LONG-WAVE SHORT-WAVE RESONANCE FOR NONLINEAR GEOMETRIC OPTICS

THIERRY COLIN and DAVID LANNES

Abstract

The aim of this paper is to study oscillatory solutions of nonlinear hyperbolic systems in the framework developed during the last decade by J.-L. Joly, G. Métivier, and J. Rauch. Here we focus mainly on rectification effects, that is, the interaction of oscillations with a mean field created by the nonlinearity. A real interaction can occur only under some geometric conditions described in [JMR1] and [L1] that are generally not satisfied by the physical models except in the 1*-dimensional case. We introduce here a new type of ansatz that allows us to obtain rectification effects under weaker assumptions. We obtain a new class of profile equations and construct solutions for a subclass. Finally, the stability of the asymptotic expansion is proved in the context of Maxwell-Bloch-type systems.*

1. Introduction

1.1. Motivations

In the study of solutions to nonlinear hyperbolic systems, many nonlinear effects can be observed. In optics, they are linked to a nonlinear response of the medium and therefore to the intensity of the incoming light. The more intense it is, the sooner these nonlinear effects occur.

This physical phenomenon encountered in optics occurs in all nonlinear hyperbolic systems; the scale of the appearance of the nonlinear effects is in inverse proportion to the size of the solution. For instance, for a semilinear hyperbolic problem

$$L^{\varepsilon}(\partial_x)\mathbf{u}^{\varepsilon} := \partial_t \mathbf{u}^{\varepsilon} + \sum_{j=0}^{d} A_j \partial_j \mathbf{u}^{\varepsilon} + \frac{L_0}{\varepsilon}\mathbf{u}^{\varepsilon} = f\big(\mathbf{u}^{\varepsilon}, \mathbf{u}^{\varepsilon}\big),$$

DUKE MATHEMATICAL JOURNAL
Vol. 107, No. 2, © 2001
Received 6 January 2000. Revision received 13 June 2000.
2000 *Mathematics Subject Classification*. Primary 35B34, 35B40, 35C20, 35L60.

nonlinear effects occur at times $O(1)$ if $\mathbf{u}^\varepsilon$ is of size $O(1)$ and at times $O(1/\varepsilon)$ if its size is $O(\varepsilon)$.

We investigate here phenomena that occur for diffractive times $O(1/\varepsilon)$, and we are interested in one particular nonlinear effect called *rectification*, which means the creation of a mean field thanks to the asymptotical nonlinear interaction of oscillating modes. It is a nonlinear interaction between the zero frequency (long waves) and nonzero frequencies (short waves).

In the first studies of rectification for times $O(1/\varepsilon)$ (in [JMR1] for the nondispersive case and in [L1] for the dispersive case), it has been shown that it can occur only if the tangent cone $\mathscr{C}_0$ to the characteristic variety $\mathscr{C}$ at (0, 0) contains a hyperplane tangent to $\mathscr{C}$. Such a condition is quite strong and seems to exclude all physical examples, unless we are in space dimension 1, since $\mathscr{C}^0$ is then a union of straight lines. But even in this case, the nonlinear coupling that should appear between the mean field and the oscillating modes remains equal to zero, as computations show (see [L1]). Such a phenomenon belongs to the *transparency* phenomena first mentioned by P. Donnat (cf. [D]) and extensively studied in [JMR2].

As said above, nonlinear effects are linked to the amplitude of the solutions we study. Since rectification does not occur at times $O(1/\varepsilon)$ when dealing with "normal" solutions of size $O(\varepsilon)$ to transparent problems, it is therefore natural to seek "abnormal" solutions of size $O(1)$. We said above that in this case nonlinear effects should occur at times $O(1)$, but because of transparency they occur only at a diffractive scale. It has been proved in this case (see [C]) that the approximate solution given by geometric optics must satisfy a Davey-Stewartson-type system that couples the leading oscillating term of the ansatz with the leading nonoscillating term.

There is therefore a nonlinear interaction between the oscillating and mean modes, but this Davey-Stewartson-type interaction is due to the algebraic structure of the system and not to asymptotical effects coming from the long-time interaction of different modes travelling at the same velocity. Hence, the nonlinear interaction in Davey-Stewartson systems cannot be called rectification.

In fact, the study made in [C] remains valid while rectification does not occur. It has been proved indeed that the classical rectification condition (i.e., $\mathscr{C}^0$ contains a hyperplane tangent to $\mathscr{C}$) is a singular case for the Davey-Stewartson system of [C]. It is not surprising since in this case the system is "more nonlinear"—since we then have to add the rectification effects to the Davey-Stewartson nonlinear effects—and the solutions are therefore more likely to explode.

There is also another singular case for this Davey-Stewartson system, which occurs when there exists a tangent plane to $\mathscr{C}$ also tangent to $\mathscr{C}^0$. This condition is close but weaker than the rectification condition. Here again, one can think that the Davey-Stewartson system becomes singular because of rectification effects.

Here lies the motivation of this paper. We want to observe rectification effects, but we are confronted by two opposite situations where their study is not possible. On the one hand (see [JMR1], [L1]), the amplitude $O(\varepsilon)$ of the solutions is too small and, because of transparency effects, rectification effects do not occur for observation times $O(1/\varepsilon)$. On the other hand (in [C]), the amplitude $O(1)$ of the solutions is too big and when rectification effects occur, solutions explode. It is therefore natural to consider solutions at an intermediate scale $O(\sqrt{\varepsilon})$ and to investigate the two cases which are singular in [C].

(i) There is in $\mathscr{C}^0$ a tangent hyperplane to $\mathscr{C}$. As said above, the only physically interesting case is when the space dimension is 1. We then seek approximate solution u^ε of the form

$$u^\varepsilon(t, y) = \sqrt{\varepsilon}\mathscr{U}^\varepsilon\Big(\varepsilon t, t, y, \frac{\tau t + \eta y}{\varepsilon}\Big),$$

where $\mathscr{U}^\varepsilon(T, t, y, \theta)$ is periodic in θ. The scale εt is the diffractive scale, while (t, y) is the scale of geometric optics and $(\tau t + \eta y)/\varepsilon$ the fast oscillating scale.

(ii) $\mathscr{C}$ has a tangent plane $\mathscr{P}$ also tangent to $\mathscr{C}^0$. The 1-dimensional case is the same as above, so that we consider only space dimension $d \geq 2$. The situation is here a bit different since if we seek approximate solutions u^ε as above, we know, thanks to [JMR1] and [L1], that there is no rectification effect (since the rectification condition is not fulfilled). Denoting $y = (y_1, \dots, y_d)$ and assuming that $\mathscr{P}$ is tangent to $\mathscr{C}^0$ along the first coordinate, a first idea is to consider approximate solutions of the form

$$u^\varepsilon(t, y) = \sqrt{\varepsilon}\mathscr{U}^\varepsilon\Big(\varepsilon t, t, y_1, \frac{\tau t + \eta y}{\varepsilon}\Big),$$

which brings us back to case (i). This is not satisfactory since we lose the dependence on $y_{II} := (y_2, \dots, y_d)$, and hence we can only study the rectification effects that occur along the first coordinate. Our problem is then quite similar to what happens when choosing the amplitude. We are indeed confronted by two opposite situations. On the one hand, rectification does not occur fast enough to be described with a dependence on y_{II} of the same scale as the dependence on y_1. On the other hand, taking a dependence on y_{II} slower or of the same scale as the diffractive scale, we miss part of it. Thus, we introduce a new scale and seek u^ε under the form

$$u^\varepsilon(t, y) = \sqrt{\varepsilon}\mathscr{U}^\varepsilon\Big(\varepsilon t, \sqrt{\varepsilon} y_{II}, t, y_1, \frac{\tau t + \eta y}{\varepsilon}\Big).$$

Throughout this paper, we investigate case (ii). The associated condition is called the long-wave short-wave resonant condition. It is easy to see that the situation described in case (i) can easily be deduced from it. We show that multidimensional nontrivial rectification occurs in this case. Since the long-wave short-wave resonance condition is likely to occur for physical systems, we suspect that this study gives a good framework to observe experimentally rectification effects. In [L2], we follow formally the theory exposed here to study rectification effects for water waves, but, unfortunately, we cannot apply directly the result of this paper to the Euler equations with free surface. Note that the system found in [L2] is also derived in the book by C. and P.-L. Sulem [Su].

1.2. Setting up the problem

We consider here a general class of hyperbolic quadratic systems. More precisely, we seek approximate solutions of size $\sqrt{\varepsilon}$ to

$$L^{\varepsilon}(\partial_x)\mathbf{u}^{\varepsilon} := \partial_t \mathbf{u}^{\varepsilon} + \sum_{j=0}^{d} A_j \partial_j \mathbf{u}^{\varepsilon} + \frac{L_0}{\varepsilon}\mathbf{u}^{\varepsilon} = f\big(\mathbf{u}^{\varepsilon}, \mathbf{u}^{\varepsilon}\big), \tag{1}$$

where the A_j are $N \times N$ symmetric real matrices, while L_0 is skew-symmetric.

We assume that the mapping $(u, v) \in \mathbb{C}^{2N} \mapsto f(u, v) \in \mathbb{C}^N$ is bilinear. Throughout this paper, $t \in \mathbb{R}^+$ denotes the time variable and $y := (y_1, \dots, y_d) \in \mathbb{R}^d$ denotes the space variables; we also write $x := (t, y) \in \mathbb{R}^{1+d}$.
For$(\tau, \eta) \in \mathbb{R}^{1+d}$, we introduce

$$L(\tau, \eta) = \tau I + \sum_{j=1}^{d} A_j \eta_j + \frac{L_0}{i},$$

as well as

$$A_{II}(\eta) := \sum_{j=2}^{d} A_j \eta_j.$$

The set of all $\beta := (\tau, \eta) \in \mathbb{R}^{1+d}$ such that $\det(L(\beta)) = 0$ is the *characteristic variety* of L and is denoted by $\mathscr{C}_L$. If β is a smooth point of $\mathscr{C}_L$, we denote by $\eta \mapsto \tau(\eta)$ a local parametrization of $\mathscr{C}_L$ in a neighborhood of β. For $\beta = (\tau, \eta) \in \mathscr{C}_L$, we finally denote by $\pi(\beta)$ the orthogonal projector on $\ker(\tau I + A(\eta) + L_0/i)$, and we denote by $L(\beta)^{-1}$ the partial inverse of $L(\beta)$. We also need to consider the characteristic variety $\mathscr{C}^0$ of the operator $\pi(0)L^{\varepsilon}\pi(0)$, which is the tangent cone to $\mathscr{C}_L$ at $(0, 0)$ (see [L1]).

As said above, we look for approximate solutions to (1) whose size is $O(\sqrt{\varepsilon})$, that is, intermediate between the normal size $O(\varepsilon)$ and the large size $O(1)$ used when deriving the Davey-Stewartson systems (see [C]). The leading oscillating term oscillates with a phase $\underline{\beta} \cdot x/\varepsilon$, where $\underline{\beta}$ satisfies the following assumption.

ASSUMPTION 1

$\underline{\beta}$ is a smooth point of $\mathscr{C}_L$, and $2\underline{\beta}$ is not on $\mathscr{C}_L$.

The short-wave long-wave resonance condition we have mentioned above, and which corresponds to the singular case for the Davey-Stewartson system with which we are dealing, states the following.

ASSUMPTION 2 (Long-wave short-wave resonance)
We say that we have a long-wave short-wave resonance if the tangent space $\mathscr{P}$ to $\mathscr{C}_L$ at $\underline{\beta}$ is the tangent space to $\mathscr{C}^0$ at $(0, 0)$.

Under this assumption, the intersection of $\mathscr{P}$ and $\mathscr{C}^0$ is a straight line passing through the origin. We denote by β^0 the point of this line with vertical coordinate equal to 1: $\beta^0 := (1, \eta^0) = (1, \eta_1^0, \dots, \eta_d^0)$.

When $\mathscr{C}_L$ is of revolution, then $\underline{\eta}$, η^0, and $\nabla\tau(\underline{\eta})$ are necessarily colinear. In the general case, this is no longer true, but we have the following proposition.

