}(0, v),$$

thus proving the direct inclusion.

(4) We now prove the reverse inclusion: let us take $u = \mathrm{d}\tau_{i_0}(0, v)$ and $x_n = \pi_{i_0}(v/n)x_n$ so that $\|x_n\| = 1$ and $(x_n)_n$ admits a limit x. We have

$$
\begin{aligned}
&P(v/n)L_1(u, v)P(v/n)x \\
&\quad = \sum_i \bigl(u - \mathrm{d}\tau_i(v/n, v)\bigr)\pi(v/n)x + \sum_{k \neq l} \pi_k(v/n) \sum_\mu v_\mu A_\mu \pi_l(v/n)x \\
&\quad = \bigl(u - \mathrm{d}\tau_{i_0}(v/n, v)\bigr)\pi_{i_0}(v/n)x + \sum_{k \neq l} \pi_k(v/n) \sum_\mu v_\mu A_\mu \pi_l(v/n)x
\end{aligned}
$$

(since $\pi_j(v/n)x = 0$ if $j \neq i_0$).

But, with the same arguments as above, we can show that

$$
\sum_{k \neq l} \pi_k(v/n) \sum_\mu v_\mu A_\mu \pi_l(v/n)x \to 0.
$$

And since $(u - \mathrm{d}\tau_{i_0}(v/n, v))\pi_{i_0}(v/n)x \to (u - \mathrm{d}\tau_{i_0}(0, v))x = 0$, one has

$$
P(v/n)L_1(u, v)P(v/n)x \to 0.
$$

By continuity this yields $\pi(0)L_1(u, v)\pi(0)x = 0$. Since $x = \pi(0)x \neq 0$, it follows that $(u, v) \in \operatorname{Char} \pi(0)L_1\pi(0)$ and the reverse inclusion is then proved.

This achieves the proof of the proposition. □

2.4. Another form of the profile equations

For $\beta \neq 0$ a smooth point in $\operatorname{Char} L$ and $n \in \mathcal{N}$ (which is a finite set thanks to Assumption 3), introduce the scalar operators $T_n(\partial_x)$:

$$
T_n(\partial_x) := \partial_t - \tau'(n\beta) \cdot \partial_y,
$$

where $\tau'(n\beta) := \tau_i'(n\eta)$ if $n\beta = (\tau_i(n\eta), n\eta)$.

We thus have $\pi(n\beta)L_1(\partial_x)\pi(n\beta) = \pi(n\beta)T_n(\partial_x)$.

For $\beta = 0$ we define

$$
T_0(\partial_x) := \pi(0)L_1(\partial_x)\pi(0),
$$

which may not be a scalar operator as were the T_n for $n \in \mathcal{N}$.

We finally define the operators

$$
R_n(\partial_y) := \pi(n\beta)L_1(\partial_x)Q(n\beta)L_1(\partial_x)\pi(n\beta),
$$

which are scalar with the expression given by Proposition 1 if $n \in \mathcal{N}$ and may be matricial if $n = 0$.

Equation (10) then reads

$$
T_n(\partial_x)\pi(n\beta)a_n = 0 \quad \text{for } n \in \mathcal{N} \cup \{0\}, \tag{21}
$$

and Eq. (12) says, for all $n \in \mathcal{N} \cup \{0\}$,

$$T_n(\partial_x)\pi(n\beta)b_n = -\mathrm{i}R_n(\partial_y)\pi(n\beta)a_n - T_n(\partial_X)\pi(n\beta)a_n - \pi(n\beta)\big(\Phi(a)_n + \Psi(a)_n\big) \tag{22}$$

(we recall that $\Psi(a) := \sum_{\mu=0}^{d} \beta_\mu \Gamma_\mu(a)\partial_\theta a$).

2.5. A remark on the equation $T_0(\partial_x)a_0 = 0$

The equations $T_n(\partial_x)\pi(n\beta)a_n = 0$ for $n \in \mathcal{N}$ are easy to understand since the $T_n(\partial_x)$ are then scalar transport operators. But when 0 is a singular point of Char L, $T_0(\partial_x)$ is not a scalar operator and the evolution law for a_0 with respect to the variables (t, y) is not as simple as for the $a_n,\ n \in \mathcal{N}$. However, since $T_0(\partial_x)$ is a non-dispersive symmetric hyperbolic system on the range of $\pi(0)$, one can decompose a_0 into simpler waves.

We first have to say a few words about Char T_0. Thanks to Proposition 2, we know that it is the tangent cone to Char L at 0, which allows us to visualize it. A new difficulty concerning the singular points arises when treating this new variety; indeed, we have supposed that 0 is at most an isolated singular point of Char L, but, in general, this property does not hold for its tangent variety Char T_0. Moreover, physical examples where Char T_0 admits non-isolated singular points exist (as in the example of ferromagnetism given in Section 5.5) preventing us from making Assumption 4 on Char T_0.

Thus, as in [7], we have to introduce the notion of good and bad wave number.

Definition 5. A wave number $\eta \in \mathbb{R}^d \setminus \{0\}$ is good when all the points of Char T_0 with $\mathbb{R}^d$ coordinate equal to η are smooth.

The complementary set consists of bad wave numbers.

These sets are respectively denoted $\mathcal{G}$ and $\mathcal{B}$.

Proposition 3. (i) *$\mathcal{B}$ is a closed conic set of measure zero in $\mathbb{R}^d \setminus \{0\}$.*

(ii) *$\mathcal{G}$ is the disjoint union of a finite family of conic connected open sets $\Omega_g \subset \mathbb{R}^d \setminus \{0\},\ g \in \mathcal{G}$.*

(iii) *If $\lambda(\eta) = -\mathbf{v} \cdot \eta$ is a root of* $\det T_0(\tau, \eta) = 0$, *then its multiplicity is independent of $\eta \in \mathcal{G}$.*

(iv) *If $\lambda(\eta)$ is a root of* $\det T_0(\tau, \eta)$ *depending smoothly on $\eta \in \Omega_g$, then either there is $\mathbf{v} \in \mathbb{R}^d$ such that $\lambda(\eta) = -\mathbf{v} \cdot \eta$ or $\lambda'' \neq 0$ almost everywhere on Ω_g.*

Proof. See [2,7] and [1]. □

We can now define the tools we will use to decompose the function a_0 into simpler waves. Before this, let us precise that in our notations the exponent 0 indicates that we refer to quantities linked to T_0 and not to L. For instance, we will parametrize Char T_0 by functions τ_i^0 and the $\pi^0(\tau_i^0(\eta), \eta)$ are the spectral projectors associated to T_0.

Definition 6. Enumerate the roots of $\det T_0(\tau, \eta) = 0$ as follows. Let $\mathcal{A}_\mathrm{f} := \{1, \dots, F\}$ denote the indices of the flat parts, and for $\alpha \in \mathcal{A}_\mathrm{f},\ \tau_\alpha^0(\eta) := -\mathbf{v}_\alpha \cdot \eta$. For $g \in \mathcal{G}$ and $\eta \in \Omega_g$, number the roots other than the $\{\tau_\alpha^0,\ \alpha \in \mathcal{A}_\mathrm{f}\}$ in increasing order $\tau_{g,1}^0(\eta) < \tau_{g,2}^0(\eta) < \cdots < \tau_{\tau,M(g)}^0$. Let $\mathcal{A}_\mathrm{c}$ denote the indices of the curved sheets

$$\mathcal{A}_\mathrm{c} := \big\{(g, j),\ g \in \mathcal{G} \text{ and } 1 \leqslant j \leqslant M(g)\big\}.$$

Let $\mathcal{A} := \mathcal{A}_{\mathrm{f}} \cup \mathcal{A}_{\mathrm{c}}$. For $\alpha \in \mathcal{A}_{\mathrm{f}}$ and $\eta \in \mathbb{R}^d$, define $E_\alpha(\eta) := \pi^0(-\mathbf{v}_j \cdot \eta, \eta)$. For $\alpha \in \mathcal{A}_{\mathrm{c}}$, define

$$E_\alpha(\eta) := \begin{cases} \pi^0(\tau_\alpha^0(\eta), \eta), & \text{for } \eta \in \Omega_g, \\ 0, & \text{for } \eta \notin \Omega_g. \end{cases}$$

We can now give, as in [7], the following proposition which decomposes an arbitrary solution of $T_0 u = 0$ as a finite sum of simpler waves.

Proposition 4. (i) *For each* $\alpha \in \mathcal{A}$, $E_\alpha(\eta) \in C^\infty(\mathcal{G})$ *is an orthogonal projection valued function positive homogeneous of degree zero.*

(ii) *For each* $\eta \in \mathcal{G}$, $\mathbb{C}^N$ *is equal to the orthogonal direct sum*

$$\mathbb{C}^N = \bigoplus_{\alpha \in \mathcal{A}} E_\alpha(\eta)(\mathbb{C}^N).$$

(iii) *The operators* $E_\alpha(D_y) := \mathcal{F}^* E(\eta) \mathcal{F}$ *are orthogonal projectors on* $H^s(\mathbb{R}^d)$*, and for each* $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ *is equal to the orthogonal direct sum*

$$H^s(\mathbb{R}^d) = \bigoplus_{\alpha \in \mathcal{A}} E_\alpha(D_y)\big(H^s(\mathbb{R}^d)\big).$$

(iv) *The solution of the initial value problem defined on range* $\pi(0)$

$$T_0(\partial_x) u = 0, \quad u|_{t=0} = f,$$

is given by the formula

$$\widehat{u}(t, \xi) = \sum_{\alpha \in \mathcal{A}} \widehat{u}_\alpha(t, \xi) := \sum_{\alpha \in \mathcal{A}} \mathrm{e}^{\mathrm{i}t\tau_\alpha^0(\xi)} E_\alpha(\xi) \widehat{f}(\xi).$$

Thus, one has $a_0 = \pi(0) a_0 := \sum_{\alpha \in \mathcal{A}} a_{0,\alpha}$, with

$$\partial_t a_{0,\alpha} = \mathrm{i}\tau_\alpha^0(D_y) a_{0,\alpha} \quad \text{and} \quad E_\alpha(D_y) a_{0,\alpha} = a_{0,\alpha}.$$

For $f \in \mathcal{A}_{\mathrm{f}}$, we will write $T_f(\partial_x) := \partial_t + \mathbf{v}_f \cdot \partial_y$.

3. The average projectors

3.1. Introducing the average projectors

In the previous section, we have made an algebraic analysis of the profiles equations. Thanks to the linear projectors π we have derived new equations which express resolubility conditions for our system.

However, we have to pursue our analysis in order to eliminate b from Eq. (22); it will involve a new and different resolubility condition. Indeed, one remembers that we want the corrector b to have a sublinear growth. It is not surprising that this constraint imposes new resolubility conditions on a.

The “average operators” introduced in this paper are constructed with this goal in mind. By the way, we will see that they simplify a lot the nonlinearities of the profile equations, which is essential for their resolution.

Let us introduce the generic pseudo-differential projector $E(D_y)$ and the function space $\mathcal{I}$.

Definition 7. Let $\mathcal{I}$ be the space of all real continuous semi-algebraic functions λ which are either flat on $\mathbb{R}^d$ (i.e., of the form $\lambda(\eta) = c \cdot \eta$ on $\mathbb{R}^d$), or never flat on a non-zero measure set.

Remark. The space $\mathcal{I}$ is defined so that it contains the functions τ_i and τ_i^0 which parametrize $\mathrm{Char}\, L$ and $\mathrm{Char}\, T_0$.

Until the end of this paper we will denote by T the generic scalar pseudo-differential operator

$$T(\partial_x) := \partial_t + \mathrm{i}\lambda(D_y),$$

where λ belongs to $\mathcal{I}$ and, in particular, is real valued.

Definition 8. For all $h > 0$ and $w \in C^0([0, T_0]_T \times \mathbb{R}_t; H^\infty(\mathbb{R}_Y^d \times \mathbb{R}_y^d \times \mathbb{T}_\theta))$, we introduce the operator

$$G_T^h w(X, t, y, \theta) := \frac{1}{h} \int_0^h \left(\int \mathrm{e}^{\mathrm{i}(y\cdot\xi + s\lambda(\xi))} \widehat{w}(t + s, \xi, X, \theta) \, \mathrm{d}\xi \right) \mathrm{d}s.$$

We also introduce

$$G_T w := \lim_{h\to\infty} G_T^h w,$$

when this limit exists in $C^0([0, T_0]_T \times \mathbb{R}_t; H^\infty(\mathbb{R}_Y^d \times \mathbb{R}_y^d \times \mathbb{T}_\theta))$.

Remark. With this definition it appears that the appellation “average operators” refers to the average we make in time t. It becomes more obvious if one takes for λ a linear function $\lambda(\eta) := v \cdot \eta$. One then gets, when this limit exists,

$$G_T w(t, y, X, \theta) = \lim_{h\to\infty} \frac{1}{h} \int_0^h w(t + s, y + sv, X, \theta) \, \mathrm{d}s,$$

and thus $G_T w$ is the average of w along the rays, with respect to the variable $x = (t, y)$.

3.2. Properties of the average operators

We now give four lemmas which describe the properties of the average operators that we will use in the elimination process.

The first one says that they have no effect on functions annihalating T.

Lemma 3. *If* $w(X, x, \theta) \in C^0([0, T_0]_T \times \mathbb{R}^d; H^\infty(\mathbb{R}_Y^d \times \mathbb{R}_y^d \times \mathbb{T}_\theta))$ *satisfies* $T(\partial_x)w = 0$, *then* $G_T w$ *exists and one has*

$$G_T w = w.$$

Proof. Since $T(\partial_x)w = 0$, one has $\widehat{w}(X,t,\xi,\theta) = \mathrm{e}^{-\mathrm{i}t\lambda(\xi)}\widehat{w_0}(X,\xi,\theta)$, where $w_0(X,y,\theta) \in C^0([0,T_0]_T;H^\infty(\mathbb{R}^d_Y \times \mathbb{R}^d_y \times \mathbb{T}_\theta))$.

Thus,

$$G^h_T w(X,t,y,\theta) = \frac{1}{h}\iint_0^h \mathrm{e}^{\mathrm{i}(y\cdot\xi+s\lambda(\xi))}\mathrm{e}^{-\mathrm{i}(t+s)\lambda(\xi)}\widehat{w_0}(X,\xi,\theta)\,\mathrm{d}\xi\,\mathrm{d}s$$
$$= \int \mathrm{e}^{\mathrm{i}y\cdot\xi}\mathrm{e}^{-\mathrm{i}t\lambda(\xi)}\widehat{w_0}(X,\xi,\theta)\,\mathrm{d}\xi = w(X,t,y,\theta). \qquad \square$$

If it happens that w does not annihilate $T(\partial_x)$ but another scalar operator $T_1(\partial_x)$, this is no longer true, as the following lemma shows.

Lemma 4. *If $w(X,x,\theta) \in C^0([0,T_0]_T \times \mathbb{R}_t;H^\infty(\mathbb{R}^d_Y \times \mathbb{R}^d_y \times \mathbb{T}_\theta))$ satisfies $T_1(\partial_x)w = 0$ with $T_1(\partial_x) := \partial_t + \mathrm{i}\lambda_1(D_y)$, and if $\lambda \neq \lambda_1$ almost everywhere on $\mathbb{R}^d$, then*

$$G_T w = 0.$$

Proof. Since $T_1(\partial_x)w = 0$, one has $\widehat{w}(X,t,\xi,\theta) = \mathrm{e}^{-\mathrm{i}t\lambda_1(\xi)}\widehat{w_0}(X,\xi,\theta)$, where $w_0(X,y,\theta) \in C^0([0,T_0]_T;H^\infty(\mathbb{R}^d_Y \times \mathbb{R}^d_y \times \mathbb{T}_\theta))$ and

$$\|\widehat{G^h_T w}(T,.,t,.,.)\|^2_{L^2(R^d_Y\times R^d_\xi\times T_\theta)} = \frac{1}{h^2}\int \left|\int_0^h \mathrm{e}^{\mathrm{i}(y\cdot\xi+s\lambda(\xi))}\mathrm{e}^{-\mathrm{i}(t+s)\lambda_1(\xi)}\widehat{w_0}(X,\xi,\theta)\,\mathrm{d}s\right|^2 \mathrm{d}Y\,\mathrm{d}\xi\,\mathrm{d}\theta$$
$$= \int \left|\widehat{w_0}(X,\xi,\theta)\right|^2 \left|\frac{1}{h}\int_0^h \mathrm{e}^{\mathrm{i}s(\lambda(\xi)-\lambda_1(\xi))}\,\mathrm{d}s\right|^2 \mathrm{d}Y\,\mathrm{d}\xi\,\mathrm{d}\theta.$$

But when $\lambda(\xi) \neq \lambda_1(\xi)$, one has

$$\frac{1}{h}\int_0^h \mathrm{e}^{\mathrm{i}s(\lambda(\xi)-\lambda_1(\xi))}\,\mathrm{d}s \to 0 \quad \text{when } h \to \infty.$$

It is then easy to conclude, thanks to Lebesgue's Dominated Convergence Theorem, that the L^2 norm tends to 0 when $h \to 0$. The same results yields for the H^s norms. $\square$

We recall that we have constructed the operators G in order to give resolubility conditions linked to the sublinear growth property.

The following lemma will be of importance to give such conditions.

Lemma 5. *If $w(X,x,\theta) \in C^0([0,T_0]_T \times \mathbb{R}_t;H^\infty(\mathbb{R}^d_Y \times \mathbb{R}^d_y \times \mathbb{T}_\theta))$ satisfies the sublinear growth property* (4), *then $G_T\,T(\partial_x)w$ is well defined and satisfies*

$$G_T T(\partial_x)w = 0.$$

Proof. By definition of G_T^h, one has

$$\begin{aligned}
G_T^h T(\partial_x)w(X,t,y,\theta) &= \frac{1}{h}\int_0^h \int \mathrm{e}^{i(y\cdot\xi+s\lambda(\xi))}\big(\partial_t\widehat{w}+\mathrm{i}\lambda(\xi)\widehat{w}\big)_{(X,t+s,\xi,\theta)}\,\mathrm{d}\xi\,\mathrm{d}s\\
&= \int \mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{1}{h}\int_0^h \frac{\mathrm{d}}{\mathrm{d}s}\big(\mathrm{e}^{\mathrm{i}s\lambda(\xi)}\widehat{w}\big)_{(X,t+s,\xi,\theta)}\,\mathrm{d}\xi\,\mathrm{d}s\\
&= \int \mathrm{e}^{\mathrm{i}y\cdot\xi}\frac{1}{h}\big(\mathrm{e}^{\mathrm{i}h\lambda(\xi)}\widehat{w}(X,t+h,\xi,\theta)-\widehat{w}(X,t,\xi,\theta)\big)\,\mathrm{d}\xi\,\mathrm{d}s\\
&= \frac{1}{h}\big(\mathrm{e}^{\mathrm{i}h\lambda(D_y)}w(X,t+h,y,\theta)-w(X,t,y,\theta)\big).
\end{aligned}$$

Thus, if w satisfies the sublinear growth property, it is quite easy to see that one has

$$G_T T(\partial_x)w = 0. \qquad \square$$

The last lemma gives an interesting property of the action of G_T on product of functions; it will be used to simplify the nonlinearities. It says that the result of the action of G_T on a product of functions is always equal to 0, except when each function of the product is annihilated by T, and T is a transport operator.

Lemma 6. *Let $T_i(\partial_x) := \partial_t + \mathrm{i}\lambda_i(D_y)$, $i = 1,\dots,m$, be m (not necessarily distinct) scalar operators ($m \in \mathbb{N}^*$), with the λ_i belonging to $\mathcal{I}$.*

Let w_i, $i = 1,\dots,m$, be m functions in $C^0([0,T_0]_T\times\mathbb{R}_t;H^\infty(\mathbb{R}^d_Y\times\mathbb{R}^d_y\times\mathbb{T}_\theta))$ such that $T_i(\partial_x)\,w_i = 0$ for all i and set $w := w_1w_2\cdots w_m$.

If $T(\partial_x)$ is a transport operator (i.e., $\lambda(\xi)$ is linear in ξ) and $T(\partial_x) = T_1(\partial_x) = \cdots = T_m(\partial_x)$, then

$$G_T(w) = w,$$

and in every other case, one has

$$G_T(w) = 0.$$

Proof. In this proof, we will omit to write the variable X and θ. As done before, one can write, for $i = 1,\dots,m$,

$$\widehat{w}_i(t,\xi) = \mathrm{e}^{-\mathrm{i}t\lambda_i(\xi)}\widehat{w}_i^0(\xi),$$

and, thus, this yields

$$G_T^h w(t,y) = \frac{1}{h}\int_0^h \int \mathrm{e}^{\mathrm{i}(y\cdot\xi_0+s\lambda(\xi_0))}\widehat{w}(t+s,\xi_0)\,\mathrm{d}\xi_0\,\mathrm{d}s.$$

But one knows that $\widehat{w} = \widehat{w}_1 * \cdots * \widehat{w}_m$. Plugging the above expressions of the $\widehat{w}_i$ into this relation gives

$$\widehat{w}(t+s,\xi_0) = \int \mathrm{e}^{-\mathrm{i}(t+s)(\lambda_1(\xi_1)+\lambda_2(\xi_2-\xi_1)+\cdots+\lambda_m(\xi_0-\xi_{m-1}))} \times \widehat{w}_1^0(\xi_1)\widehat{w}_2^0(\xi_2-\xi_1)\cdots\widehat{w}_m^0(\xi_0-\xi_{m-1})\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2\cdots\,\mathrm{d}\xi_{m-1}.$$

Introduce the functions

$$W(\xi_0,\ldots,\xi_{m-1}) := \widehat{w}_1^0(\xi_1)\widehat{w}_2^0(\xi_2-\xi_1)\cdots\widehat{w}_m^0(\xi_0-\xi_{m-1})\,\mathrm{d}\xi_1\cdots\,\mathrm{d}\xi_{m-1},$$

$$\mathbb{T}heta(\xi_0,\ldots,\xi_{m-1}) := \lambda(\xi_0)-\lambda_1(\xi_1)-\lambda_2(\xi_2-\xi_1)-\cdots-\lambda_m(\xi_0-\xi_{m-1}).$$

We have

$$G_T^h w(t,y) = \int \mathrm{e}^{\mathrm{i}(y\cdot\xi_0+t(\mathbb{T}heta(\xi_0,\ldots,\xi_{m-1})-\lambda(\xi_0)))} W(\xi_0,\ldots,\xi_{m-1}) \times \frac{1}{h}\int_0^h \mathrm{e}^{\mathrm{i}s\mathbb{T}heta(\xi_0,\ldots,\xi_{m-1})}\,\mathrm{d}s\,\mathrm{d}\xi_0\cdots\,\mathrm{d}\xi_{m-1}.$$

The strategy is to show that the average on $[0,h]$ that appears in the above equation tends to 0 when $h\to\infty$ or is a constant equal to 1, and then to apply Lebesgue's Dominated Convergence Theorem.