PROPOSITION 1
If $\nabla\tau(\underline{\eta}) \cdot \eta^0 \neq 0$, then we can be brought back to the case where $\nabla\tau(\underline{\eta})$ and the contact direction η^0 are colinear.

The proof of this proposition is postponed to Section 6.1.

In order to be in this framework, we make the following assumption, satisfied by all the physical examples we have encountered.

ASSUMPTION 3
One has $\nabla\tau(\underline{\eta}) \cdot \eta^0 \neq 0$.

Convention
Under the above assumption, we can assume from now on that $\eta^0 = (\eta_1^0, 0, \dots, 0)$ and $\tau'(\underline{\eta}) = (\partial_1\tau(\underline{\eta}), 0, \dots, 0)$.

1.3. The ansatz
In diffractive optics (see [D], [JMR1], [L1], [C]), ansatzes with three scales are used, and the approximate solutions are therefore of the form

$$u^\varepsilon(x) = \varepsilon^p \mathscr{U}\left(\varepsilon, \varepsilon t, x, \frac{\underline{\beta} \cdot x}{\varepsilon}\right),$$

where the profile $\mathscr{U}(\varepsilon, T, x, \theta)$ is periodic in θ.

The scale $O(1/\varepsilon)$ is the fast scale associated to the oscillations, and the intermediate scale $O(1)$ is the scale of geometric optics, that is, the scale for which propagation of oscillations along rays furnishes a good approximation. The last scale $O(\varepsilon)$ is the slow scale we have to introduce in order to take into account the diffractive modifications one has to make to the non-space-time dispersive propagation along rays.

As said in the introduction, we introduce here a fourth scale $O(\sqrt{\varepsilon})$ in order to take into account the rectification effects in the transverse directions. Still supposing that η^0 and $-\tau'(\underline{\eta})$ are along the first coordinate, we seek approximate solutions of the form

$$u^\varepsilon(x) = \sqrt{\varepsilon}\mathscr{U}\left(\varepsilon, \varepsilon t, \sqrt{\varepsilon} y_{II}, t, y_1, \frac{\underline{\beta} \cdot x}{\varepsilon}\right), \tag{2}$$

where y_1 is thus the direction of η^0 and $\tau'(\underline{\eta})$, and $y_{II} := (y_2, \ldots, y_d)$.
The profile $\mathscr{U}(\varepsilon, T, Y, t, y_1, \theta)$ is chosen of the form

$$\mathscr{U}\big(\varepsilon, T, Y, t, y_1, \theta\big) := \big(\mathscr{U}_1 + \sqrt{\varepsilon}\mathscr{U}_2 + \varepsilon\mathscr{U}_3 + \varepsilon^{3/2}\mathscr{U}_4 + \varepsilon^2\mathscr{U}_5\big)\big(\varepsilon, T, Y, t, y_1, \theta\big), \tag{3}$$

where the $\mathscr{U}_i$ are smooth functions of their arguments and are periodic in θ.

Since the above expansion is used for times $O(1/\varepsilon)$, we have to control the growth of the profiles in t. In order for the correctors $\mathscr{U}_i$, $i = 2, \ldots, 5$, to remain smaller than the leading term $\mathscr{U}_1$ for such times, we must have $\mathscr{U}_2 = o(\sqrt{t})$, $\mathscr{U}_3 = o(t)$, $\mathscr{U}_4 = o(t^{3/2})$, and $\mathscr{U}_5 = o(t^2)$. We impose the following stronger conditions.

- The first corrector $\mathscr{U}_2$ remains bounded,

$$\exists C > 0, \quad \sup_{t \in \mathbb{R}^+} \big\| \mathscr{U}_2(\cdot, \cdot, t, \cdot, \cdot) \big\|_{L^\infty([0,\underline{T}] \times \mathbb{R}^d_{Y,y_1} \times \mathbb{T})} \leq C. \tag{4}$$

- The other correctors $\mathscr{U}_i$, $i = 3, 4, 5$, satisfy the sublinear growth condition introduced in [JMR1],

$$\lim_{t \to \infty} \frac{1}{t} \big\| \mathscr{U}_i(\cdot, \cdot, t, \cdot, \cdot) \big\|_{L^\infty([0,\underline{T}] \times \mathbb{R}^d_{Y,y_1} \times \mathbb{T})} = 0, \quad i = 3, 4, 5. \tag{5}$$

1.4. Outline of the results

In Section 2 we derive the profile equations using the techniques of geometrical optics. However, the size of the solution considered here is too big to allow a standard derivation, and we have to make a *transparency assumption*. To our knowledge (see [JMR2], [L1]), this assumption is satisfied by all the physical systems of the form (1). The profile equations found in this case are given in Theorem 1. In particular, one can notice that the evolution equations of the oscillating and mean modes are coupled.

In Section 3 we assume the existence of a solution to the profile equations and prove a few properties of the approximate solutions associated to these profiles. In Proposition 7 we show that the residual that one obtains when plugging these approximate solutions into (1) is small.

Section 4 is devoted to the study of a particular subclass of systems (see (1)), the Maxwell-Bloch systems. This class of problems has been extensively studied in [JMR2]. Under a *strong transparency assumption*, we prove that the nonlinearity appearing in the evolution equation of the mean mode vanishes. In this case, the existence of a solution to the profile equations is proved. Moreover, we prove in Theorem 2 that the associated approximate solutions are stable, that is, remain close to an exact solution of (1).

The 1-dimensional case is another framework in which we can prove the existence of a solution to the profile equations, as we show in Section 5.

Finally, we prove in Section 6 an existence theorem for the profile equations in two dimensions, without doing the strong transparency assumption. Though the system we consider in this version is simplified with regard to the profile equations given in Theorem 1, it is of particular interest since this is the system obtained by Sulem and Sulem [Su] when studying the long-wave short-wave resonance for water waves.

2. Derivation of the equations

2.1. Equations for the profiles

As usual in geometric optics, we expand $L^\varepsilon u^\varepsilon - f(u^\varepsilon, u^\varepsilon)$ (where u^ε is the approximate solution given by equations (2) and (3)) in powers of ε. One finds

$$\begin{aligned} L^\varepsilon u^\varepsilon - f\big(u^\varepsilon, u^\varepsilon\big) = \Big(&\varepsilon^{-1/2} iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_1 + \varepsilon^0 iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_2 + \varepsilon^{1/2} iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_3 \\ &+ \big(\partial_t + A_1\partial_{y_1}\big)\mathcal{U}_1 + \varepsilon^1 iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_4 + \big(\partial_t + A_1\partial_{y_1}\big)\mathcal{U}_2 \\ &+ A_{II}(\partial_Y)\mathcal{U}_1 - f\big(\mathcal{U}_1, \mathcal{U}_1\big) + \varepsilon^{3/2} iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_5 \\ &+ \big(\partial_t + A_1\partial_{y_1}\big)\mathcal{U}_3 + A_{II}(\partial_Y)\mathcal{U}_2 + \partial_T\mathcal{U}_1 - 2f\big(\mathcal{U}_1, \mathcal{U}_2\big) \\ &+ \varepsilon^2\big(\partial_t + A_1\partial_{y_1}\big)\mathcal{U}_4 + A_{II}(\partial_Y)\mathcal{U}_3 + \partial_T\mathcal{U}_2 - f\big(\mathcal{U}_2, \mathcal{U}_2\big) \\ &- 2f\big(\mathcal{U}_1, \mathcal{U}_3\big) + \varepsilon^{5/2}\mathcal{R}^\varepsilon\Big)\Big|_{T=\varepsilon t, Y=\sqrt{\varepsilon} y_{II}, \theta=\underline{\beta}\cdot x/\varepsilon}, \end{aligned} \tag{6}$$

where we recall that $A_{II}(\partial_Y) := \sum_{j=2}^{d} A_j(\partial_{Y_j})$ and where $D_\theta := \partial_\theta / i$.

We want to choose profiles $\mathcal{U}_i$ in order to cancel the first terms in the above expansion. This yields the following profile equations:

$$iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_1 = 0, \tag{7}$$

$$iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_2 = 0, \tag{8}$$

$$iL\big(\underline{\beta} D_\theta\big)\mathcal{U}_3 + \big(\partial_t + A_1\partial_{y_1}\big)\mathcal{U}_1 = 0, \tag{9}$$

$$iL\big(\underline{\beta}D_\theta\big)\mathscr{U}_4 + \big(\partial_t + A_1\partial_{y_1}\big)\mathscr{U}_2 + A_{II}(\partial_Y)\mathscr{U}_1 - f\big(\mathscr{U}_1, \mathscr{U}_1\big) = 0, \qquad (10)$$

$$iL\big(\underline{\beta}D_\theta\big)\mathscr{U}_5 + \big(\partial_t + A_1\partial_{y_1}\big)\mathscr{U}_3 + A_{II}(\partial_Y)\mathscr{U}_2 + \partial_T\mathscr{U}_1 - 2f\big(\mathscr{U}_1, \mathscr{U}_2\big) = 0, \qquad (11)$$

and

$$\big(\partial_t + A_1\partial_{y_1}\big)\mathscr{U}_4 + A_{II}(\partial_Y)\mathscr{U}_3 + \partial_T\mathscr{U}_2 - f\big(\mathscr{U}_2, \mathscr{U}_2\big) - 2f\big(\mathscr{U}_1, \mathscr{U}_3\big) = 0. \qquad (12)$$

2.2. Algebraic analysis of equations (7)–(12)

For the principal term of our ansatz, we choose

$$\mathscr{U}_1(T, Y, t, y_1, \theta) := \mathscr{U}_{11}(T, Y, t, y_1)e^{i\theta} + \text{c. c. (complex conjugate)},$$

which means that we exclude nonoscillating terms from the principal term. This is realistic since the nonoscillating terms, which are created by rectification effects, cannot reach the same amplitude as the main oscillating terms.

In order to deduce conditions on $\mathscr{U}_{11}$ from (7)–(12) and throughout this section, we need the following algebraic lemma.

LEMMA 1

Let $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$*, and let* $\beta \in \mathbb{R}^{1+d}$*. The following two assertions are then equivalent:*

(i) $L(\beta)\mathbf{a} = \mathbf{b}$,

(ii) $\pi(\beta)\mathbf{b} = 0$ *and* $(I - \pi(\beta))\mathbf{a} = L(\beta)^{-1}\mathbf{b}$.

Thanks to this lemma, equation (7) is then equivalent to the *polarization condition*

$$\pi(\underline{\beta})\mathscr{U}_{11} = \mathscr{U}_{11}. \qquad (13)$$

Contrary to what has been done for $\mathscr{U}_1$, we allow nonoscillating terms for the first corrector $\mathscr{U}_2$. We take therefore

$$\mathscr{U}_2\big(T, Y, t, y_1, \theta\big) := \mathscr{U}_{20}\big(T, Y, t, y_1\big) + \mathscr{U}_{21}\big(T, Y, t, y_1\big)e^{i\theta} + \text{c. c.}$$

We first decompose (8) into its Fourier modes and then apply Lemma 1 to find that (8) is equivalent to

$$\pi(\underline{\beta})\mathscr{U}_{21} = \mathscr{U}_{21} \qquad (14)$$

and

$$\pi(0)\mathscr{U}_{20} = \mathscr{U}_{20}. \qquad (15)$$

Pursuing our analysis, we now want to find necessary conditions from (9). We search a $\mathscr{U}_3$ of the form

$$\mathscr{U}_3\big(T, Y, t, y_1, \theta\big) := \mathscr{U}_{30}\big(T, Y, t, y_1\big) + \mathscr{U}_{31}\big(T, Y, t, y_1\big)e^{i\theta} + \text{c. c.},$$

so that the nonoscillating Fourier coefficient of (9) reads

$$L(0)\mathscr{U}_{30} = 0.$$

Thanks to Lemma 1, this is equivalent to

$$\pi(0)\mathscr{U}_{30} = \mathscr{U}_{30}. \tag{16}$$

The first mode of the Fourier expansion of (9) reads

$$iL(\underline{\beta})\mathscr{U}_{31} + \left(\partial_t + A_1\partial_{y_1}\right)\mathscr{U}_{11} = 0.$$