The phase that appears in our average integral is $\mathbb{T}heta$, and thus is a continuous semi-algebric function on $(\mathbb{R}^d)^m$.

One knows (cf. [1,2]) that $(\mathbb{R}^d)^m$ is the union of a set of measure 0 in $(\mathbb{R}^d)^m$ and of a finite number of open connected subsets U_k where $\mathbb{T}heta$ is a Nash function (that is, a C^∞ semi-algebraic function).

It is known ([2, Proposition 8.1.14]) that on each U_k one has the following alternative:

$$\mathbb{T}heta = 0 \quad \text{or} \quad \dim Z(\mathbb{T}heta) < \dim U_k,$$

where $Z(\mathbb{T}heta)$ is the zero-variety of $\mathbb{T}heta$.

Let us examine the first point of this alternative: $\mathbb{T}heta = 0$ on U_k means

$$\lambda(\xi_0)-\lambda_1(\xi_1)-\lambda_2(\xi_2-\xi_1)-\cdots-\lambda_m(\xi_0-\xi_{m-1}) = 0 \quad \text{on } U_k. \tag{23}$$

If we differentiate this equality twice with respect to ξ_i and ξ_{i+1}, we get

$$\lambda_i''(\xi_i-\xi_{i-1}) = 0 \quad \text{on } U_k,\ \text{for } i=2,\ldots,m$$

(one takes $\xi_m := \xi_0$).

Knowing this fact, we now differentiate twice with respect to ξ_0 or ξ_1 to find

$$\lambda''(\xi_0) = \lambda_1''(\xi_1) = 0 \quad \text{on } U_k.$$

Thus, on each U_k, one gets

$$\lambda_i(\xi_i-\xi_{i-1}) = c_i\cdot(\xi-\xi_{i-1}), \quad i\geqslant 2,$$

and

$$\lambda(\xi_0) = c \cdot \xi_0, \quad \lambda_1(\xi_1) = c_1 \cdot \xi_1,$$

where the c, c_i are vectors of $\mathbb{R}^d$.

We want to show that $c = c_1 = \cdots = c_m$.

Equality (23) now reads

$$\xi_0 \cdot (c - c_m) + \xi_1 \cdot (c_2 - c_1) + \cdots + \xi_{m-1} \cdot (c_m - c_{m-1}) = 0 \quad \text{on } U_k.$$

But since U_k is open, one can choose, for instance, $\xi_i \cdot (c_{i+1} - c_i) = 0$ for $i \geqslant 1$. This yields $\xi_0 \cdot (c - c_m) = 0$ on U_k and thus $c = c_m$.

The same method applied to the other indices yields the desired relation $c = c_1 = \cdots = c_m$.

Thus, on the open set U_k,

$$\lambda_i(\xi_i - \xi_{i-1}) = c \cdot (\xi - \xi_{i-1}), \quad i \geqslant 2,$$

and

$$\lambda(\xi_0) = c \cdot \xi_0, \quad \lambda_1(\xi_1) = c \cdot \xi_1.$$

Since λ and λ_i belong to $\mathcal{I}$, we can then conclude that $\lambda(\xi) = \lambda_1(\xi) = \cdots = \lambda_m(\xi) = c \cdot \xi$ on $\mathbb{R}^d$.

We then have, using Lemma 3, $G_T w = w$.

Let us now examine the second point of the alternative: in this case, the set $\{(\xi_0, \ldots, \xi_{m-1}),\ \lambda(\xi_0) - \lambda(\xi_1) - \cdots - \lambda_m(\xi_0 - \xi_{m-1}) = 0\}$ is of measure zero.

As in Lemma 4, we can conclude, thanks to Lebesgue's Dominated Convergence Theorem, that one has $G_T w = 0$ and the lemma follows. $\square$

3.3. *Consequences for the profile equations*

We are looking for resolubility conditions for Eqs (22).

We first treat the case of Eqs (22) when $n \in \mathcal{N}$. They read

$$T_n(\partial_x)\pi(n\beta)b_n = -\mathrm{i}R_n(\partial_y)\pi(n\beta)a_n - T_n(\partial_X)\pi(n\beta)a_n - \pi(n\beta)\big(\Phi(a)_n + \Psi(a)_n\big).$$

Let us introduce $G_n := G_{T_n}$.

Thanks to Lemma 5 we are now able to say that the sublinear growth of b imposes $G_n\pi(n\beta)T_n(\partial_x) \times b_n = 0$, for all $n \in \mathcal{N}$, thus yielding the necessary condition

$$G_n\big\{-\mathrm{i}R_n(\partial_y)\pi(n\beta)a_n - T_n(\partial_X)\pi(n\beta)a_n - \pi(n\beta)\big(\Phi(a)_n + \Psi(a)_n\big)\big\} = 0.$$

From (21) and Lemma 3 it follows that G_n leaves the first two factors of this equation invariant.

We also know how acts G_n on the nonlinearities thanks to Lemma 6. The nonlinearities are products of the components a_n and $a_{0,\alpha}$ of a, and the action of G_n only preserves the products where all the factors are transported by T_n. Thus, the term $G_n\pi(n\beta)(\Phi(a)_n + \Psi(a)_n)$ is a J- or K-linear function of the a_i

with $i \in \mathcal{N}$ so that $T_i(\partial_x) = T_n(\partial_x)$, and of the $a_{0,f}$ with $f \in \mathcal{A}_\mathrm{f}$ so that $T_f(\partial_x) = T_n(\partial_x)$. Hence, we can write $G_n\pi(n\beta)(\Phi(a)_n + \Psi(a)_n)$ under the form

$$G_n\pi(n\beta)\big(\Phi(a)_n + \Psi(a)_n\big) := \pi(n\beta)g_n\big(a_i, i \in \mathcal{N}, a_{0,\alpha}, \alpha \in \mathcal{A}_\mathrm{f}, T_n(\partial_x)a_i = T_n(\partial_x)a_{0,\alpha} = 0\big).$$

Our resolubility condition is thus

$$\begin{aligned} &-\mathrm{i}R_n(\partial_y)\pi(n\beta)a_n - T_n(\partial_X)\pi(n\beta)a_n \\ &\quad - \pi(n\beta)g_n\big(a_i, a_{0,\alpha}, T_n(\partial_x)a_i = T_n(\partial_x)a_{0,\alpha} = 0\big) = 0. \end{aligned} \tag{24}$$

We now consider the case of Eq. (22) with $n = 0$,

$$T_0(\partial_x)\pi(0)b_0 = -\mathrm{i}R_0(\partial_y)\pi(0)a_0 - T_0(\partial_X)\pi(0)a_0 - \pi(0)\big(\Phi(a)_0 + \Psi(a)_0\big).$$

We first decompose T_0 into its spectral components, as done in Section 2.5, and then apply the lemmas of the above section.

Let us introduce $G_\alpha := G_{T_\alpha} \circ E_\alpha(D_y)$ for $\alpha \in \mathcal{A}$.

As above, we deduce the following resolubility condition from the sublinear growth of b and from Lemmas 3–6:

$$E_\alpha(D_y)\big\{R_0(\partial_y)\pi(0)a_{0,\alpha} - T_0(\partial_X)\pi(0)a_{0,\alpha}\big\} - G_\alpha\big(\pi(0)\big(\Phi(a)_0 + \Psi(a)_0\big)\big) = 0.$$

Lemma 6 says we have to consider two cases:

(i) If $\alpha = f$ is in $\mathcal{A}_\mathrm{f}$ then the term $G_f\pi(0)(\Phi(a)_0 + \Psi(a)_0)$ has to be treated as the term $G_n\pi(n\beta) \times (\Phi(a)_n + \Psi(a)_n)$ above and takes the form

$$\begin{aligned} &G_f\pi(0)\big(\Phi(a)_0 + \Psi(a)_0\big) \\ &\quad := E_f(D_y)\pi(0)g_f\big(a_i, i \in \mathcal{N}, a_{0,\alpha}, \alpha \in \mathcal{A}_\mathrm{f}, T_f(\partial_x)a_i = T_f(\partial_x)a_{0,\alpha} = 0\big). \end{aligned}$$

Our resolubility condition is then

$$\begin{aligned} &E_f(D_y)\big\{-\mathrm{i}R_0(\partial_y)\pi(0)a_{0,f} - T_0(\partial_X)\pi(0)a_{0,f} \\ &\quad - \pi(0)g_f\big(a_i, a_{0,\alpha}, T_f(\partial_x)a_i = T_f(\partial_x)a_{0,\alpha} = 0\big)\big\} = 0. \end{aligned} \tag{25}$$

(ii) If $\alpha = c$ is in $\mathcal{A}_\mathrm{c}$ then Lemma 6 says that G_c annihilates the nonlinearities, thus yielding the resolubility condition

$$E_c(D_y)\big\{-\mathrm{i}R_0(\partial_y)\pi(0)a_{0,c} - T_0(\partial_X)\pi(0)a_{0,c}\big\} = 0. \tag{26}$$

Remark. We first have to notice that Eq. (26), which gives $a_{0,c}$ for $c \in \mathcal{A}_\mathrm{c}$ is always linear.

On the other hand, Eqs (24) and (26) which give a_n for $n \in \mathcal{N}$ and $a_{0,f}$ for $f \in \mathcal{A}_\mathrm{f}$ may be nonlinear.

If nonlinearities are present in Eq. (25), there may be rectification, i.e., creation of a non-oscillating component by the interaction of two or more oscillating components. Looking at the interacting factors and using Proposition 2 yields that this can occur only if the tangent variety at 0 of Char L (that is, Char T_0) contains a flat part which is also a tangent plane to Char L at a smooth point.

4. The approximate solution u^ε and its properties

4.1. Equations (24)–(26) determine a

In Lemma 1, we have given the expression of the terms r_0, r_1 and r_2 appearing in the expansion in powers of ε of the residual. Equations (6)–(8) must be satisfied if we want to have $r_0 = r_1 = r_2 = 0$.

From these equations, we have derived necessary conditions on a, b, c. These necessary conditions essentially came from resolubility conditions expressed with the operators $\pi(\beta)$ and G_T.

We have thus obtained the necessary conditions (24)–(26); we now prove that these conditions, together with necessary conditions (21), completely determine a.