Using Lemma 1 and (13), this is equivalent to the following two equations:

$$\pi(\underline{\beta})\left(\partial_t + A_1\partial_{y_1}\right)\pi(\underline{\beta})\mathscr{U}_{11} = 0 \tag{17}$$

and

$$\left(I - \pi(\underline{\beta})\right)\mathscr{U}_{31} = iL(\underline{\beta})^{-1}\left(\partial_t + A_1\partial_{y_1}\right)\pi(\underline{\beta})\mathscr{U}_{11};$$

that is, since $L(\underline{\beta})^{-1}\pi(\underline{\beta}) = 0$,

$$\left(I - \pi(\underline{\beta})\right)\mathscr{U}_{31} = iL(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{11}. \tag{18}$$

Since (10) is nonlinear quadratic, we have to look for a $\mathscr{U}_4$ with the second harmonic

$$\begin{aligned}\mathscr{U}_4\left(T, Y, t, y_1, \theta\right) := \mathscr{U}_{40}\left(T, Y, t, y_1\right) &+ \mathscr{U}_{41}\left(T, Y, t, y_1\right)e^{i\theta} + \text{c. c.}\\ &+ \mathscr{U}_{42}\left(T, Y, t, y_1\right)e^{2i\theta} + \text{c. c.}\end{aligned}$$

With the same method as above and using (15), we obtain the following equivalent equations to (10):

$$\pi(0)\left(\partial_t + A_1\partial_{y_1}\right)\pi(0)\mathscr{U}_{20} = 2\Re\left(\pi(0)f\left(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\right)\right) \tag{19}$$

and

$$(I - \pi(0))\mathscr{U}_{40} = iL(0)^{-1}A_1\partial_{y_1}\pi(0)\mathscr{U}_{20} - 2i\Re\left(L(0)^{-1}f\left(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\right)\right), \tag{20}$$

as far as the nonoscillating mode is concerned, and

$$\pi(\underline{\beta})\left(\partial_t + A_1\partial_{y_1}\right)\pi(\underline{\beta})\mathscr{U}_{21} + \pi(\underline{\beta})A_{II}(\partial_Y)\pi(\underline{\beta})\mathscr{U}_{11} = 0 \tag{21}$$

and

$$\left(I - \pi(\underline{\beta})\right)\mathscr{U}_{41} = iL(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{21} + iL(\underline{\beta})^{-1}A_{II}(\partial_Y)\mathscr{U}_{11} \tag{22}$$

for the first oscillating mode, and finally

$$\mathscr{U}_{42} = -iL\left(2\underline{\beta}\right)^{-1}f\left(\mathscr{U}_{11}, \mathscr{U}_{11}\right) \tag{23}$$

for the second harmonic, since $L(2\underline{\beta})$ is invertible thanks to Assumption 1.

Since (11) is also nonlinear quadratic, we look for a $\mathscr{U}_5$ of the same kind as $\mathscr{U}_4$,

$$\mathscr{U}_5\big(T, Y, t, y_1, \theta\big) := \mathscr{U}_{50}\big(T, Y, t, y_1\big) + \mathscr{U}_{51}\big(T, Y, t, y_1\big)e^{i\theta} + \text{c. c.} \\ + \mathscr{U}_{52}\big(T, Y, t, y_1\big)e^{2i\theta} + \text{c. c.},$$

and we obtain the following equivalent conditions:

$$\pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\pi(0)\mathscr{U}_{30} + \pi(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{20} \\ = 4\Re\big(\pi(0)f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{21}}\big)\big) \tag{24}$$

and

$$\big(I - \pi(0)\big)\mathscr{U}_{50} = iL(0)^{-1}A_1\partial_{y_1}\pi(0)\mathscr{U}_{30} \\ + iL(0)^{-1}A_{II}(\partial_Y)\mathscr{U}_{20} - 4iL(0)^{-1}\Re\big(f\big(\mathscr{U}_{11}, \overline{\mathscr{U}_{21}}\big)\big), \tag{25}$$

as far as the nonoscillating mode is concerned, and

$$\pi(\underline{\beta})A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{31} + \pi(\underline{\beta})A_{II}(\partial_Y)\pi(\underline{\beta})\mathscr{U}_{21} + \partial_T\pi(\underline{\beta})\mathscr{U}_{11} = 2\pi(\underline{\beta})f\big(\mathscr{U}_{11}, \mathscr{U}_{20}\big) \tag{26}$$

and

$$\big(I - \pi(\underline{\beta})\big)\mathscr{U}_{51} = iL(\underline{\beta})^{-1}\big(\partial_t + A_1\partial_{y_1}\big)\mathscr{U}_{31} + iL(\underline{\beta})^{-1}A_{II}(\partial_Y)\mathscr{U}_{21} \\ - 2iL(\underline{\beta})^{-1}f\big(\mathscr{U}_{11}, \mathscr{U}_{20}\big) \tag{27}$$

for the first order term of the Fourier expansion. The second harmonic $\mathscr{U}_{52}$ is obtained in the same way as $\mathscr{U}_{42}$,

$$\mathscr{U}_{52} = -2iL\big(2\underline{\beta}\big)^{-1}f\big(\mathscr{U}_{11}, \mathscr{U}_{21}\big). \tag{28}$$

Equation (26) involves $\mathscr{U}_{31}$, which can be split under the form $\mathscr{U}_{31} = \pi(\underline{\beta})\mathscr{U}_{31} + (I - \pi(\underline{\beta}))\mathscr{U}_{31}$. Plugging this decomposition into (26) and using the expression of $(I - \pi(\underline{\beta}))\mathscr{U}_{31}$ given by (18) yields

$$\partial_T\pi(\underline{\beta})\mathscr{U}_{11} + i\pi(\underline{\beta})A_1\partial_{y_1}L(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{11} \\ + \pi(\underline{\beta})\big(\partial_t + A_1\partial_{y_1}\big)\pi(\underline{\beta})\mathscr{U}_{31} + \pi(\underline{\beta})A_{II}(\partial_Y)\pi(\underline{\beta})\mathscr{U}_{21} \\ = 2\pi(\underline{\beta})f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \pi(0)\mathscr{U}_{20}\big). \tag{29}$$

We finally consider (12). In fact, we do not solve it entirely, but only its projection onto the range of $\pi(0)$. The equation thus obtained reads, thanks to (15)–(16),

$$\pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\mathscr{U}_{40} + \pi(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{30} + \pi(0)\partial_T\pi(0)\mathscr{U}_{20} \\ = \pi(0)\big(f\big(\pi(0)\mathscr{U}_{20}, \pi(0)\mathscr{U}_{20}\big)L + 2f\big(\pi(\underline{\beta})\mathscr{U}_{21}, \overline{\pi(\underline{\beta})\mathscr{U}_{21}}\big)\big) \\ + 4\Re\big(\pi(0)f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\mathscr{U}_{31}}\big)\big). \tag{30}$$

2.3. The transparency condition

Without any additional information, the equations found in the above section cannot be solved. We recall indeed that the scaling of our solutions is bigger than the normal scaling, so that the nonlinear effects should occur too soon to allow a study over large times. As said in the introduction, these nonlinear effects do not occur in many cases, provided that the following transparency condition is fulfilled.

ASSUMPTION 4 (Transparency)
For any $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$*, one has*

$$\pi(0) f\big(\pi(\underline{\beta})\mathbf{a}, \overline{\pi(\underline{\beta})\mathbf{b}}\big) = 0.$$

Under this assumption, (19) becomes linear,

$$\pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\pi(0)\mathscr{U}_{20} = 0, \tag{31}$$

and so does (24), which reads

$$\pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\pi(0)\mathscr{U}_{30} + \pi(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{20} = 0. \tag{32}$$

We finally consider (30). Under the transparency assumption, it reads

$$\begin{aligned}&\partial_T\pi(0)\mathscr{U}_{20} + \pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\mathscr{U}_{40} + \pi(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{30}\\ &\quad = \pi(0) f\big(\pi(0)\mathscr{U}_{20}, \pi(0)\mathscr{U}_{20}\big) + 4\Re\big(\pi(0) f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{(I-\pi(\underline{\beta}))\mathscr{U}_{31}}\big)\big).\end{aligned}$$

We can now use the expression of $(I-\pi(\underline{\beta}))\mathscr{U}_{31}$ given by (18), decompose $\mathscr{U}_{40}$ under the form $\mathscr{U}_{40} = \pi(0)\mathscr{U}_{40} + (I-\pi(0))\mathscr{U}_{40}$, and use the expression of $(I-\pi(0))\mathscr{U}_{40}$ given by (20) to find

$$\begin{aligned}&\partial_T\pi(0)\mathscr{U}_{20} + i\pi(0)A_1\partial_{y_1}L(0)^{-1}A_1\partial_{y_1}\pi(0)\mathscr{U}_{20}\\ &\quad + \pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\pi(0)\mathscr{U}_{40} + \pi(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{30}\\ &\quad\quad = \pi(0) f\big(\pi(0)\mathscr{U}_{20}, \pi(0)\mathscr{U}_{20}\big)\\ &\quad\quad\quad + 4\Re\big(\pi(0) f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{iL(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{11}}\big)\big)\\ &\quad\quad\quad + 2i\pi(0)A_1\partial_{y_1}L(0)^{-1}\Re\big(f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)\big).\end{aligned} \tag{33}$$

2.4. Transport at the group velocity

In this section, we review some of the profiles which are transported at the group velocity, since these profiles play the essential part in the asymptotic study. The first proposition we give is a simple consequence of the classical property of transport along rays.

PROPOSITION 2

When $\underline{\beta}$ is a smooth point of $\mathscr{C}_L$ and under Proposition 1, one has

$$\pi(\underline{\beta})\big(\partial_t + A_1\partial_{y_1}\big)\pi(\underline{\beta}) = \big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi(\underline{\beta})$$

and

$$\pi(\underline{\beta})A_{II}(\partial_Y)\pi(\underline{\beta}) = 0.$$

Proof

It is known that for all j one has $\pi(\underline{\beta})A_j\pi(\underline{\beta}) = -\partial_j\tau(\underline{\eta})\pi(\underline{\beta})$. Since in the present case we have $\partial_j\tau(\underline{\eta}) = 0$ when $j \geq 2$, the results of the proposition follow. □

Using this proposition, together with (17) and (21), yields

$$\big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi(\underline{\beta})\mathscr{U}_{11} = 0$$

and

$$\big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi(\underline{\beta})\mathscr{U}_{21} = 0, \tag{34}$$

so that both $\pi(\underline{\beta})\mathscr{U}_{11}$ and $\pi(\underline{\beta})\mathscr{U}_{21}$ are transported at the group velocity, since we recall that the group velocity reads $-\tau'(\underline{\eta}) = (\partial_1\tau(\underline{\eta}), 0, \ldots, 0)$ in our coordinates.

We finally prove that a component of $\pi(0)\mathscr{U}_{20}$ also travels at the group velocity. We recall that, thanks to (31), one has

$$\pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\pi(0)\mathscr{U}_{20} = 0.$$

Since $\pi(0)(\partial_t + A_1\partial_{y_1})\pi(0)$ is a hyperbolic symmetric operator of dimension 1, we can decompose it under the form

$$\pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\pi(0) = \sum_{j=1}^{p}\big(\partial_t + v_j\partial_{y_1}\big)\pi^j(0), \tag{35}$$

where the v_j are the distinct eigenvalues of $\pi(0)A_1\pi(0)$ and where $\pi^j(0)$ is the associated orthogonal projector defined on the range of $\pi(0)$.

Each component $\pi^j(0)\mathscr{U}_{20}$ of $\pi(0)\mathscr{U}_{20}$ is therefore transported at the velocity v_j, with respect to the variables t and y_1. The following lemma says that one of these components is tranported at the group velocity.

LEMMA 2

The group velocity $-\partial_1\tau(\underline{\eta})$ is an eigenvalue of $\pi(0)A_1\pi(0)$.

Proof

The vector $(1, -\tau'(\underline{\eta}))$ is by definition normal to the tangent plane $\mathscr{P}$ to $\mathscr{C}_L$ at $\underline{\beta}$. We recall that $\beta^0 = (1, \eta^0)$ is on the contact line between $\mathscr{P}$ and $\mathscr{C}^0$; thanks to

Assumption 2, we thus know that $(1, -\tau'(\underline{\eta}))$ is also normal to $\mathscr{C}^0$ at β^0.

Denoting by $\tau^0(\eta)$ a local parametrization of $\mathscr{C}^0$ in a neighborhood of β^0, we have therefore $\tau^{0'}(\eta^0) = \tau'(\underline{\eta})$ and $\tau^0(\eta^0) = 1$. But since $\mathscr{C}^0$ is conic, τ^0 is homogenous of degree 1, and Euler's formula yields $\tau^0(\eta^0) = \tau^{0'}(\eta^0) \cdot \eta^0$. It follows that $1 = \tau'(\underline{\eta}) \cdot \eta^0$.

Since in our coordinates we have $\eta^0 = (\eta_1^0, 0, \dots, 0)$, this last equality reads $1 = \partial_1 \tau(\underline{\eta})\eta_1^0$, and β^0 thus reads $\beta^0 = (\partial_1 \tau(\underline{\eta})\eta_1^0, \eta_1^0, 0, \dots, 0)$. We have therefore $L(\beta^0) = \eta_1^0(\partial_1 \tau(\underline{\eta}) + A_1)$.

Since $\beta^0 \in \mathscr{C}^0$, the endomorphism $\pi(0)L(\beta^0)\pi(0)$ is not invertible on the range of $\pi(0)$, and hence neither is $\pi(0)(\partial_1\tau(\underline{\eta}) + A_1)\pi(0)$, thanks to the expression just found for $L(\beta^0)$. This means that $-\partial_1\tau(\underline{\eta})$ is an eigenvalue of $\pi(0)A_1\pi(0)$, and the lemma is thus proved. □

Convention

In other words, the lemma says that there exists j such that $v_j = -\partial_1\tau(\underline{\eta})$. Up to a change of indices, we suppose from now on that $v_1 = -\partial_1\tau(\underline{\eta})$, so that $\pi^1(0)\mathscr{U}_{20}$ travels at the group velocity.

2.5. Averaging

We now use the average projectors introduced in [L1] to obtain new equations that describe the asymptotic behavior of the solution for long times. We first recall the definition of the average projector in the case in which we are interested.

Definition 1 (Average projector)

Let $T(\partial_x) := \partial_x + v\partial_{y_1}$ be a transport operator. The average projector associated to T is the operator $\mathscr{G}_T$ defined on smooth functions on $\mathbb{R}^2_{t,y_1}$ as

$$\mathscr{G}_T w(t, y_1) = \lim_{h\to\infty} \frac{1}{h} \int_0^h w\big(t+s, y_1 + vs\big)\, ds$$

when this limit exists.

If $v = -\partial_1\tau(\underline{\eta})$ and if the function $\mathscr{G}_T w$ exists, it is denoted by $\langle w\rangle$.

The following proposition gives the properties of $\mathscr{G}_T$ which we need in this paper.

PROPOSITION 3

(i) *Let* $T(\partial_x) = \partial_t + v\partial_{y_1}$.

- *If w is a smooth function of $(t, y_1) \in \mathbb{R}^2$ such that $T(\partial_x)w = 0$, then we have*

$$\mathscr{G}_T w = w;$$

- *If w has a sublinear growth, that is, if* $\lim_{t\to\infty}(1/t)\|w(t,\cdot)\|_\infty = 0$, *then*

$$\mathcal{G}_T\big(T(\partial_x)w\big) = 0.$$

(ii) *Let* $v_1 \neq v_2$, $T_1(\partial_x) := \partial_t + v_1\partial_{y_1}$, *and* $T_2(\partial_x) := \partial_t + v_2\partial_{y_1}$. *If w is such that* $T_1(\partial_x)w = 0$, *then*

$$\mathcal{G}_{T_2} w = 0.$$

(iii) *Let* $T_1(\partial_x) := \partial_t + v_1\partial_{y_1}$, $T_2(\partial_x) := \partial_t + v_2\partial_{y_1}$, *and* $T(\partial_x) := \partial_t + v\partial_{y_1}$, *and suppose that* $T_1(\partial_x)w_1 = T_2(\partial_x)w_2 = 0$. *Then*

$$\mathcal{G}_T f(w_1, w_2) = 0$$

unless $v = v_1 = v_2$, *in which case*

$$\mathcal{G}_T f(w_1, w_2) = f(w_1, w_2).$$

(iv) *If w has a sublinear growth,* $\langle w\rangle$ *is well defined, and* $v \neq -\partial_1\tau(\underline{\eta})$, *then one has*

$$\big\langle(\partial_t + v\partial_{y_1})w\big\rangle = \big(\partial_1\tau(\underline{\eta}) + v\big)\partial_{y_1}\langle w\rangle.$$

Proof

We only prove (iv) since all the other assertions of the proposition can be found in [L1].

One has

$$\begin{aligned}
&\frac{1}{h}\int_0^h \big[\big(\partial_t + v\partial_{y_1}\big)w\big]\big(t+s, y-\partial_1\tau(\underline{\eta})s\big)\,ds \\
&\quad = \frac{1}{h}\int_0^h \partial_s\big(w\big(t+s, y-\partial_1\tau(\underline{\eta})s\big)\big) \\
&\qquad\qquad + \big[\big(v+\partial_1\tau(\underline{\eta})\big)\partial_{y_1}w\big]\big(t+s, y-\partial_1\tau(\underline{\eta})s\big)\,ds \\
&\quad = \frac{1}{h}\big[w\big(t+h, y-\partial_1\tau(\underline{\eta})h\big) - w(t,y)\big] \\
&\qquad + \frac{1}{h}\int_0^h \big(v+\partial_1\tau(\underline{\eta})\big)\partial_{y_1}w\big(t+s, y-\partial_1\tau(\underline{\eta})s\big)\,ds \\
&\quad = \frac{1}{h}\big[w\big(t+h, y-\partial_1\tau(\underline{\eta})h\big) - w(t,y)\big] \\
&\qquad + \big(v+\partial_1\tau(\underline{\eta})\big)\partial_{y_1}\frac{1}{h}\int_0^h w\big(t+s, y-\partial_1\tau(\underline{\eta})s\big)\,ds.
\end{aligned}$$

Since w has a sublinear growth, the first of these two terms tends to zero when $h \to \infty$. The second of these terms tends toward $(\partial_1\tau(\underline{\eta}) + v)\partial_{y_1}\langle w\rangle$ since $\langle w\rangle$ is well defined. The assertion of the proposition is thus proved. □

We first use these results to solve (32), which reads

$$\pi(0)\big(\partial_t + A_1\partial_{y_1}\big)\pi(0)\mathscr{U}_{30} + \pi(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{20} = 0.$$

There is not uniqueness of the solution to this equation; the following lemma gives the most natural.

LEMMA 3
As a solution to (32), one can take

$$\pi(0)\mathscr{U}_{20} = \pi^1(0)\mathscr{U}_{20}$$

and

$$\pi(0)\mathscr{U}_{30} = \big\langle\pi(0)\mathscr{U}_{30}\big\rangle = -\sum_{j=2}^{p}\frac{1}{\partial_1\tau(\underline{\eta}) + v_j}\partial_{y_1}^{-1}\pi^j(0)A_{II}(\partial_Y)\pi^1(0)\mathscr{U}_{20}$$

(where the v_j, $j \geq 2$, are the eigenvalues of $\pi(0)A_1\pi(0)$ distinct from $-\partial_1\tau(\underline{\eta})$).

Proof
Using decomposition (35), equation (32) writes

$$\sum_{j=1}^{p}\big(\partial_t + v_j\partial_{y_1}\big)\pi^j(0)\mathscr{U}_{30} + \pi(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{20} = 0,$$

with $v_1 = -\partial_1\tau(\underline{\eta})$ and $v_j \neq v_1$ for $j \geq 2$. We also recall that $\pi(0)\mathscr{U}_{20} = \sum_{j=1}^{p}\pi^j(0)\mathscr{U}_{20}$ with $(\partial_t + v_j\partial_{y_1})\pi^j(0)\mathscr{U}_{20} = 0$ for all j, so that

$$\sum_{j=1}^{p}\big(\partial_t + v_j\partial_{y_1}\big)\pi^j(0)\mathscr{U}_{30} + \pi(0)A_{II}(\partial_Y)\sum_{j=1}^{p}\pi^j(0)\mathscr{U}_{20} = 0.$$

Multiplying this equation on the left by $\pi^j(0)$, with $1 \leq j \leq p$, yields

$$\big(\partial_t + v_j\partial_{y_1}\big)\pi^j(0)\mathscr{U}_{30} + \pi^j(0)A_{II}(\partial_Y)\sum_{k=1}^{p}\pi^k(0)\mathscr{U}_{20} = 0.$$

Let us introduce the operator $T_j(\partial_x) := \partial_t + v_j\partial_{y_1}$. Since we impose that $\mathscr{U}_{30}$ has a sublinear growth, we can apply the average projector $\mathscr{G}_{T_j}$ to the above equation and use Proposition 3 to find

$$\pi^j(0)A_{II}(\partial_Y)\pi^j(0)\mathscr{U}_{20} = 0. \qquad (36)_j$$

When $j \geq 2$, the operator $\pi^j(0)A_{II}(\partial_Y)\pi^j(0)$ is in general not equal to zero, so that we take $\pi^j(0)\mathscr{U}_{20} = 0$ as a solution to $(36)_j$.

When $j = 1$, things are different since $\pi^1(0)A_{II}(\partial_Y)\pi^1(0) = 0$, as we now prove. This is done in two steps

(i) One has $\ker \pi(0)(\partial_1 \tau(\underline{\eta})I + A_1)\pi(0) = \ker \pi(0)(I + A_1\eta_1^0)\pi(0)$. Indeed, one has $\ker \pi(0)(\partial_1 \tau(\underline{\eta})I + A_1)\pi(0) = \ker \pi(0)(\partial_1 \tau(\underline{\eta})\eta_1^0 I + A_1\eta_1^0)\pi(0)$, and $1 = \partial_1 \tau(\underline{\eta})\eta_1^0$, as we have seen in the proof of Lemma 2.

(ii) As in the proof of Lemma 2, denote by $\tau^0(\eta)$ a local parametrization of $\mathscr{C}^0$ in a neighborhood of β^0. Denote by $\pi^0(\eta)$ the orthogonal projector on $\ker \pi(0)L(\tau^0(\eta), \eta)$ $\pi(0)$. Thanks to (i), we know that $\pi^0(\eta^0) = \pi^1(0)$. We thus have

$$\pi^1(0)A_j\pi^1(0) = -\partial_j \tau^0(\eta^0),$$

and since $\tau^{0'}(\eta^0) = \tau'(\underline{\eta}) = (\partial_1 \tau(\underline{\eta}), 0, \dots, 0)$, we have $\pi^1(0)A_j\pi^1(0) = 0$ for all $j \geq 2$, and therefore

$$\pi^1(0)A_{II}(\partial_Y)\pi^1(0) = 0,$$

as wanted.

Therefore, $(36)_1$ does not impose any condition, so that the choice of $\pi(0)\mathscr{U}_{20} = \pi^1(0)\mathscr{U}_{20}$ is free.

Before giving an expression for $\pi(0)\mathscr{U}_{30}$, first remark that if $\mathscr{U}_{20}$ is regular enough, $\pi^j(0)\mathscr{U}_{30}$, for $j \geq 2$, is a sum of regular functions that travel at velocity v_1 or v_j, so that $\langle \pi^j(0)\mathscr{U}_{30} \rangle$ exists. Thanks to Proposition 3, applying $\mathscr{G}_{T_1}$ on $(36)_j$ yields

$$\big(\partial_1 \tau(\underline{\eta}) + v_j\big)\partial_{y_1}\big\langle \pi^j(0)\mathscr{U}_{30}\big\rangle = -\pi^j(0)A_{II}(\partial_Y)\pi^1(0)\mathscr{U}_{20}.$$

It is then easy to see that the function given in the lemma indeed solves (32). □

Remark 1

As one can see in the proof of Lemma 3, the solution given by the lemma is not the only possible one, but it is the simplest and most natural.