Theorem 3. *Let $h(Y, y, \theta)$ be a trigonometric polynomial in θ with Fourier coefficients $h_n = \pi(n\beta)h_n \in H^\infty(\mathbb{R}^d \times \mathbb{R}^d)$.*

There exists $T_* > 0$ and a unique $a(T, Y, t, y, \theta)$ such that $\mathbf{\Pi}(\beta)a = a$, satisfying the initial condition $a(0, Y, 0, y, \theta) = h(Y, y, \theta)$ at $T = t = 0$ together with the evolution equations

$$T_m(\partial_x)a_m = 0 \quad \text{for all } m \in \mathcal{N},$$

and

$$\begin{aligned}
&T_n(\partial_X)\pi(n\beta)a_n + \mathrm{i}R_n(\partial_y)\pi(n\beta) + \pi(n\beta)g_n(a_j, a_{0,f}) = 0,\\
&\{\partial_T + v_f \cdot \partial_Y + \mathrm{i}E_f(D_y)R_0(\partial_y)\}E_f(D_y)a_{0,f} + E_f(D_y)g_f(a_j, a_{0,f}) = 0,\\
&\{\partial_T + \tau_c^0(D_Y) + \mathrm{i}E_c(D_y)R_0(\partial_y)\}E_c(D_y)a_{0,c} = 0,
\end{aligned}$$

for $n \neq 0$, $f \in \mathcal{A}_\mathrm{f}$ and $c \in \mathcal{A}_\mathrm{c}$.

Moreover, a satisfies, for all $0 < \underline{T} < T_*$ and all $\alpha \in \mathbb{N}^{2d+3}$,

$$\sup_{0\leqslant T\leqslant \underline{T}} \sup_{t\in R} \left\| \partial^\alpha_{T,Y,t,y,\theta} a(T, ., t, ., .)\right\|_{L^2(\mathbb{R}^d_Y \times \mathbb{R}^d_y \times \mathbb{T})} < \infty. \tag{27}$$

Remark. Each coefficient of a thus satisfies 2 equations, one in t and one in T, together with an initial condition at $T = t = 0$. This is quite unusual since such a Cauchy problem is normally overdetermined and admits no solution. Applying the average projectors G to the nonlinearities, we have indeed removed the terms which would have prevented us to find a common solution to these equations.

Proof. We want to find solutions to Eqs (24)–(26), so that one has

$$T_n(\partial_x)a_n = 0 \quad \text{for } n \in \mathcal{N}$$

and

$$\partial_t a_{0,\alpha} + \mathrm{i}\tau^0_\alpha(D_y)a_{0,\alpha} = 0 \quad \text{for } \alpha \in \mathcal{A}.$$

We thus seek solutions to this system under the form $a_n(X, t, y) = \mathsf{a}_n(X, y - tv_n)$ for $n \in \mathcal{N}$, $a_{0,f}(X, t, y) = \mathsf{a}_{0,f}(X, y - tv_f)$ for $f \in \mathcal{A}_\mathrm{f}$, and $a_{0,c}(X, t, y) = \mathrm{e}^{\mathrm{i}t\tau_c^0(D_y)}\mathsf{a}_{0,c}(X, y)$ for $c \in \mathcal{A}_\mathrm{c}$.

We have seen that the nonlinearity appearing in Eq. $(24)_n$ and denoted g_n is a product of terms a_j and $a_{0,f}$ all transported by T_n. Thus we can write $g_n(a_j(X,t,y), a_{0,f}(X,t,y))$ under the form $g_n(\mathsf{a}_j, \mathsf{a}_{0,f})(X, y - tv_n)$.

Equation $(24)_n$ is thus equivalent to

$$T_n(\partial_X)\pi(n\beta)\mathsf{a}_n + \mathrm{i}R_n(\partial_y)\pi(n\beta)\mathsf{a}_n + \pi(n\beta)g_n(\mathsf{a}_j, \mathsf{a}_{0,f}) = 0. \tag{28}$$

For the same reason, Eq. $(25)_f$ can read

$$\begin{aligned}&\partial_T E_f(D_y)\mathsf{a}_{0,f} + v_f \cdot \partial_Y E_f(D_y)\mathsf{a}_{0,f}\\ &\quad + \mathrm{i}E_f(D_y)R_0(\partial_y)E_f(D_y)\mathsf{a}_{0,f} + E_f(D_y)g_f(\mathsf{a}_j, \mathsf{a}_{0,f}) = 0,\end{aligned} \tag{29}$$

and Eq. $(26)_c$ becomes

$$\partial_T E_c(D_y)\mathsf{a}_c + \tau_c^0(D_y)E_c(D_y)\mathsf{a}_{0,c} + \mathrm{i}E_c(D_y)R_0(D_y)E_c(D_y)\mathsf{a}_{0,c} = 0. \tag{30}$$

Some of these equations are coupled; indeed, the nonlinearities in $(28)_n$ and $(29)_f$ involve the same unknown functions a_j and $\mathsf{a}_{0,\alpha}$ if $T_n = T_f$.

Let us solve a system of coupled equations; it is of the form

$$\begin{cases}\partial_T\pi(n\beta)\mathsf{a}_n - \tau'(n\beta)\cdot\partial_Y\pi(n\beta)\mathsf{a}_n + \mathrm{i}R_n(\partial_y)\pi(n\beta)\mathsf{a}_n + \pi(n\beta)g_n = 0,\\ \partial_T E_f(D_y)\mathsf{a}_{0,f} + v_f\cdot\partial_Y E_f(D_y)\mathsf{a}_{0,f} + \mathrm{i}E_f R_0(\partial_y)E_f\mathsf{a}_{0,f} + E_f(D_y)g_f = 0,\end{cases}$$

for all n and f such that $T_n = T_f$.

We will solve the following system derived from the precedent by omitting the polarizing terms $\pi(n\beta)$ and $E_f(D_y)$ in factor of ∂_T:

$$\begin{cases}\partial_T\phi_n - \pi(n\beta)\tau'(n\beta)\cdot\partial_Y\phi_n + \mathrm{i}R_n(\partial_y)\phi_n + \pi(n\beta)g_n = 0,\\ \partial_T\phi_{0,f} + v_f\cdot\partial_Y E_f(D_y)\phi_{0,f} + \mathrm{i}E_f(D_y)R_0(\partial_y)E_f(D_y)\phi_{0,f} + E_f(D_y)g_f = 0.\end{cases}$$

Simple Picard iterates for this system with evolution in variable T lead us to conclude to the local existence and unicity of a solution $\phi := (\phi_n, \ldots, \phi_{0,f})$ to the Cauchy problem; by the way, it is easy to see that $\partial_T(\phi_n - \pi(n\beta)\phi_n) = 0$ and $\partial_T(\phi_{0,f} - E_f(D_y)\phi_{0,f}) = 0$, so that if the initial value h is well polarized, so is ϕ. Taking $\mathsf{a}_n = \phi_n$ and $\mathsf{a}_{0,f} = \phi_{0,f}$ yields the desired solution. We can do this for all $n \in \mathcal{N}$ and $f \in \mathcal{A}_\mathrm{f}$.

It remains to prove the result for the uncoupled Eqs (30), which is quite straightforward.

The proof is thus complete. □

4.2. *Finding b and c*

We have seen that our necessary conditions (24)–(26) were sufficient to determine the leading term a.

We now determine the correctors b and c in order to satisfy Eqs (6)–(8). Thanks to (10), one has $(I - \mathbf{\Pi}(\beta))b = \mathrm{i}\mathbf{Q}(\beta)L_1(\partial_x)a$.

In order to find the missing component $\mathbf{\Pi}(n\beta)b$, we have to solve Eq. (12). Each component of this equation being a symmetric hyperbolic system, this is possible and $\mathbf{\Pi}(\beta)b$ is thus determined.

We then take c so that $c = (I - \mathbf{\Pi}(\beta))c$. It is thus given by Eq. (11), since we now know a and b.

4.3. $\varepsilon b + \varepsilon^2 c$ *is effectively a corrector*

The most important thing to prove is that b as determined in the above section satisfies the desired sublinear growth condition. It is indeed the case, as the following proposition shows.

Proposition 5. *The term b determined in the above section verifies the sublinear growth condition*

$$\lim_{t\to\infty} \frac{1}{t} \left\| \partial^{\gamma}_{X,x,\theta} b(T,Y,t,y,\theta) \right\|_{L^2([0,\underline{T}]\times\mathbb{R}^d_Y\times\mathbb{R}^d_y\times\mathbb{T})} = 0, \tag{31}$$

for all $\gamma \in \mathbb{N}^{2d+3}$.

Proof. We recall that we have

$$\sup_{0\leqslant T\leqslant \underline{T}} \sup_{t\in R} \left\| \partial^{\alpha}_{T,Y,t,y,\theta} a(T,.,t,.,.) \right\|_{L^2(\mathbb{R}^{2d}\times\mathbb{T})} < \infty.$$

Thanks to Eq. (10), it is easy to see that the component $(I - \mathbf{\Pi}(\beta))b$ has sublinear growth.

It remains to show that the growth of $\mathbf{\Pi}(\beta)b$ is also sublinear.

For $n \in \mathcal{N}$, Eq. (10) reads

$$T_n(\partial_x)\pi(n\beta)b_n(T,Y,t,y,\theta) = F(T,Y,t,y,\theta), \tag{32}$$

where F depends on a, which is already determined.

By the way, the term F verifies $G_n F = 0$, since this is the resolubility condition that we have satisfied in Section 4.1 when solving Eqs (24), (25).

Since $T_n(\partial_x) = \partial_t - \tau'(n\beta)\cdot\partial_y$, one can integrate Eq. (32):

$$\mathrm{e}^{-\mathrm{i}t\tau'(n\beta)\cdot\xi}\pi(n\beta)\widehat{b}_n(X,t,\xi,\theta) = \int_0^t \mathrm{e}^{-\mathrm{i}s\tau'(n\beta)\xi}\widehat{F}(X,t,\xi,\theta)\,\mathrm{d}s$$

and, thus,

$$\frac{1}{t}\mathrm{e}^{-\mathrm{i}t\tau'(n\beta)\cdot D_y}\pi(n\beta)b_n(X,t,y,\theta) = \frac{1}{t}\int_0^t \int \mathrm{e}^{\mathrm{i}(y\cdot\xi - s\tau'(n\beta)\xi)}\widehat{F}(X,t,\xi,\theta)\,\mathrm{d}\xi\,\mathrm{d}s,$$

which reads also

$$\frac{1}{t}\mathrm{e}^{-\mathrm{i}t\tau'(n\beta)\cdot D_y}\pi(n\beta)b_n(X,t,y,\theta) = \pi(n\beta)G^t_{T_n,\pi(n\beta)}F.$$

Since $G_n F = 0$, the right-hand side of the above relation tends to 0 in $C^0([0,T_0]_T\times\mathbb{R}_t; H^\infty(\mathbb{R}^d_Y\times\mathbb{R}^d_y\times\mathbb{T}_\theta))$ when $t\to\infty$, thus yielding the desired sublinear relation on $\pi(n\beta)b_n$ for $n\in\mathcal{N}$.