We now use the average projector to obtain two new equations equivalent to (29). We recall that $\pi(\underline{\beta})\mathscr{U}_{11}$, $\pi(\underline{\beta})\mathscr{U}_{21}$, and $\pi^1(0)\mathscr{U}_{20}$ are transported at the group velocity and are therefore left invariant by the action of $\langle\ \rangle$. Using also Proposition 2 and the fact that $\mathscr{U}_{31}$ has a sublinear growth, one then finds, after applying $\langle\rangle$ to (29), that this equation is equivalent to the couple of equations

$$\partial_T \pi(\underline{\beta})\mathscr{U}_{11} + i\pi(\underline{\beta})A_1\partial_{y_1}L(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{11} = 2\pi(\underline{\beta})f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \pi^1(0)\mathscr{U}_{20}\big) \tag{37}$$

and

$$\pi(\underline{\beta})\big(\partial_t + A_1\partial_{y_1}\big)\pi(\underline{\beta})\mathscr{U}_{31} = 2\pi(\underline{\beta})f\left(\pi(\underline{\beta})\mathscr{U}_{11}, \sum_{j=2}^{p}\pi^j(0)\mathscr{U}_{20}\right).$$

Since one has $\pi(0)\mathscr{U}_{20} = \pi^1(0)\mathscr{U}_{20}$ from Lemma 3, this last equation reads

$$\pi(\underline{\beta})\big(\partial_t + A_1\partial_{y_1}\big)\pi(\underline{\beta})\mathscr{U}_{31} = 0, \tag{38}$$

which is equivalent to saying that $\pi(\underline{\beta})\mathscr{U}_{31}$ is transported at the group velocity, thanks to Proposition 2.

In the evolution equation for $\mathscr{U}_{11}$ given by (37), the corrector $\mathscr{U}_{20}$ also appears, and we therefore need another profile equation in order to determine $\mathscr{U}_{11}$ and $\mathscr{U}_{20}$. This second equation is derived from (33). In fact, we do not solve (33), but only its spectral component on the range of $\pi^1(0)$. It is obtained by multiplying (33) on the left by $\pi^1(0)$. Using decomposition (35) and Lemma 3, this reads

$$\begin{aligned}
&\partial_T\pi^1(0)\mathscr{U}_{20} + i\pi^1(0)A_1\partial_{y_1}L(0)^{-1}A_1\partial_{y_1}\pi^1(0)\mathscr{U}_{20}\\
&\quad + \big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi^1(0)\mathscr{U}_{40} + \pi^1(0)A_{II}(\partial_Y)\pi(0)\mathscr{U}_{30}\\
&\qquad = \pi^1(0)f\big(\pi^1(0)\mathscr{U}_{20},\pi^1(0)\mathscr{U}_{20}\big)\\
&\qquad\quad + 4\Re\big(\pi^1(0)f\big(\pi(\underline{\beta})\mathscr{U}_{11},\overline{iL(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{11}}\big)\big)\\
&\qquad\quad + 2i\pi^1(0)A_1\partial_{y_1}L(0)^{-1}f\big(\pi(\underline{\beta})\mathscr{U}_{11},\overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big).
\end{aligned}$$

Since $\mathscr{U}_{20}$, $\mathscr{U}_{30}$, and $\mathscr{U}_{11}$ travel at the group velocity and since we impose that $\mathscr{U}_{40}$ has a sublinear growth, we can see by applying the average projector $\mathscr{G}$ to this equation that it is equivalent to

$$\begin{aligned}
&\partial_T\pi^1(0)\mathscr{U}_{20} + i\pi^1(0)A_1\partial_{y_1}L(0)^{-1}A_1\partial_{y_1}\pi^1(0)\mathscr{U}_{20}\\
&\quad - \pi^1(0)A_{II}(\partial_Y)\sum_{j=2}^{p}\frac{1}{\partial_1\tau(\underline{\eta})+v_j}\partial_{y_1}^{-1}\pi^j(0)A_{II}(\partial_Y)\pi^1(0)\mathscr{U}_{20}\\
&\qquad = \pi^1(0)f\big(\pi^1(0)\mathscr{U}_{20},\pi^1(0)\mathscr{U}_{20}\big)\\
&\qquad\quad + 4\Re\big(\pi^1(0)f\big(\pi(\underline{\beta})\mathscr{U}_{11},\overline{iL(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{11}}\big)\big)\\
&\qquad\quad + 2i\pi^1(0)A_1\partial_{y_1}L(0)^{-1}f\big(\pi(\underline{\beta})\mathscr{U}_{11},\overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)
\end{aligned} \tag{39}$$

and

$$\big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi^1(0)\mathscr{U}_{40} = 0.$$

Equation (39) is the coupled equation on $\mathscr{U}_{11}$ and $\mathscr{U}_{20}$ for which we were looking.

2.6. The evolution system for $\pi(\underline{\beta})\mathscr{U}_{11}$ and $\pi^1(0)\mathscr{U}_{20}$

We simplify here (37) and (39), which yields a system on $\pi(\underline{\beta})\mathscr{U}_{11}$ and $\pi^1(0)\mathscr{U}_{20}$ that is easier to handle. We first need the following proposition.

PROPOSITION 4

(i) *One has*

$$\pi(\underline{\beta})A_1\partial_{y_1}L(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta}) = \frac{1}{2}\partial_1^2\tau(\underline{\eta})\pi(\underline{\beta})\partial_{y_1}^2$$

and

$$\pi^1(0)A_1\partial_{y_1}L(0)^{-1}A_1\partial_{y_1}\pi^1(0) = 0.$$

(ii) *For any* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^N$*, we have*

$$\pi^1(0)f\big(\pi^1(0)\mathbf{a}, \pi^1(0)\mathbf{b}\big) = 0.$$

(iii) *The first quadratic term in* $\mathscr{U}_{11}$ *in (39) is a derivative:*

$$4\Re\big(\pi^1(0)f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{iL(\underline{\beta})^{-1}A_1\partial_{y_1}\pi(\underline{\beta})\mathscr{U}_{11}}\big)\big) \\ = -2i\partial_{y_1}\pi^1(0)f\big(\partial_1\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big).$$

Proof

(i) The first assertion of this point is very classical and can, for instance, be found in [DJMR]. We now prove the second assertion.

For any $\beta_1 := (\tau, \eta_1) \in \mathbb{R}^2$, introduce $L_I(\tau, \eta_1) := \tau I + A_1\eta_1 + L_0/i$. The associated characteristic variety $\mathscr{C}_I$ is parametrized by $(\tau_I^j(\eta_1))_{j=1,\dots,r}$, where, up to a change of indices, $\tau_I^1, \dots, \tau_I^s$ denotes the τ_I^j such that $\tau_I^j(0) = 0$.

Since we are in dimension 1, the τ_I^j are analytic functions (cf. [K]) and are odd when $j \le s$ (because A_1 and L_0 are real). We also denote by $\pi_I^j(\eta_1)$ the projector on $\ker L_I(\tau_I^j(\eta_1), \eta_1)$ when β_1 is smooth. These functions can be analytically extended to $\mathbb{R}$ (cf. [K]).

(a) We prove here that $\pi_I^j(0) = \pi^j(0)$ for $1 \le j \le s$ (where $\pi_I^j(0)$ denotes the analytic extension of $\pi_I^j(\eta_1)$ to zero).

We know that the characteristic variety $\mathscr{C}_I^0$ defined as $\{(\tau, \eta_1), \det(\pi(0)(\tau I + A_1\eta_1 + L_0/i)\pi(0)) = 0\}$ is the tangent cone to $\mathscr{C}_I$ at $(0,0)$ (see [L1]); as we are in space dimension 1, it is a union of straight lines. But, thanks to decomposition (35), we can write

$$\pi(0)A_1\pi(0) = \sum_{j=1}^{p} v_j\pi^j(0),$$

so that

$$\mathscr{C}_I^0 = \big\{\tau + v_j\eta_1 = 0,\ j = 1, \dots, p\big\}. \tag{40}$$

On the other hand, since $\mathscr{C}_I$ is the union of the analytic curves τ_I^j, the tangent cone $\mathscr{C}_I^0$ is given by

$$\mathscr{C}_I^0 = \big\{\tau - \tau_I^{j'}(0)\eta_1 = 0,\ j = 1, \dots, s\big\}. \tag{41}$$

Thanks to (40) and (41), we know that $p = s$ and that, up to a change of indices, $v_j = -\tau_I^{j'}(0)$.

We also have

$$\pi_I^j(\eta_1)\Big(\tau_I^j(\eta_1) + A_1\eta_1 + \frac{1}{i}L_0\Big) = 0, \quad \forall \eta_1.$$

Differentiating this equality yields

$$\pi_I{}'(\eta_1)\Big(\tau_I^j(\eta_1) + A_1\eta_1 + \frac{1}{i}L_0\Big) + \pi_I^j(\eta_1)\Big(\tau_I^{j'}(\eta_1) + A_1\Big) = 0. \tag{42}$$

Taking the limit of this equality when $\eta_1 \to 0$ and multiplying on the right by $\pi_I^j(0)$ yields

$$\pi_I^j(0)A_1\pi_I^j(0) = -\tau_I^{j'}(0)\pi_I^j(0).$$

Since $-\tau_I^{j'}(0) = v_j$, this means that $\pi_I^j(0)$ is the eigenprojector associated to the eigenvalue v_j of $\pi_I^j(0)A_1\pi_I^j(0)$, and therefore $\pi_I^j(0) = \pi^j(0)$.

(b) We now introduce

$$\phi(\eta_1) = \frac{\big(\pi_I^1(\eta_1) + \cdots + \pi_I^p(\eta_1)\big)A_1\big(\pi_I^1(\eta_1) + \cdots + \pi_I^p(\eta_1)\big) - \pi(0)A_1\pi(0)}{\eta_1},$$

and we prove that

$$\lim_{\eta_1\to 0}\phi(\eta_1) = -\sum_{j=1}^{p}\tau_I^{j'}(0)\pi_I^{j'}(0) + \sum_{j,k,k\neq j}\big(\tau_I^{k'}(0) - \tau_I^{j'}(0)\big)\pi_I^{j'}(0)\pi_I^k(0). \tag{43}$$

We know that $\pi_I^j(\eta_1)A_1\pi_I^j(\eta_1) = -\tau_I^{j'}(\eta_1)\pi_I^j(\eta_1)$ and, as we have seen in (a), that $\pi(0)A_1\pi(0) = -\sum_{j=1}^{p}\tau_I^{j'}(0)\pi_I^j(0)$. We therefore have

$$\begin{aligned}\phi(\eta_1) &= \sum_{j=1}^{p}\frac{\tau_I^{j'}(0) - \tau_I^{j'}(\eta_1)}{\eta_1}\pi_I^j(\eta_1)\\ &\quad + \sum_{j=1}^{p}\tau_I^{j'}(0)\frac{\pi_I^j(0) - \pi_I^j(\eta_1)}{\eta_1} + \sum_{i\neq j}\frac{\pi_I^j(\eta_1)A_1\pi_I^k(\eta_1)}{\eta_1}\\ &:= A + B + C.\end{aligned}$$

One then has

- $A \to \sum_j \tau_I^{j''}(0)\pi_I^j(0)$ when $\eta_1 \to 0$, and therefore $A \to 0$ since the τ_I^j are odd for $j \leq p$;
- $B \to -\sum_j \tau_I{}'(0)\pi_I^{j'}(0)$ when $\eta_1 \to 0$;
- $C \to \sum_{j\neq k}(\tau_I^{k'}(0) - \tau_I^{j'}(0))\pi_I^{j'}(0)\pi_I^k(0)$ when $\eta_1 \to 0$.

In order to prove this result, we first multiply (42) on the right by $\pi_I^k(\eta_1)$, for $k \neq j$,

$$\pi_I{}'(\eta_1)\Big(\tau_I^j(\eta_1)+A_1\eta_1+\frac{1}{i}L_0\Big)\pi_I^k(\eta_1)+\pi_I^j(\eta_1)A_1\pi_I^k(\eta_1)=0,$$

and thus,

$$\big(\tau_I^j(\eta_1)-\tau_I^k(\eta_1)\big)\pi_I^{j'}(\eta_1)\pi_I^k(\eta_1)+\pi_I^j(\eta_1)A_1\pi_I^k(\eta_1)=0.$$

We have therefore

$$\sum_{i\neq j}\pi_I^j(\eta_1)A_1\pi_I^k(\eta_1)=\sum_{i\neq j}\big(\tau_I^k(\eta_1)-\tau_I^j(\eta_1)\big)\pi_I^{j'}(\eta_1)\pi_I^k(\eta_1).$$

We just have to divide this equality by η_1 and take the limit when $\eta_1\to 0$ to obtain the desired result.