The same methods also yields the result for $\pi(0)b_0$ which achieves the proof of the proposition. □

The term c does not rise any problem: since it is a function of a, b and their derivatives we deduce from (27) and (31) that $\varepsilon^2 c$ is indeed a corrector. We can therefore give the following proposition:

Proposition 6. *In our approximate solution $a + \varepsilon b + \varepsilon^2 c$, the term $\varepsilon b + \varepsilon^2 c$ is a corrector for times* $\mathrm{O}(1/\varepsilon)$*, that is,* $\varepsilon b + \varepsilon^2 c = \mathrm{o}(1)$ *as* $\varepsilon \to 0$.

4.4. Estimate for the residual

The ansatz $u^\varepsilon = \varepsilon^p(a + \varepsilon b + \varepsilon^2 c)$ is only an approximate solution: it does not satisfy (1) exactly. We have already computed the residual in Section 1.2 and we now want to prove that it is small.

Proposition 7. *With the approximate solution $u^\varepsilon = \varepsilon^p u$, the residual*

$$k(\varepsilon, X, x, \theta) := L\left(\varepsilon^p u, \varepsilon\partial_X + \partial_x + \frac{\beta}{\varepsilon}\partial_\theta\right)\varepsilon^p u + F(\varepsilon^p u)$$

satisfies the following estimates for $\gamma \in \mathbb{N}^{2d+3}$ and $\underline{T} < T_*$:

$$\sup_{0\leqslant t\leqslant \underline{T}/\varepsilon} \left\|\partial^\gamma_{X,x,\theta} k(\varepsilon, X, x, \theta)\right\|_{L([0,\underline{T}]\times\mathbb{R}^{2d}\times\mathbb{T})} = \mathrm{o}\left(\varepsilon^{p+1}\right),$$

as $\varepsilon \to 0$.

Sketch of proof. A detailed proof of a similar proposition can be found in [7].

Since we have taken a, b and c in order to have $r_0 = r_1 = r_2 = 0$, we have the following expression of the residual k:

$$\begin{aligned} k(\varepsilon, X, x, \theta) = {} & L_1(\varepsilon^p u, \varepsilon\partial_X)\varepsilon^{p+1} b + L_1(\varepsilon\partial_X + \partial_x)\varepsilon^{p+2} c \\ & + \sum_\mu \left[A_\mu(\varepsilon^p u) - A_\mu(0)\right]\partial_x \varepsilon^p a + F(\varepsilon^p u) - \Phi(\varepsilon^p a) \\ & + \sum_\mu \left[A_\mu(\varepsilon^p u) - a_\mu(0) - \Gamma_\mu(\varepsilon^p a)\right]\beta_\mu/\varepsilon\partial_\theta\varepsilon^p u. \end{aligned} \tag{33}$$

Thanks to estimates (27), (31) and to the expression of c given by (11), it is easy to see that the first two terms of (33) are $\mathrm{o}(\varepsilon^{p+1})$.

This property also holds for the other terms of (33): to prove it, one has to use estimates (27), (31), Taylor's theorem and Schauder's lemma.

The residual is therefore $\mathrm{o}(\varepsilon^{p+1})$ which achieves this sketch of proof. □

4.5. A stability result

At this point, we have shown that there exists an approximate solution to (1) over times $\mathrm{O}(1/\varepsilon)$ and whose residual is small. But this does not say that it is close to an exact solution of (1). We thus have to prove that problem (1) admits an exact solution over times $\mathrm{O}(1/\varepsilon)$ which remains close to our approximate solution over such time intervals.

Theorem 4. *We recall that $\beta = (\tau, \eta)$ and that our family of approximate solutions is given by $u^\varepsilon = \varepsilon^p u(\varepsilon, \varepsilon x, x, \beta \cdot x/\varepsilon)$, with u given by $u = a + \varepsilon b + \varepsilon^2 c$.*

Let v^ε be the exact solution to the initial value problem

$$L^\varepsilon\left(v^\varepsilon, \partial_x\right)v^\varepsilon + F(v^\varepsilon) = 0, \quad v^\varepsilon|_{t=0} = u^\varepsilon|_{t=0}.$$

Then, there exists $T_- > 0$ so that

(i) *For each* $T \in]0, T_-[$ *and for* ε *small enough,* v^ε *exists and is smooth on* $[0, T/\varepsilon] \times \mathbb{R}^d$.
(ii) *We can write* $v^\varepsilon = \varepsilon^p \mathcal{V}^\varepsilon(t, y, y \cdot \eta/\varepsilon)$ *and* $u^\varepsilon = \varepsilon^p \mathcal{U}^\varepsilon(t, y, y \cdot \eta/\varepsilon)$, *with* $\mathcal{U}^\varepsilon, \mathcal{V}^\varepsilon \in C^\infty([0, T/\varepsilon]; H^\infty(\mathbb{R}^d \times \mathbb{T}))$, *and one has, for all* s,

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant t \leqslant T/\varepsilon} \left\| \mathcal{U}^\varepsilon(t) - \mathcal{V}^\varepsilon(t) \right\|_{H^s(\mathbb{R}^d \times \mathbb{T})} = 0.$$

Remark. The theorem remains true if v^ε is an exact solution of a small perturbation of problem (1), in the sense that

$$L^\varepsilon(v^\varepsilon, \partial_x) v^\varepsilon + F(v^\varepsilon) = \varepsilon^{p+1} l^\varepsilon(x, y \cdot \eta/\varepsilon), \quad v^\varepsilon|_{t=0} = u^\varepsilon|_{t=0} + \varepsilon^p g^\varepsilon(y, y \cdot \eta/\varepsilon),$$

where for all $s \in \mathbb{R}$ and $j \in \mathbb{N}$,

$$\|g^\varepsilon\|_{H^s(\mathbb{R}^d \times \mathbb{T})} \to 0, \qquad \sup_{0 \leqslant t \leqslant T_*/\varepsilon} \left\| (\varepsilon \partial_t)^j l^\varepsilon(t) \right\|_{H^s(\mathbb{R}^d \times \mathbb{T})} \to 0.$$

Proof. The general proof for the quasilinear case involves technical difficulties. We thus refer to [7] for a general proof.

However, we give a proof for the semilinear case, when the semilinearity is an homogeneous polynomial Φ of degree J. The proof is then very simple.

Problem (1) thus reads

$$L^\varepsilon(\partial_x) v^\varepsilon + \Phi(v^\varepsilon) = 0.$$

If we seek solutions of size ε^p, $v^\varepsilon(x) = \varepsilon^p v(\varepsilon, x)$, with the same scaling as in Section 1.1, the problem becomes

$$L^\varepsilon(\partial_x) v + \varepsilon \Phi(v) = 0.$$

This is the formulation of the problem we will use in this proof, our approximate solution thus beeing $u = a + \varepsilon b + \varepsilon^2 c$.

We first prove the existence of an exact solution for times $O(1/\varepsilon)$.

The form of our approximate solution invites us to search an exact solution under the form $v(\varepsilon, x) = \mathcal{V}^\varepsilon(x, \beta \cdot x/\varepsilon)$, with $\mathcal{V}^\varepsilon(x, \theta)$ periodic in θ.

It is sufficient for this that $\mathcal{V}^\varepsilon$ satisfies

$$\left[L_1(\partial_x) + \frac{1}{\varepsilon}\big(L_1(\beta)\partial_\theta + L_0\big) \right] \mathcal{V}^\varepsilon + \varepsilon \Phi(\mathcal{V}^\varepsilon) = 0. \tag{34}$$

Introducing $G^\varepsilon(\partial_y, \partial_\theta) := \sum_\mu A_\mu \partial_\mu + (1/\varepsilon)(L_1(\beta)\partial_\theta + L_0)$, this equation reads

$$\left[\partial_t + G^\varepsilon(\partial_y, \partial_\theta) \right] \mathcal{V}^\varepsilon + \varepsilon \Phi(\mathcal{V}^\varepsilon) = 0. \tag{35}$$

The change of time scale $t' := \varepsilon t$ transforms system (35) into

$$\left[\partial_{t'} + \frac{1}{\varepsilon} G^\varepsilon(\partial_y, \partial_\theta) \right] \mathcal{V}^\varepsilon + \Phi(\mathcal{V}^\varepsilon) = 0.$$

For ε fixed, this is a semilinear conservative symmetric hyperbolic system with constant coefficients. Since the H^s estimates are independant of ε, we deduce the existence and unicity of a solution for times $t' \in [0, T_0]$, with T_0 independant of ε.

In our previous time scale, this gives us a solution for times $t \in [0, T_0/\varepsilon]$ to problem (34), which is what we were looking for.

For T such that both the approximate and the exact solutions exist on $[0, T/\varepsilon]$, we want to prove that these functions remain close if their initial values are close.

We thus choose the initial conditions so that

$$\mathcal{U}^\varepsilon - \mathcal{V}^\varepsilon|_{t=0} = \mathrm{o}(1) \quad \text{in } H^\infty(\mathbb{R}^d \times \mathbb{T}).$$

The function $\mathcal{U}^\varepsilon$ satisfies

$$\left[\partial_t + G^\varepsilon(\partial_y, \partial_\theta)\right]\mathcal{U}^\varepsilon + \varepsilon \Phi(\mathcal{U}^\varepsilon) = \mathrm{o}(\varepsilon),$$

and substracting Eq. (35) from this one gives the following equation on $\mathcal{W}^\varepsilon := \mathcal{U}^\varepsilon - \mathcal{V}^\varepsilon$:

$$\left[\partial_t + G^\varepsilon(\partial_y, \partial_\theta)\right]\mathcal{W}^\varepsilon + \varepsilon \phi^\varepsilon(x, \theta)\mathcal{W}^\varepsilon = \mathrm{o}(\varepsilon),$$

where ϕ^ε is smooth.

Doing again the change of time scale $t' := \varepsilon t$ yields

$$\left[\partial_{t'} + \frac{1}{\varepsilon} G^\varepsilon(\partial_y, \partial_\theta)\right]\mathcal{V}^\varepsilon + \phi^\varepsilon(t'/\varepsilon, y, \theta)\mathcal{W}^\varepsilon = \mathrm{o}(1).$$

Usual Picard iterates then yields

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant t' \leqslant T} \left\|\mathcal{W}^\varepsilon(t'/\varepsilon)\right\|_{H^s(R^d \times T)} = 0.$$

Returning to our previous time scales gives the desired result

$$\lim_{\varepsilon \to 0} \sup_{0 \leqslant t \leqslant T/\varepsilon} \left\|\mathcal{U}^\varepsilon(t) - \mathcal{V}^\varepsilon(t)\right\|_{H^s(\mathbb{R}^d \times \mathbb{T})} = 0.$$

The theorem is thus proved in the semilinear case. □

5. Examples

5.1. A qualitative result

We have seen that each component of a is entirely determined. But the components $a_{0,c}$ for $c \in \mathcal{A}_\mathrm{c}$ can be neglected, as shows the following proposition, and that is why we may omit to compute them in the following examples.