Since $\phi(\eta_1)=A+B+C$, equality (43) is proved.

(c) Let us introduce $\Pi(\eta_1):=(\pi_I^1(\eta_1)+\cdots+\pi_I^p(\eta_1))$. One has $\Pi(0)=\pi(0)$, and Π is an analytic function of η_1; we prove here the equality

$$L(0)^{-1}A_1\pi(0)+(I-\pi(0))\Pi'(0)=0. \tag{44}$$

In order to prove this relation, first notice that

$$\prod_{j=1}^{p}\Big(\tau_I^j(\eta_1)+A_1\eta_1+\frac{1}{i}L_0\Big)\Pi(\eta_1)=0.$$

Differentiating this equality with respect to η_1 yields

$$\sum_{k=1}^{p}\prod_{j<k}\Big(\tau_I^j(\eta_1)+A_1\eta_1+\frac{1}{i}L_0\Big)\big(\tau_I^{k'}(\eta_1)+A_1\big)\prod_{j>k}\Big(\tau_I^j(\eta_1)+A_1\eta_1+\frac{1}{i}L_0\Big)\Pi(\eta_1)$$
$$+\prod_{j=1}^{p}\Big(\tau_I^j(\eta_1)+A_1\eta_1+\frac{1}{i}L_0\Big)\Pi'(\eta_1)=0.$$

Taking the limit of this expression when $\eta_1\to 0$ yields

$$\sum_{k=1}^{p}\prod_{j<k}\frac{1}{i}L_0\big(\tau_I^{k'}(0)+A_1\big)\prod_{j>k}\frac{1}{L_0}\Pi(0)+\prod_{j=1}^{p}\frac{1}{i}L_0\Pi'(0)=0;$$

that is,

$$\Big(\frac{L_0}{i}\Big)^{s-1}A_1\Pi(0)+\Big(\frac{L_0}{i}\Big)^{s}\Pi'(0)=0.$$

Multiplying this equality on the left by $(L(0)^{-1})^s$ then yields equality (44).

(d) We prove here that

$$\lim_{\eta_1\to 0}\phi(\eta_1)=-2\pi(0)A_1L(0)^{-1}A_1\pi(0) \tag{45}$$

$$- \Pi'(0) \sum_j \tau_I^{j\,\prime}(0)\pi_I^j(0) - \sum_j \tau_I^{j\,\prime}(0)\pi_I^j(0)\Pi'(0).$$

Indeed, one has

$$\phi(\eta_1) = \frac{\big(\Pi(\eta_1) - \pi(0)\big)A_1\Pi(\eta_1) + \pi(0)A_1\big(\Pi(\eta_1) - \pi(0)\big)}{\eta_1},$$

and therefore

$$\lim_{\eta_1\to 0} \phi(\eta_1) = \Pi'(0)A_1\Pi(0) + \Pi'(0)A_1\Pi(0).$$

But thanks to (c), it is easy to see that

$$\begin{aligned}\Pi'(0)A_1\Pi(0) &= \Pi'(0)\big(I - \Pi(0)\big)A_1\Pi(0) + \Pi'(0)\Pi(0)A_1\Pi(0)\\ &= -\Pi(0)A_1L(0)^{-1}A_1\Pi(0) - \Pi'(0)\sum_j \tau_I^{j\,\prime}(0)\pi_I^j(0).\end{aligned}$$

We just have to transpose this equality to find $\Pi(0)A_1\Pi'(0)$, and thus (45) is proved.

(e) Thanks to (43) and (45), we find an expression for $\pi(0)A_1L(0)^{-1}A_1\pi(0)$. It is then easy to see that if we multiply this expression on both sides by $\pi^1(0)$, we find zero, so that the second assertion of point (i) of the proposition is proved.

(ii) Thanks to what we have seen in the proof of (i), we can write

$$\pi^1(0)f\big(\pi^1(0)\mathbf{a}, \pi^1(0)\mathbf{b}\big) = \lim_{\eta_1\to 0} \pi_I^1(0)f\big(\pi_I^1(\eta_1)\mathbf{a}, \overline{\pi_I^1(\eta_1)\mathbf{b}}\big).$$

Since $\pi_I^1(\eta_1) = \pi(\beta_1)$, with $\beta_1 := (\tau_I^1(\eta_1), \eta_1, 0, \dots, 0) \in \mathbb{R}^{1+d}$, the right-hand side of the above equation is equal to zero, thanks to Assumption 4, and the result follows.

(iii) The proof of this point can be found in [C]. □

We have thus proved the following theorem.

THEOREM 1

Suppose that u^ε, given by

$$u^\varepsilon(x) = \sqrt{\varepsilon}\,\mathscr{U}\left(\varepsilon, \varepsilon t, \sqrt{\varepsilon}y_{II}, t, y_1, \frac{\underline{\beta}\cdot x}{\varepsilon}\right),$$

with

$$\mathscr{U} := \mathscr{U}_1 + \sqrt{\varepsilon}\,\mathscr{U}_2 + \varepsilon\,\mathscr{U}_3 + \varepsilon^{3/2}\mathscr{U}_4 + \varepsilon^2\mathscr{U}_5$$

is the approximate solution to (1) given by geometric optics. If $\mathscr{U}_1 = \mathscr{U}_{11}e^{i\theta} + \mathrm{c.c.}$ and $\mathscr{U}_2 = \mathscr{U}_{20} + \mathscr{U}_{21}e^{i\theta} + \mathrm{c.c.}$, then one has

$$\pi(\underline{\beta})\mathscr{U}_{11} = \mathscr{U}_{11}, \qquad \mathscr{U}_{21} = 0, \qquad \textit{and} \qquad \pi(0)\mathscr{U}_{20} = \mathscr{U}_{20}.$$

Moreover, $\pi(\underline{\beta})\mathscr{U}_{11}$ *and* $\pi^1(0)\mathscr{U}_{20} = \langle\pi(0)\mathscr{U}_{20}\rangle$ *are transported at the group velocity* $-\partial_1\tau(\underline{\eta})$*, that is,*

$$\big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi(\underline{\beta})\mathscr{U}_{11} = \big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi^1(0)\mathscr{U}_{20} = 0,$$

and must also satisfy

$$\partial_T\pi(\underline{\beta})\mathscr{U}_{11} + \frac{i}{2}\partial_1^2\tau(\underline{\eta})\partial_{y_1}^2\pi(\underline{\beta})\mathscr{U}_{11} = 2\pi(\underline{\beta})f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \pi^1(0)\mathscr{U}_{20}\big) \tag{46}$$

and

$$\begin{aligned}\partial_T\pi^1(0)\mathscr{U}_{20} &- \pi^1(0)A_{II}(\partial_Y)\sum_{j=2}^{p}\frac{1}{\partial_1\tau(\underline{\eta})+v_j}\partial_{y_1}^{-1}\pi^j(0)A_{II}(\partial_Y)\pi^1(0)\mathscr{U}_{20}\\ &= -2i\partial_{y_1}\pi^1(0)f\big(\partial_1\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)\\ &\quad + 2i\pi^1(0)A_1\partial_{y_1}L(0)^{-1}f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big).\end{aligned} \tag{47}$$

3. The approximate solution u^ε and its properties

3.1. The leading terms of the ansatz

We want to know the leading term $\mathscr{U}_1$ of ansatz (3). We have seen that $\mathscr{U}_{11} = \mathscr{U}_{11}e^{i\theta} +$ c. c. and that $\mathscr{U}_{11}(T, Y, t, y_1)$ satisfies the polarization condition $\mathscr{U}_{11} = \pi(\underline{\beta})\mathscr{U}_{11}$, together with the transport equation

$$\big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\mathscr{U}_{11} = 0,$$

so that $\mathscr{U}_{11}(T, Y, t, y_1)$ may be written under the form $\mathscr{U}_{11}(T, Y, \zeta)$, where $\zeta := y_1 + t\partial_1\tau(\underline{\eta})$.

The second term of the ansatz writes $\mathscr{U}_2 = \mathscr{U}_{20} + \mathscr{U}_{21}e^{i\theta} +$ c. c., and its nonoscillating mode $\mathscr{U}_{20}$ satisfies the polarization condition $\mathscr{U}_{20} = \pi^1(0)\mathscr{U}_{20}$ together with the same transport equation as $\mathscr{U}_{11}$, so that we can also write $\mathscr{U}_{20}(T, Y, t, y_1)$ under the form $\mathscr{U}_{20}(T, Y, \zeta)$.

We have seen that the slow evolutions of $\mathscr{U}_{11}$ and $\mathscr{U}_{20}$ are coupled by (46) and (47). Such a system presents many difficulties. In this paper, we assume that it admits sufficiently regular solutions and pursue the analysis.

ASSUMPTION 5

Let $\mathscr{U}_{11}^0 = \pi(\underline{\beta})\mathscr{U}_{11}^0$ *and* $\mathscr{U}_{20}^0 = \pi^1(0)\mathscr{U}_{20}^0$ *be in* $H^\infty(\mathbb{R}^d_{Y,\zeta})$*. There exists a* $\underline{T} > 0$*, an integer* s *sufficiently large, and a unique couple of profiles* $\mathscr{U}_{11}, \mathscr{U}_{20} \in C([0, \underline{T}]; H^s(\mathbb{R}^d_{Y,\zeta}))$ *satisfying*

$$(\mathbf{S})\begin{cases}\partial_T\pi(\underline{\beta})\mathscr{U}_{11}+\dfrac{i}{2}\partial_1^2\tau(\underline{\eta})\partial_\zeta^2\pi(\underline{\beta})\mathscr{U}_{11}=2\pi(\underline{\beta})f\big(\pi(\underline{\beta})\mathscr{U}_{11},\pi^1(0)\mathscr{U}_{20}\big),\\ \partial_T\pi^1(0)\mathscr{U}_{20}-\pi^1(0)A_{II}(\partial_Y)\displaystyle\sum_{j=2}^{p}\frac{1}{\partial_1\tau(\underline{\eta})+v_j}\partial_\zeta^{-1}\pi^j(0)A_{II}(\partial_Y)\pi^1(0)\mathscr{U}_{20}\\ \qquad=-2i\partial_\zeta\pi^1(0)f\big(\partial_1\pi(\underline{\beta})\mathscr{U}_{11},\overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)\\ \qquad\quad+2i\pi^1(0)A_1L(0)^{-1}\partial_\zeta f\big(\pi(\underline{\beta})\mathscr{U}_{11},\overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big),\end{cases}$$

together with the polarization conditions

$$\mathscr{U}_{11}=\pi(\underline{\beta})\mathscr{U}_{11}\qquad \textit{and}\qquad \mathscr{U}_{20}=\pi^1(0)\mathscr{U}_{20}$$

and with the initial conditions

$$\mathscr{U}_{11}\big|_{T=0}=\mathscr{U}_{11}^0\qquad \textit{and}\qquad \mathscr{U}_{20}\big|_{T=0}=\mathscr{U}_{20}^0.$$

Remark 2
In Section 4 we prove that this assumption can be proved for Maxwell-Bloch systems satisfying a strong transparency condition. We also prove that this assumption is satisfied in the 1-dimensional case in Section 5. Finally, we give in Section 6 an existence theorem for a simplified system arising also in the study of water waves.

Under this assumption, the profiles $\mathscr{U}_{11}$ and $\mathscr{U}_{20}$ may be determined, and the other terms follow, as we now see.

3.2. Corrector terms of the ansatz

In this section, we suppose that $\mathscr{U}_{11}$ and $\mathscr{U}_{20}$ are known, and we construct the missing terms of ansatz (3) in accordance with the equations found in Section 2.