Proposition 8. *For $c \in \mathcal{A}_c$, one has*

$$\lim_{t\to\infty} \|a_{0,c}\|_{L^\infty} = 0.$$

Proof. We give here the main steps of the proof of [7].

For an initial value w_c such that $\widehat{w}_c \in C_0^\infty(\Omega_g)$, one has

$$a_{0,c}(t,y) = \int_{\Omega_g} \mathrm{e}^{\mathrm{i}(\tau_c^0(\xi)t + y\cdot\xi)} \widehat{w}_c(\xi)\,\mathrm{d}\xi.$$

Thanks to partitions of unity, we can suppose that

$$\frac{\partial^2 \tau_c^0}{\partial \xi_1^2} \neq 0 \quad \text{on supp}\,\widehat{w}_c.$$

Thanks to Lebesgue's Dominated Convergence Theorem, it suffices to show that

$$\forall \xi_2,\ldots,\xi_d \in \mathbb{R}^{d-1}, \quad \lim_{t\to\infty} \sup_{z\in R} \left| \int \mathrm{e}^{\mathrm{i}(t\tau_c^0(\xi) + z\cdot\xi_1)} \widehat{w}_c(\xi)\,\mathrm{d}\xi_1 \right| = 0.$$

Thanks to non-stationnary and stationnary phase theorems, we show that this is true and that the above integral is in fact $\mathrm{O}(t^{-1/2})$.

This achieves this sketch of proof. □

Remark. The component $a_{0,c}$ for $c \in \mathcal{A}_c$ are therefore negligible. One could think that they should be in the corrector b, thus yielding an indetermination on these components $a_{0,c}$. But we have seen that our necessary conditions completely determine a and b; thus, there is no indetermination on the $a_{0,c}$. In fact, the above proposition shows that $a_{0,c}$ are correctors for the L^∞ norm, but not for the norms H^s. For these norms, the terms $a_{0,c}$ can no longer be considered as correctors and, therefore, do not suffer any kind of indermination.

5.2. *One-dimensional case*

We want to solve $L^\varepsilon(v, \partial_x)v + F(v) = 0$ in the one-dimensional case.

Our operator $L^\varepsilon(\partial_x)$ then has the form

$$L^\varepsilon(., \partial_x) = \partial_t I + A(.)\partial_x + L_0/\varepsilon.$$

In this case, Eqs (25), (26) take a peculiar form.

Indeed, there are no curved part in $\mathrm{Char}\,\pi(0)L_1\pi(0)$ and the projectors $E_f(D_y)$ are in fact constant projectors that we will denote by π_f^0.

The decomposition of a_0 seen in Section 2.5 is thus the following:

$$a_0 = \sum_f a_{0,f},$$

where $a_{0,f}$ has the form $a_{0,f}(X,t,y) = \mathsf{a}_{0,f}(X, y - v_f t)$ and verifies $\pi_f^0 a_{0,f} = a_{0,f}$. Thanks to Propositon 2, we know that the v_f are given by the slope of the distinct sheets of Char L at the origin.

Equation (25) thus takes the following form:

$$\partial_T \pi_f^0 \mathsf{a}_{0,f} + v_f \partial_Y \pi_f^0 \mathsf{a}_{0,f} + \mathrm{i}\pi_f^0 R_0(\partial_y)\pi_f^0 \mathsf{a}_{0,f} + \pi_f^0 g_f = 0.$$

Therefore, even if the initial value of the non-oscillating component is zero, the presence of $g_f(\mathsf{a}_i, \mathsf{a}_{0,\alpha}, T_f a_i = T_f a_{0,\alpha} = 0)$ induces rectification and can create a non-oscillating field a_0.

5.3. *Maxwell–Lorentz equations*

The Maxwell–Lorentz equations are given by

$$\begin{cases} \varepsilon \mathbf{E}_t = \varepsilon \operatorname{curl} \mathbf{B} - \varepsilon \mathbf{P}_t, \\ \varepsilon \mathbf{B}_t = -\varepsilon \operatorname{curl} \mathbf{E}, \\ \varepsilon \mathbf{P}_t = \mathbf{Q}, \\ \varepsilon \mathbf{Q}_t = \gamma \mathbf{E} - \mathbf{P}. \end{cases}$$

We can symmetrize it under the form

$$\begin{cases} \varepsilon \mathbf{E}_t = \varepsilon \operatorname{curl} \mathbf{B} - \varepsilon \mathbf{P}_t, \\ \varepsilon \mathbf{B}_t = -\varepsilon \operatorname{curl} \mathbf{E}, \\ \varepsilon \mathbf{P}_t/\gamma = \mathbf{Q}/\gamma, \\ \varepsilon \mathbf{Q}_t/\gamma = \mathbf{E} - \mathbf{P}/\gamma. \end{cases}$$

It is a symmetric hyperbolic system which satisfies Assumption 1. As indicated in Section 1.1, we can consider a new system where the coefficient of ∂_t is I.

This system reads $L^\varepsilon(\partial_x)U := \partial_t U + A(\partial_y)U + L_0/\varepsilon U = 0$ with

$$A(\partial_y) = \begin{pmatrix} 0 & -\operatorname{curl} & 0 & 0 \\ \operatorname{curl} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad L_0 = \begin{pmatrix} 0 & 0 & 0 & \sqrt{\gamma}\, I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ -\sqrt{\gamma}\, I & 0 & I & 0 \end{pmatrix}.$$

This system describes the Lorentz model. It gives a good description of the physical phenomenons for very weaks fields. But when the fields get stronger, nonlinearities arise since the polarization P responds to the electric field in a nonlinear way. There are two standard models which describe such effects: the anharmonic oscillator model and the instantaneous nonlinear response (see [4] and [6], for instance). For each of these two models, the Lorentz model appears to be the linearization at $U = 0$.

The anharmonic oscillator model induces a semilinearity, and the instantaneous nonlinear response model a quasilinearity. That is why we now consider the system

$$L^\varepsilon(\partial_x)U := \partial_t U + A(U, \partial_y)U + L_0/\varepsilon U + F(U) = 0,$$

where $A(0, \partial_y) = A(\partial_y)$ and $F(0) = 0$.

The characteristic variety is given by

$$\tau^2\big(\tau^2-1-\gamma\big)\big[\big(\tau^2-1\big)\big(\tau^2-|\eta|^2\big)-\gamma\tau^2\big]^2=0.$$

There are 3 singular points: 0 and $\pm\sqrt{1+\gamma}$. They are isolated and located on $\mathbb{R}_\tau\times\{0\}_{R^d}$ and Assumption 4 is thus satisfied.

This characteristic variety has 7 sheets; 3 of them have 0 value in 0 and are given by

$$\tau_0(\eta)=0,$$

and

$$\tau_\pm(\eta):=\pm\tau(\eta):=\pm\frac{1}{2}\Big(\sqrt{1+\gamma+|\eta|^2+2|\eta|}-\sqrt{1+\gamma+|\eta|^2-2|\eta|}\Big).$$

The other four are given by

$$\tau_{1\pm}(\eta):=\pm\tau_1(\eta):=\pm\sqrt{1+\gamma}$$

and

$$\tau_{2\pm}(\eta):=\pm\tau_2(\eta):=\pm\frac{1}{2}\Big(\sqrt{1+\gamma+|\eta|^2+2|\eta|}+\sqrt{1+\gamma+|\eta|^2-2|\eta|}\Big).$$

Proposition 2 shows that there is no flat part in Char T_0 unless we are in dimension $d=1$. Therefore, rectification effects can only occur in dimension 1.

In dimension $d>1$, Eq. (25) then says that there is no interaction possible between the oscillating terms and, therefore, no rectification. We have thus proved that rectification never occurs for these models in dimension $d>1$, so that if the mean field has initial value 0, it remains equal to 0.

Let us take $\beta:=(\tau_+(\eta),\eta)\neq 0\in$ graph τ_+, for instance; one then has $\mathcal{N}=\{1,-1\}$. Since there is no rectification, we just have to solve Eq. (24) to determine a_1 and a_{-1}.

We first notice that $\tau_\pm(\eta)$ is of the form $\tau_\pm(\eta)=\pm f(|\eta|)$. Therefore, one has

$$\tau_\pm'(\eta)=\frac{\pm f'(\eta)}{|\eta|}\eta \quad\text{and}\quad \tau_\pm''(\eta)(x,y)=\frac{\pm f''(\eta)}{|\eta|^2}\sum_{i,j}\eta_i\eta_j x_i y_j.$$

Thus, the operators $T_{\pm 1}$ and $R_{\pm 1}$ take the form

$$T_{\pm 1}(\partial_x)=\partial_t-\frac{f'(|\eta|)}{|\eta|}\eta\cdot\partial_y$$

and

$$R_{\pm 1}(\partial_y)=\frac{1}{2}\pi(\pm\beta)\frac{\pm f''(|\eta|)}{|\eta|^2}\sum_{i,j}\eta_i\eta_j\partial^2_{i,j}.$$

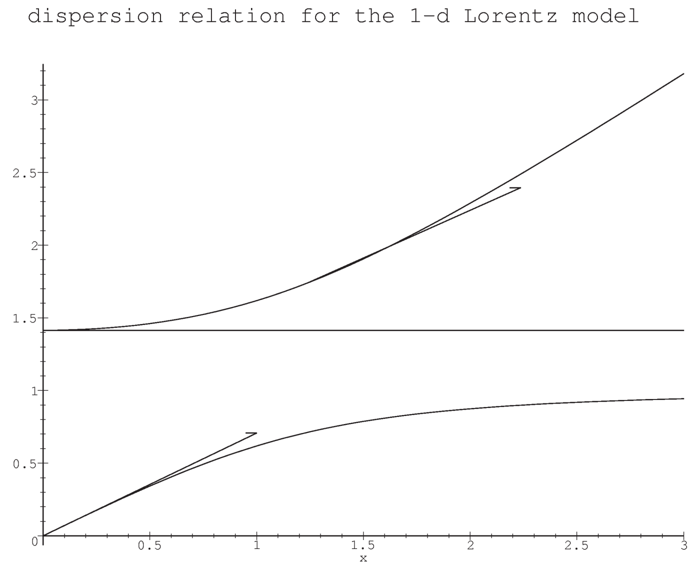

Fig. 1. Maxwell–Lorentz.

Thus, introducing $v(\eta) := -(f'(|\eta|)/|\eta|)\eta$ and $\alpha(\eta) := f''(|\eta|)/|\eta|^2$, Eqs (24) read

$$\left\{\partial_T + v(\eta)\cdot\partial_Y \pm \frac{\mathrm{i}}{2}\alpha(\eta)\sum_{i,j}\eta_i\eta_j\partial^2_{y_i,y_j}\right\}\pi(\pm\beta)a_{\pm 1} + \pi(\pm\beta)g_\pm(a_{-1},a_1) = 0.$$

If one has a bilinear semilinearity, this equation is in fact linear since the interaction of a_{-1} and a_1 cannot contribute to the first mode of the Fourier expansion.

In dimension $d = 1$, $\operatorname{Char} T_0$ has only flat parts and, therefore, one can expect rectification effects.

We first compute the slope of the demi-tangent to graph τ_+ at 0: it is equal to $1/\sqrt{1+\gamma}$ (see Fig. 1).