The leading term $\mathscr{U}_1$ is already known since $\mathscr{U}_1=\mathscr{U}_{11}e^{i\theta}+\text{c. c.}$, but we still have to find $\mathscr{U}_{21}$ to determine the first corrector $\mathscr{U}_2$. The only conditions found so far on $\mathscr{U}_{21}$ are the polarization condition (14) and the transport equation (34). We can therefore take $\mathscr{U}_{21}=0$.

The second corrector $\mathscr{U}_3$ writes $\mathscr{U}_3=\mathscr{U}_{30}+\mathscr{U}_{31}e^{i\theta}+\text{c. c.}$ The nonoscillating component satisfies the polarization condition (16), that is, $\mathscr{U}_{30}=\pi(0)\mathscr{U}_{30}$, and is therefore given by Lemma 3.

The component $\pi(\underline{\beta})\mathscr{U}_{31}$ of the oscillating mode must only satisfy the transport equation (38) and can therefore be taken equal to zero. The component $(I-\pi(\underline{\beta}))\mathscr{U}_{11}$ is given in terms of $\mathscr{U}_{11}$ by (18).

For the corrector $\mathscr{U}_4=\mathscr{U}_{40}+\mathscr{U}_{41}e^{i\theta}+\text{c. c.}+\mathscr{U}_{42}e^{2i\theta}+\text{c. c.}$, we obtain similarly $(I-\pi(0))\mathscr{U}_{40}$, thanks to (20), and we can take $\pi(0)\mathscr{U}_{40}=0$. The component $(I-$

$\pi(\underline{\beta}))\mathscr{U}_{41}$ of the first oscillating mode is given by (22), and we can take $\pi(\underline{\beta})\mathscr{U}_{41} = 0$. The second harmonic is found using (23).

Finally, for the last corrector $\mathscr{U}_5 = \mathscr{U}_{50} + \mathscr{U}_{51}e^{i\theta} + \text{c. c.} + \mathscr{U}_{52}e^{i\theta} + \text{c. c.}$, we obtain $(I - \pi(0))\mathscr{U}_{50}$, thanks to (25), and we can take $\pi(0)\mathscr{U}_{50} = 0$. The component $(I - \pi(\underline{\beta}))\mathscr{U}_{51}$ is given by (27), while $\pi(\underline{\beta})\mathscr{U}_{51}$ can also be taken equal to zero. The second harmonic is given by (28) and is therefore equal to zero, since $\mathscr{U}_{21} = 0$.

All the components of the ansatz (3),

$$\mathscr{U}\big(\varepsilon, T, Y, t, y_1, \theta\big) := \big(\mathscr{U}_1 + \sqrt{\varepsilon}\mathscr{U}_2 + \varepsilon\mathscr{U}_3 + \varepsilon^{3/2}\mathscr{U}_4 + \varepsilon^2\mathscr{U}_5\big)\big(\varepsilon, T, Y, t, y_1, \theta\big),$$

are therefore known, once Assumption 5 is made. The dependence on t and y_1 of all these profiles is indeed a dependence on $\zeta = y_1 + \partial_1\tau(\underline{\eta})t$ since they are all transported at the group velocity. We now give explicitly the expression of the ansatz we have found:

$$\begin{aligned}
\mathscr{U}\big(\varepsilon, T, Y, \zeta, \theta\big) = {}& \pi(\underline{\beta})\mathscr{U}_{11}(T, Y, \zeta)e^{i\theta} + \text{c. c.} + \sqrt{\varepsilon}\pi^1(0)\mathscr{U}_{20}(T, Y, \zeta)\\
&+ \varepsilon \sum_{j=2}^{p} \frac{-1}{\partial_1\tau(\underline{\eta}) + v_j}\partial_\zeta^{-1}\pi^j(0)A_{II}(\partial_Y)\pi^1(0)\mathscr{U}_{20}\\
&+ iL(\underline{\beta})^{-1}A_1\partial_\zeta\pi(\underline{\beta})\mathscr{U}_{11}e^{i\theta} + \text{c. c.}\\
&+ \varepsilon^{3/2} iL(0)^{-1}A_1\partial_\zeta\pi^1(0)\mathscr{U}_{20} - 2iL(0)^{-1}f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)\\
&+ iL(\underline{\beta})^{-1}A_{II}(\partial_Y)\pi(\underline{\beta})\mathscr{U}_{11}e^{i\theta} + \text{c. c.}\\
&- iL(2\underline{\beta})^{-1}f\big(\mathscr{U}_{11}, \mathscr{U}_{11}\big)e^{2i\theta} + \text{c. c.}\\
&+ \varepsilon^2 - iL(0)^{-1}A_1 \sum_{j=2}^{p} \frac{1}{\partial_1\tau(\underline{\eta}) + v_j}\pi^j(0)A_{II}(\partial_Y)\pi^1(0)\mathscr{U}_{20}\\
&- \big(L(\underline{\beta})^{-1}\big(\partial_1\tau(\underline{\eta}) + A_1\big)L(\underline{\beta})^{-1}A_1\partial_\zeta^2\pi(\underline{\beta})\mathscr{U}_{11}\\
&\quad - 2iL(\underline{\beta})^{-1}f\big(\mathscr{U}_{11}, \mathscr{U}_{20}\big)\big)e^{i\theta} + \text{c. c.}
\end{aligned}$$

3.3. Properties of ansatz (3)

Now that we have found the ansatz we were looking for, we give a few properties. The first one concerns regularity.

PROPOSITION 5

If $\mathscr{U}_{11}$ and $\mathscr{U}_{20}$ are in $C([0, \underline{T}]; H^s(\mathbb{R}^d_{Y,\zeta}))$ as asserted in Assumption 5, then all the Fourier coefficients $\mathscr{U}_{ij}$, $i = 1, \ldots, 5$ and $j = 0, \ldots, 2$, are in $C([0, \underline{T}]; H^{s-2}(\mathbb{R}^d_{Y,\zeta}))$.

Proof
Thanks to the expression of $\mathscr{U}$ given above, the only difficulty is to prove that

$$\pi(0)\mathscr{U}_{30} = -\sum_{j=2}^{p} \frac{1}{\partial_1 \tau(\underline{\eta}) + v_j} \partial_\zeta^{-1} \pi^j(0) A_{II}(\partial_Y) \pi^1(0) \mathscr{U}_{20}$$

is in $C([0, \underline{T}]; H^{s-2}(\mathbb{R}^d_{Y,\zeta}))$.

The crucial point is that the nonlinearity of the second equation of (**S**) is a derivative with respect to ζ. If $\pi^1(0)\mathscr{U}_{20}$ and $\pi(\underline{\beta})\mathscr{U}_{11}$ solve (**S**), then $\mathscr{W} := \partial_\zeta^{-1}\pi^1(0)\mathscr{U}_{20}$ solves

$$\begin{aligned}
&\partial_T \mathscr{W} - \pi^1(0) A_{II}(\partial_Y) \sum_{j=2}^{p} \frac{1}{\partial_1 \tau(\underline{\eta}) + v_j} \partial_\zeta^{-1} \pi^j(0) A_{II}(\partial_Y) \mathscr{W} \\
&\quad = 2i\pi^1(0) f\big(\partial_1 \pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big) \\
&\qquad + 2i\pi^1(0) A_1 L(0)^{-1} \Re\big(f\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)\big).
\end{aligned}$$

Since the second member of this equation is in $C([0, \underline{T}]; H^s(\mathbb{R}^d_{Y,\zeta}))$ (if s is large enough), then $\mathscr{W}$ is also in this space. Since $\pi(0)\mathscr{U}_{30}$ writes

$$\pi(0)\mathscr{U}_{30} = -\sum_{j=2}^{p} \frac{1}{\partial_1 \tau(\underline{\eta}) + v_j} \pi^j(0) A_{II}(\partial_Y) \mathscr{W},$$

it is therefore in $C([0, \underline{T}]; H^{s-1}(\mathbb{R}^d_{Y,\zeta}))$. □

We now prove that the corrector term $\sqrt{\varepsilon}\mathscr{U}_2 + \cdots + \varepsilon^2\mathscr{U}_5$ remains smaller than the leading term $\mathscr{U}_1$ for times $O(1/\varepsilon)$. In order to do this, we show that the boundedness condition (4) and the sublinear growth conditions (5) are satisfied.

PROPOSITION 6
The profile $\mathscr{U}_2$ satisfies the boundedness condition (4):

$$\exists C > 0, \quad \sup_{t \in \mathbb{R}^+} \big\| \mathscr{U}_2(\cdot, \cdot, t, \cdot, \cdot) \big\|_{L^\infty([0,\underline{T}] \times \mathbb{R}^d_{Y,y_1} \times \mathbb{T})} \le C.$$

The other correctors $\mathscr{U}_i$, $i = 3, 4, 5$, also satisfy this boundedness condition, so that the sublinear growth condition (5) is a fortiori satisfied.

Proof
We recall that $\mathscr{U}_2 = \pi^1(0)\mathscr{U}_{20}$, so that the fact that $\mathscr{U}_2$ satisfies the boundedness condition (4) is a mere consequence of Assumption 5 if s is large enough.

Thanks to the expressions already given, it is also easy to see that the other correctors $\mathscr{U}_i$, $i = 3, 4, 5$, are also bounded. □

Remark 3

(1) As seen in Proposition 6, all the profiles are bounded, so that the sublinear growth condition may seem too strong. But it was not a priori obvious that this would be the case. What happens here is that all the profiles considered travel at the velocity $-\partial_1\tau(\underline{\eta})$, while sublinear growth occurs when other velocities are present. To be more precise, if w_1 and w_2 are two functions such that

$$\big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)w_1 = w_2 \qquad \text{and} \qquad \langle w_2\rangle = 0,$$

then w_1 has a sublinear growth. In [L1], we can find a second member w_2 that travels at a different velocity than $-\partial_1\tau(\underline{\eta})$. One then has $\langle w_2\rangle = 0$ but $w_2 \neq 0$, and w_1 has therefore a sublinear growth but is not bounded. In this paper, the second member w_2 is always equal to zero, so that w_1 is bounded.

(2) The fact that all the profiles are bounded suggest an improvement of the precision of our approximation, as we see in the next sections.

3.4. Estimate for the residual

In this section, we prove that the approximate solution (defined thanks to the ansatz we have found) is almost a solution of problem (1) since it provides a small residual. We first give a regularity result for the residual.

LEMMA 4

To the approximate solution $u^\varepsilon = \sqrt{\varepsilon}\mathscr{U}(\varepsilon, \varepsilon t, \sqrt{\varepsilon}y_{II}, t, y_1, \underline{\beta}\cdot x/\varepsilon)$ *corresponds the residual*

$$L^\varepsilon(\partial_x)u^\varepsilon - f\big(u^\varepsilon, u^\varepsilon\big) = k^\varepsilon(x),$$

which may be written under the form

$$k^\varepsilon(x) = \mathscr{K}\left(\varepsilon, \varepsilon t, \sqrt{\varepsilon}y_{II}, y_1 + \partial_1\tau(\underline{\eta})t, \frac{\underline{\beta}\cdot x}{\varepsilon}\right),$$

with

$$\mathscr{K}\big(\varepsilon, T, Y, \zeta, \theta\big) = \sum_{j=-4}^{4} \mathscr{K}_j\big(\varepsilon, T, Y, \zeta\big)e^{ij\theta},$$

and the $\mathscr{K}_j$ *are in* $C([0,\underline{T}]; H^{s-4}(\mathbb{R}^d_{Y,\zeta}))$ *if* $\mathscr{U}_{11}$ *and* $\mathscr{U}_{20}$ *are in* H^s *as asserted by Assumption 5.*

Proof

The proof of this lemma is straightforward, once we have proved that the derivatives $\partial_T\mathscr{U}_1$, $\partial_T\mathscr{U}_2$, $\partial_T\mathscr{U}_3$, $\partial_T\mathscr{U}_4$, and $\partial_T\mathscr{U}_5$ which appear in the residual are in $C([0,\underline{T}]; H^{s-4}(\mathbb{R}^d_{Y,\zeta}))$.

This is clear for $\partial_T \mathscr{U}_1$, thanks to the first equation of (**S**).