We now want to find an $\eta_0 \in \mathbb{R}$ so that $\tau'_{2+}(\eta_0) = 1/\sqrt{1+\gamma}$. By simple computation, one finds the following solution:

$$\eta_0 = \frac{-1+\sqrt{\gamma+(1+\gamma)^2}}{\gamma}.$$

We thus choose $\beta_0 := (\tau_{2+}(\eta_0),\eta_0)$. One then has $\mathcal{N} = \{1,-1\}$.

Because of the rectification effects, one has to compute here three coefficients: $a_1,\ a_{-1}$ and a_0.

Equation (24) gives a_1 and a_{-1}:

$$\left\{\partial_T + \tau'_2(\eta_0)\partial_Y \pm \frac{\mathrm{i}}{2}\tau''_2(\eta_0)\partial^2_y\right\}\pi(\pm\beta_0)a_{\pm 1} + \pi(\pm\beta_0)g_\pm(a_{-1},a_1,a_0) = 0, \tag{36}$$

and, since $R_0 = 0$, Eq. (24) which gives a_0 reads, using Section 5.2,

$$\left\{\partial_T + \frac{1}{\sqrt{1+\gamma}}\partial_Y\right\}\pi^0_+ a_0 + \pi^0_+ h(a_1,a_{-1},a_0) = 0. \tag{37}$$

Remark. We have thus proved that for a model whose linearization at $U = 0$ is the Lorentz model, rectification may occur because of the presence of the nonlinear term $\pi_+^0 h(a_1, a_{-1}, a_0)$ in Eq. (37). But sometimes, this nonlinear term vanishes because of transparency properties of the model. We now discuss nonlinear effects for two models: the anharmonic oscillator model and the instantaneous nonlinear response model.

The anharmonic oscillator model: The change here is to model the response of the polarization as an anharmonic oscillator. This means that the equation $\varepsilon^2\partial_t^2\mathbf{P} + \mathbf{P} = \gamma\mathbf{E}$ is replaced by a nonlinear differential equation

$$\varepsilon^2\partial_t^2\mathbf{P} + \nabla V(\mathbf{P}) = \gamma\mathbf{E}.$$

As said above, the Lorentz model is a good approximation of this model for weak fields (see [4,6]).

This model is semi-linear with a quadratic semi-linearity B. Simple computations yield that one has $\pi(0)B = 0$. The nonlinearity in Eq. (37) thus vanishes and there is no rectification. The evolution equation for a_0 is thus linear and that is why a_0 remains equal to 0 if its initial value is 0. Therefore, the nonlinearities in Eq. (36) also vanish and the evolution equations for $a_{\pm 1}$ are also linear. Therefore, there is no difference between the equation of this particular case (dimension $d = 1$ and $\beta := \beta_0$) where rectification was a priori possible and the general case (dimension $d \geqslant 1$ and any β).

The instantaneous nonlinear response model: The change here is to model the response of the nonlinear polarization as an instantaneous response (see [4,6]). We will take

$$\mathbf{P}_N = \alpha|\mathbf{E}|^2\mathbf{E},$$

where the constant α is O(1). The global polarization is the sum of this instantaneous cubic term and the term of Lorentz.

The system reads

$$\begin{cases} \varepsilon\mathbf{E}_t = \varepsilon\operatorname{curl}\mathbf{B} - \varepsilon\partial_t(\mathbf{P} + \alpha|\mathbf{E}|^2\mathbf{E}), \\ \varepsilon\mathbf{B}_t = -\varepsilon\operatorname{curl}\mathbf{E}, \\ \varepsilon\mathbf{P}_t = \mathbf{Q}, \\ \varepsilon\mathbf{Q}_t = \gamma\mathbf{E} - \mathbf{P}. \end{cases}$$

After symmetrization and a change of dependent variable in order to have $A_0(0) = I$, this system reads as the quasilinear system

$$L^\varepsilon(\partial_x)U := A_0(U)\partial_t U + A(\partial_y)U + \frac{1}{\varepsilon}L_0 U,$$

where

$$A_0(\mathbf{E}, \mathbf{B}, \mathbf{P}, \mathbf{Q}) := \begin{pmatrix} I + \alpha(2\mathbf{E}\otimes\mathbf{E} + |\mathbf{E}|^2 I) & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

$E \otimes E$ being the 3×3 matrix of coefficients $E_i E_j$.

With the notations used in Section 1, the nonlinearity appearing in the profile equations is given by

$$\Psi(\mathbf{E}, \mathbf{B}, \mathbf{P}, \mathbf{Q}) = \alpha\tau \begin{pmatrix} (2\mathbf{E} \otimes \mathbf{E} + |\mathbf{E}|^2 I)\partial_\theta \mathbf{E} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

We recall that we must have $\pi(\beta)U_1 = U_1$ and $\pi(0)U_0 = U_0$; therefore, $\mathbf{E}_0$ and $\mathbf{E}_1$ are of the form

$$\mathbf{E}_0 := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \mathbf{E}_1 := \begin{pmatrix} 0 \\ K \\ L \end{pmatrix},$$

where reality imposes $a, b, c \in \mathbb{R}$.

Since $\mathcal{N} = \{-1, 1\}$, the contribution of the non-zero component of the nonlinearity to the nonoscillating mode of the Fourier expansion is given by

$$n_0(\mathbf{E}) := -4\alpha\tau \Im\{(\mathbf{E}_0 \otimes \mathbf{E}_1 + \mathbf{E}_1 \otimes \mathbf{E}_0 + \mathbf{E}_0 \cdot \mathbf{E}_1 I)\overline{\mathbf{E}}_1\},$$

and it is easy to see that it is equal to 0.

Therefore, rectification does not occur here and, as in the previous model, the nonoscillating term a_0 remains equal to 0 if its initial value is 0.

In the anharmonic oscillator model, the nonlinearity was quadratic, and the nullity of a_0 implied that the evolution equations for the oscillating components $a_{\pm 1}$ were linear. In this model, the nonlinearity is cubic and this phenomenon does not occur any more.

Remark. Since rectification effects do not occur for $\beta := \beta_0$, what follows remains true for every $\beta \neq 0$ in Char L. We could also work in dimension greater than one, but the computations would be far heavier.

The contribution of the non-zero component of the nonlinearity to the first mode of the Fourier expansion is given by

$$n_1(\mathbf{E}) := 2\mathrm{i}\alpha\tau(\mathbf{E}_1 \otimes \overline{\mathbf{E}}_1 + \overline{\mathbf{E}}_1 \otimes \mathbf{E}_1 + |\mathbf{E}_1|^2 I)\mathbf{E}_1.$$

One then have

$$n_1(\mathbf{E}) = \begin{pmatrix} 0 \\ u(K, L) := (3|K|^2 + 2|L|^2)K + \overline{K}L^2 \\ v(K, L) := (2|K|^2 + 3|L|^2)L + K^2\overline{L} \end{pmatrix}.$$

The nonlinearity we need to compute is $\pi(\beta)\Psi(\mathbf{E},\mathbf{B},\mathbf{P},\mathbf{Q})_1 = \pi(\beta)(n_1(\mathbf{E}),0,0,0)$. Thus it remains to compute $\pi(\beta)$. For $\beta \in$ graph τ_+, one finds for the first three columns of $\pi(\beta)$ (the only ones we need):

$$\frac{1}{N^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \eta/\tau \\ 0 & -\eta/\tau & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau^2} & 0 \\ 0 & 0 & -\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau^2} \\ 0 & 0 & 0 \\ 0 & -\mathrm{i}\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau} & 0 \\ 0 & 0 & -\mathrm{i}\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau} \end{pmatrix},$$

with $N^2 := (\gamma(\tau^2+\eta^2) + 2(\tau^2-\eta^2)^2)/(\gamma\tau^2)$.

And, therefore, $U_1 = \pi(\beta)U_1 := (J, K, L, \mathbf{B}, \mathbf{P}, \mathbf{Q})$ is found solving

$$\Big\{\partial_T + \tau'(\eta)\partial_Y + \frac{\mathrm{i}}{2}\tau''(\eta)\partial_y^2\Big\}U = -\frac{1}{N^2}\begin{pmatrix} 0 \\ u(K,L) \\ v(K,L) \\ 0 \\ (\eta/\tau)v(K,L) \\ -(\eta/\tau)u(K,L) \\ 0 \\ -\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau^2}u(K,L) \\ -\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau^2}v(K,L) \\ 0 \\ -\mathrm{i}\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau}u(K,L) \\ -\mathrm{i}\frac{\tau^2-\eta^2}{\sqrt{\gamma}\tau}v(K,L) \end{pmatrix}.$$

Reality imposes $U_{-1} = \overline{U}_1$ and, thus, we have fully determined U.

5.4. Maxwell–Bloch equations

A rough quantum model is given by the Maxwell–Bloch equations (see [4,9]), introducing another unknown variable n which represents the difference between the number of atoms in the ground state and those in the excited state. We suppose that one has the thermodynamical equilibrium at time $t = 0$, and we denote N_0 the value of n at this time. Denoting $N := N_0 - n$, the equations read

$$\begin{cases} \mathbf{E}_t - \operatorname{curl}\mathbf{B} + \mathbf{Q}/\varepsilon = 0, \\ \mathbf{B}_t + \operatorname{curl}\mathbf{E} = 0, \\ \mathbf{P}_t - \mathbf{Q}/\varepsilon = 0, \\ \mathbf{Q}_t - (1/\varepsilon)N_0\mathbf{E} + (1/\varepsilon)\mathbf{P} = -(1/\varepsilon)N\mathbf{E}, \\ N_t = (1/\varepsilon)\mathbf{E}\cdot\mathbf{Q}. \end{cases}$$

Looking for solutions of size ε^2 of this system yields a problem similar to problem (1).

This problem is quite similar to those treated in the above section, and that is why we will only focus on the problem of rectification. The characteristic variety of this system is the same as the characteristic variety for the Maxwell–Lorentz model. We have thus already proved that rectification can only occur in dimension $d = 1$.

As in the previous section, computations yield that the nonlinearities predicted by Eq. (25) vanish because of the polarisation conditions. We are thus in a situation very similar to the anharmonic oscillator model, where rectification does not occur, and where all the equations determining the leading term a of the ansatz are linear.

5.5. Ferromagnetism

Maxwell equations in a ferromagnetic medium read for the Landau–Lifshitz model

$$\begin{cases} \partial_t E - \operatorname{curl} H = 0, \\ \partial_t H + \operatorname{curl} E + \partial_t M = 0, \\ \partial_t M = M \wedge H. \end{cases} \tag{38}$$

Let (E_0, H_0, M_0) be a constant solution of this system. Inspired by [8], we seek solutions which are small perturbations of this reference state under the form $U = U_0 + \varepsilon u(\varepsilon x)$, where $U := (E, H, M)$.

Since U_0 is a solution of (38), H_0 and M_0 must be colinear and we will denote $H_0 = \alpha M_0$, with $\alpha > 0$.

We obtain the following system on $u = (e, h, m)$:

$$\begin{cases} \partial_t e - \operatorname{curl} h = 0, \\ \partial_t h + \operatorname{curl} e + (1/\varepsilon)M_0 \wedge h + (1/\varepsilon)m \wedge H_0 = -m \wedge h, \\ \partial_t m - (1/\varepsilon)M_0 \wedge h - (1/\varepsilon)m \wedge H_0 = m \wedge h. \end{cases}$$

Therefore, $u' = (e', h', m') := (e, h, \sqrt{\alpha}m)$ must annihilate the following semilinear symmetric hyperbolic system:

$$\partial_t u' + \begin{pmatrix} 0 & -\operatorname{curl} & 0 \\ \operatorname{curl} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} u' + \frac{1}{\varepsilon}\begin{pmatrix} 0 & 0 & 0 \\ 0 & M_0\wedge & -\sqrt{\alpha}M_0\wedge \\ 0 & -\sqrt{\alpha}M_0\wedge & \alpha M_0\wedge \end{pmatrix} u' = B(u'),$$

where B is the quadratic form defined by $B(e', h', m') := (0, -(1/\sqrt{\alpha})m' \wedge h', m' \wedge h')$.

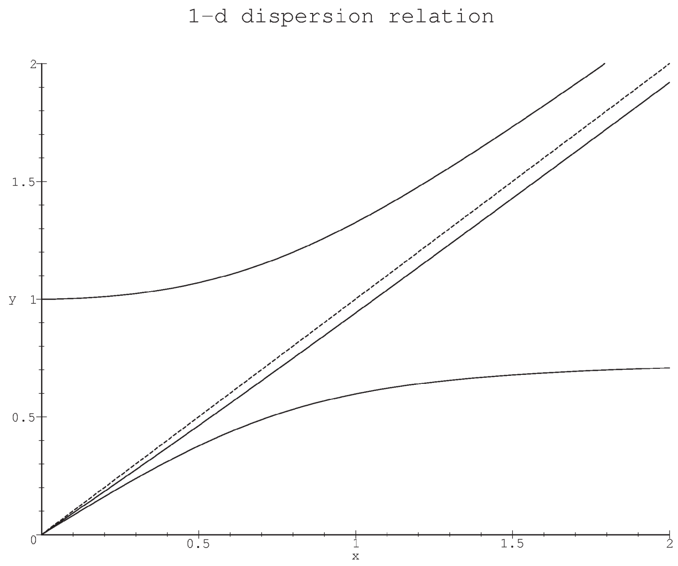

Fig. 2. Ferromagnetism.

We choose our axes so that M_0 is in the plane (y_1, y_2); thus, M_0 has coordinates $M_0 = (m\cos\varphi, m\sin\varphi, 0)$. For $\eta = (u, v, w)$, one then finds the following dispersion relation:

$$\begin{aligned}\tau^3\big(&-\tau^6 + \big((1+\alpha)^2 m^2 + 2|\eta|^2\big)\tau^4\\ &+ \big(-|\eta|^4 - 2\alpha(1+\alpha)m^2|\eta|^2 - (1+\alpha)m^2\big(w^2 + (u\sin\varphi - v\cos\varphi)^2\big)\big)\tau^2\\ &+ \alpha m^2|\eta|^2\big(\alpha|\eta|^2 + w^2 + (u\sin\varphi - v\cos\varphi)^2\big)\big) = 0.\end{aligned}$$

Here, the origin in $\mathbb{R}^{1+d}$ is the only singular point and is isolated so that Assumption 4 is satisfied.

This variety has 6 non-zero sheets that will be denoted by $\tau_1, \tau_2, \tau_3, -\tau_1, -\tau_2, -\tau_3$, where $\tau_1(0) = \tau_2(0) = 0$ and $\tau_3(0) = m(1+\alpha)$.

This characteristic variety is by the way quite singular since its tangent variety at the origin consists of two cones which coincide on two generators. Therefore, the origin is not an isolated singular point, which illustrates the need of Section 2.5.

In dimension $d > 1$, there is no flat part in $\operatorname{Char} T_0$; therefore, the equations for the mean field are only given by (26) and, thanks to Proposition 8, we may omit to compute them.

For $\beta := (\tau_i(\eta), \eta) \neq 0$ such that $\mathcal{N} = \{-1, +1\}$, we just have to compute a_1 and a_{-1} using (24):

$$\left\{\partial_T - \tau_i'(\eta)\cdot\partial_Y \pm \frac{\mathrm{i}}{2}\tau_i''(\eta)(\partial_y, \partial_y)\right\}\pi(\pm\beta)a_{\pm 1} = 0$$

(the nonlinearity vanishes since the contribution from a bilinear term to the first mode in the Fourier expansion can only come from the interaction of a_0 and a_1. But it is also required that both term be transported by T_1, which is obviously not the case for a_0.)

In dimension $d = 1$, $\operatorname{Char} T_0$ has only flat parts and, therefore, one can expect rectification effects.

The dispersion relation (see Fig. 2) is here

$$\tau^3\big(-\tau^6 + \big(2u^2 + (1+\alpha)^2 m^2\big)\tau^4 - \big(u^4 + (1+\alpha)(2\alpha + \sin^2\varphi)u^2\big)\tau^2 + \big(\alpha + \sin^2\varphi\big)u^2\big) = 0,$$

which also reads

$$(\tau^2 - u^2)\left[(1+\alpha)\tau^2 - \alpha u^2\right](1+\alpha)m^2 \sin^2\varphi + \left[(1+\alpha)\tau^2 - \alpha u^2\right]^2 m^2 \cos^2\varphi = (\tau^2 - u^2)^2 \tau^2.$$

We may suppose that for $u > 0$, one has $\tau_1(u) < \tau_2(u) < \tau_3(u)$. The slope of the demi-tangent to graph τ_1 and graph τ_2 at 0 are, respectively, $\sqrt{\alpha/(\alpha+1)}$ and $\sqrt{(\alpha + \sin^2\varphi)/(\alpha+1)}$.

The slope of the demi-tangent to graph τ_3 at 0 is zero and tends towards 1 when $u \to 0$; therefore, there exists two value η_1 and η_2 for which one has $\tau_3'(\eta_1) = \tau_1'(0^+)$ and $\tau_3'(\eta_2) = \tau_2'(0^+)$.

We thus choose $\beta := (\tau_3(\eta_i), \eta_i)$, where $i = 1$ or 2. One then has $\mathcal{N} = \{1, -1\}$.

Because of the rectification effects, one may have to compute here four coefficients: $a_1,\ a_{-1},\ a_{0,1}$ and $a_{0,2}$.

As usual, Eq. (24) gives a_1 and a_{-1}:

$$\left\{\partial_T - \tau_3'(\eta_i)\partial_Y \pm \frac{\mathrm{i}}{2}\tau_3''(\eta_i)\partial_y^2\right\}\pi(\pm\beta)a_{\pm 1} + \pi(\pm\beta)g_{\pm}(a_{-1}, a_1, a_{0,i}) = 0. \tag{39}$$

As said above, we might have nonlinear equations for $a_{0,1}$ and $a_{0,2}$ because of rectification effects. If $\sin^2\varphi < 1$, then the slopes of the demi-tangents at the origin are different and only $a_{0,i}$ will be affected by rectification. When $\sin^2\varphi = 1$, the slopes are the same and rectification affects both coefficients. When $\sin^2\varphi < 1$, the evolution equation where rectification occurs is given by Eq. (25) and Section 4.1:

$$\left\{\partial_T + v_i\partial_Y + \mathrm{i}\pi_i^0 R_0(\partial_y)\right\}\pi_i^0 a_{0,i} + \pi_i^0 g_i(a_{-1}, a_1, a_{0,i}) = 0.$$

When $\sin^2\varphi = 1$, rectifications occurs for the evolution equations of both $a_{0,1}$ and $a_{0,2}$:

$$\left\{\partial_T + v_1\partial_Y + \mathrm{i}\pi_1^0 R_0(\partial_y)\right\}\pi_1^0 a_{0,1} + \pi_1^0 g_1(a_{-1}, a_1, a_{0,1}, a_{0,2}) = 0,$$

$$\left\{\partial_T + v_2\partial_Y + \mathrm{i}\pi_2^0 R_0(\partial_y)\right\}\pi_2^0 a_{0,2} + \pi_2^0 g_2(a_{-1}, a_1, a_{0,1}, a_{0,2}) = 0.$$

We have already computed v_1 and v_2:

$$v_1 = \sqrt{\frac{\alpha}{\alpha+1}} \quad \text{and} \quad v_2 = \sqrt{\frac{\alpha + \sin^2\varphi}{\alpha+1}}.$$

We now have to find the expression of the nonlinearities. Simple computations yield that the nonlinearities involved in the equations for the nonoscillating component are equal to 0. Therefore, there is no rectification, and the nonoscillating component remains equal to 0 if its initial value is 0.

Since these nonoscillating component appear in the expression of the nonlinearities in Eq. (39), these nonlinearities are also equal to 0.

The approximate solution is thus given by solving the linear equations

$$\left\{\partial_T - \tau_3'(\eta_i)\partial_Y \pm \frac{\mathrm{i}}{2}\tau_3''(\eta_i)\partial_y^2\right\}\pi(\pm\beta)\mathsf{a}_{\pm 1} = 0,$$

where $a_{\pm 1}(X, t, y, \theta) := \mathsf{a}_{\pm 1}(X, y - \tau_3'(\eta_i), \theta)$.

Acknowledgement

The author wants to thank warmly Professor J.-L. Joly for all his very fruitful remarks and suggestions, as well as for suggesting this subject, and Professor T. Colin and G. Métivier for their fruitful remarks about this work.

References

[1] R. Benedetti and J.-J. Risler, *Real Algebraic and Semi-Algebraic Sets*, Actualités Mathématiques, Hermann, 1990.
[2] J. Bochnak, M. Coste and M.-F. Roy, *Géometrie Algébrique Réelle*, Ergebnisse der Mathematik, Springer, 1987.
[3] G. Bouligand, *Introduction à la Géométrie Infinitésimale Directe*, Gauthier-Villars, 1932.
[4] P. Donnat, Quelques contributions mathématiques en optique non linéaire, Thèse doctorale, Ecole Polytechnique, 1994.
[5] P. Donnat, J.-L. Joly, G. Métivier and J. Rauch, Diffractive nonlinear geometric optics I, in: *Séminaire Equations aux Dérivées Partielles*, Ecole Polytechnique, Palaiseau, 1995–1996, pp. XVII 1–XVII 23; revised version available at www.math.lsa.umich.edu/rauch.
[6] P. Donnat and J. Rauch, Dispersive nonlinear geometric optics, *J. Math. Phys.* **38**(3) (1997), 1484–1523.
[7] J.-L. Joly, G. Métivier and J. Rauch, Diffractive nonlinear geometric optics with rectification, Preprint, August 1997; available at www.math.lsa.umich.edu/rauch.
[8] H. Leblond and M. Manna, Benjamin–Feir-type instability in a saturated ferrite: Transition between focusing and defocusing regimes for polarized electromagnetic waves, *Phys. Rev. E* **50**(3) (September 1994), 2275–2286.
[9] R.H. Pantell and H.E. Puthoff, *Electronique Quantique en Vue des Applications*, Dunod, 1973.