We have already seen in the proof of Proposition 5 that $\mathscr{W} = \partial_\zeta^{-1} \pi^1(0) \mathscr{U}_{20}$ is in $C([0, \underline{T}]; H^s(\mathbb{R}^d_{Y,\zeta}))$. Thanks to the second equation of (**S**), $\partial_T \mathscr{U}_2$ is thus in $C([0, \underline{T}]; H^{s-2}(\mathbb{R}^d_{Y,\zeta}))$.

Differentiating the second equation of (**S**) with respect to T and using the same method as in the proof of Proposition 5 then yields that $\partial_T \partial_\zeta^{-1} \mathscr{U}_{20}$ is in $C([0, \underline{T}]; H^{s-2}(\mathbb{R}^d_{Y,\zeta}))$. Thanks to the expression given by Lemma 3, we can then conclude that $\partial_T \mathscr{U}_3$ is in $C([0, \underline{T}]; H^{s-3}(\mathbb{R}^d_{Y,\zeta}))$.

The proof that $\partial_T \mathscr{U}_4$ and $\partial_T \mathscr{U}_5$ are in $C([0, \underline{T}]; H^{s-4}(\mathbb{R}^d_{Y,\zeta}))$ is left to the reader. □

Knowing in which spaces things are living, we can give estimates on the residual.

PROPOSITION 7

(i) *The Fourier coefficients of the profile $\mathscr{K}$ of the residual satisfy*

$$\|K_j\|_{L^\infty([0,\underline{T}], H^{s-4}(\mathbb{R}^d_{Y,\zeta}))} = O(\varepsilon^2), \quad \textit{for } j = -4, \ldots, 4.$$

(ii) *We have a better estimate for the component $\pi^1(0)\mathscr{K}_0$ of the nonoscillating mode*

$$\left\|\pi^1(0) K_0\right\|_{L^\infty([0,\underline{T}], H^{s-4}(\mathbb{R}^d_{Y,\zeta}))} = O\left(\varepsilon^{5/2}\right).$$

Proof

This proposition is a direct consequence of the method we have used to find our approximate solution, since we have cancelled the terms of expansion (6) up to the power $\varepsilon^{3/2}$. We also have cancelled the component polarized along $\pi^1(0)$ of the term in ε^2, which yields the improvement stated in (ii). □

Remark. (i) If the profiles $\mathscr{U}_3$, $\mathscr{U}_4$, and $\mathscr{U}_5$ had a sublinear growth instead of being bounded, then we would have $\mathscr{K}_j = o(\varepsilon)$ and $\pi^1(0)\mathscr{K}_0 = o(\varepsilon^{3/2})$ instead of $O(\varepsilon^2)$ and $O(\varepsilon^{5/2})$, respectively.

(ii) Proposition 7(ii) is of crucial importance in the proof of the stability result of the next section.

4. The case of Maxwell-Bloch systems

In the previous section, we proved that our approximate solution u^ε is almost a solution of problem (1). But the most important thing is to prove that u^ε remains close to the exact solution $\mathbf{u}^\varepsilon$. Such a stability property is very difficult to prove because of resonances (see [JMR2]). The general case remains at the moment out of reach, and, as done in [C], we limit ourselves to a smaller class of problems than those of type (1). Under a strong transparency assumption we also prove that the nonlinearity in System (**S**) vanishes, so that Assumption 5 can be proved in this case.

4.1. General Maxwell-Bloch systems

We now look for solutions of size $O(\sqrt{\varepsilon})$ to systems of the form

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^{d_1} A_j \partial_{y_j} \mathbf{u}^\varepsilon + \frac{L_0}{\varepsilon} \mathbf{u}^\varepsilon = f\big(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon\big), \\ \partial_t \mathbf{v}^\varepsilon + \sum_{j=1}^{d_2} B_j \partial_{y_j} \mathbf{v}^\varepsilon + \frac{M_0}{\varepsilon} \mathbf{v}^\varepsilon = g\big(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon\big), \end{cases} \tag{48}$$

where A_j and B_j denote symmetric real-valued matrices, while L_0 and M_0 are skew-symmetric.

The mappings f and g are bilinear mappings and g is symmetric.

For $(\tau, \eta) \in \mathbb{R}^{1+d}$, we recall that

$$L(\tau, \eta) = \tau I + \sum_{j=1}^{d_1} A_j \eta_j + \frac{L_0}{i},$$

as well as

$$A_{II}(\eta) := \sum_{j=2}^{d_1} A_j \eta_j.$$

We similarly define

$$M(\tau, \eta) = \tau I + \sum_{j=1}^{d_2} B_j \eta_j + \frac{M_0}{i},$$

as well as

$$B_{II}(\eta) := \sum_{j=2}^{d_2} B_j \eta_j.$$

The set of all $\beta := (\tau, \eta) \in \mathbb{R}^{1+d}$ such that $\det(L(\beta)) = 0$ is the characteristic variety of L and is denoted by $\mathscr{C}_L$. Similarly, $\mathscr{C}_M$ denotes the characteristic variety of M. For any $\eta \in \mathbb{R}^d$, we denote by $(-\tau_L^l(\eta))_{l=1,\dots,p_1}$ the eigenvalues of $A(\eta) + L_0/i$ and by $(-\tau_M^l(\eta))_{l=1,\dots,p_2}$ those of $B(\eta) + M_0/i$, thus providing a parametrization of $\mathscr{C}_L$ and $\mathscr{C}_M$. Up to a renumbering, we can suppose that $\underline{\beta} = \tau_L^1(\underline{\eta})$.

We also denote by $\pi_L(\beta)$ and $\pi_M(\beta)$ the orthogonal projectors on $\ker(\tau I + A(\eta) + L_0/i)$ and $\ker(\tau I + B(\eta) + M_0/i)$, respectively, and we denote by $L(\beta)^{-1}$ and $M(\beta)^{-1}$ the partial inverses of $L(\beta)$ and $M(\beta)$. Similarly, $\pi_L(0)$ and $\pi_M(0)$ are the orthogonal projectors on the kernel of $L(0)$ and $M(0)$, and $L(0)^{-1}$ and $M(0)^{-1}$ their partial inverses.

We finally denote by $\mathscr{C}_L^0$ and $\mathscr{C}_M^0$ the characteristic varieties of the operators $\pi_L(0)L^\varepsilon(\partial_x)\pi_L(0)$ and $\pi_M(0)M^\varepsilon(\partial_x)\pi_M(0)$, respectively. Thanks to Lemma 2, we know that $\pi_M(0)B_1\pi_M(0)$ admits $-\partial_1\tau_L^1(\underline{\eta})$ as an eigenvalue. The associated eigenprojector is denoted by $\pi_M^1(0)$, while the projectors associated to the other eigenvalues v_j are denoted $\pi_M^j(0)$, $j \geq 2$.

Assumption 1 on the choice of $\underline{\beta}$ and Assumption 2 on the long-wave short-wave resonance are replaced in this new framework by the following assumption.

ASSUMPTION 6

(i) *The point* $\underline{\beta}$ *of* $\mathscr{C}_L$ *is smooth and* $2\underline{\beta} \notin \mathscr{C}_L$*; neither* $\underline{\beta}$ *nor* $2\underline{\beta}$ *is in* $\mathscr{C}_M$.

(ii) *The plane* $\mathscr{P}$ *tangent to* $\mathscr{C}_L$ *at* $\underline{\beta}$ *is tangent to* $\mathscr{C}_M^0$ *at* $(0,0)$.

As systems like (48) are a subclass of systems like (1), all the results proved above remain valid. In particular, we can construct approximate solutions $(u^\varepsilon, v^\varepsilon)$ to (48) under the form

$$u^\varepsilon(x) = \mathscr{U}_a^\varepsilon\Big(\varepsilon t, \sqrt{\varepsilon} y_{II}, y_1 + \partial_1\tau(\underline{\eta})t, \frac{\underline{\beta}\cdot x}{\varepsilon}\Big),$$

$$v^\varepsilon(x) = \mathscr{V}_a^\varepsilon\Big(\varepsilon t, \sqrt{\varepsilon} y_{II}, y_1 + \partial_1\tau(\underline{\eta}), \frac{\underline{\beta}\cdot x}{\varepsilon}\Big),$$

where the profiles $\mathscr{U}_a^\varepsilon$ and $\mathscr{V}_a^\varepsilon$ are given by the formulas

$$\begin{aligned}\mathscr{U}_a^\varepsilon(T,Y,\zeta,\theta) = {}& \sqrt{\varepsilon}\big(\pi_L(\underline{\beta})\mathscr{U}_{11}e^{i\theta} + \text{c. c.}\big)\\ &+ \varepsilon^{3/2}\big(iL(\underline{\beta})^{-1}A_1\partial_\zeta\pi_L(\underline{\beta})\mathscr{U}_{11}e^{i\theta} + \text{c. c.}\big)\\ &+ \varepsilon^2\big(iL(\underline{\beta})^{-1}A_{II}(\partial_Y)\pi_L(\underline{\beta})\mathscr{U}_{11}e^{i\theta} + \text{c. c.}\big)\\ &+ \varepsilon^{5/2}\Big(-\Big(L(\underline{\beta})^{-1}\big(\partial_1\tau^L(\underline{\eta}) + A_1\big)L(\underline{\beta})^{-1}A_1\partial_\zeta^2\pi_L(\underline{\beta})\mathscr{U}_{11}\\ &\qquad + 2iL(\underline{\beta})^{-1}f\big(\mathscr{U}_{11}, \mathscr{V}_{20}\big)\Big)e^{i\theta} + \text{c. c.}\Big)\end{aligned}$$

and

$$\begin{aligned}\mathscr{V}_a^\varepsilon(T,Y,\zeta,\theta) = {}& \varepsilon\pi_M^1(0)\mathscr{V}_{20}\\ &- \varepsilon^{3/2}\sum_{j=2}^{p}\frac{1}{\partial_1\tau^L(\underline{\eta}) + v_j}\partial_\zeta^{-1}\pi_M^j(0)B_{II}(\partial_Y)\pi_M^1(0)\mathscr{V}_{20}\\ &+ \varepsilon^2 iM(0)^{-1}B_1\partial_\zeta\pi_M^1(0)\mathscr{V}_{20} - 2iM(0)^{-1}g\big(\pi(\underline{\beta})\mathscr{U}_{11}, \overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)\\ &- iM\big(2\underline{\beta}\big)^{-1}g\big(\mathscr{U}_{11}, \mathscr{U}_{11}\big)e^{2i\theta} + \text{c. c.}\end{aligned}$$

$$- \varepsilon^{5/2} i M(0)^{-1} B_1 \sum_{j=2}^{p} \frac{1}{\partial_1 \tau^L(\underline{\eta}) + v_j} \pi_M^j(0) B_{II}(\partial_Y) \pi_M^1(0) \mathscr{V}_{20}.$$

Remark 4
One can notice that $\mathscr{U}_a^\varepsilon = O(\sqrt{\varepsilon})$ while $\mathscr{V}_a^\varepsilon = O(\varepsilon)$, so that u'^ε and v'^ε defined as $u^\varepsilon = \sqrt{\varepsilon} u'^\varepsilon$ and $v^\varepsilon = \varepsilon v'^\varepsilon$ are of size $O(1)$. Instead of looking for solutions of size $(O(\sqrt{\varepsilon}), O(\varepsilon))$ to (48), we could therefore look for solutions $\mathbf{u}'^\varepsilon$ and $\mathbf{v}'^\varepsilon$ of size $O(1)$ to

$$\begin{cases} \partial_t \mathbf{u}'^\varepsilon + \sum_{j=1}^{d_1} A_j \partial_{y_j} \mathbf{u}'^\varepsilon + \frac{L_0}{\varepsilon} \mathbf{u}'^\varepsilon = \varepsilon f\big(\mathbf{u}'^\varepsilon, \mathbf{v}'^\varepsilon\big), \\ \partial_t \mathbf{v}'^\varepsilon + \sum_{j=1}^{d_2} B_j \partial_{y_j} \mathbf{v}'^\varepsilon + \frac{M_0}{\varepsilon} \mathbf{v}'^\varepsilon = g\big(\mathbf{u}'^\varepsilon, \mathbf{u}'^\varepsilon\big). \end{cases} \tag{49}$$

Such a system belongs to the general class of Maxwell-Bloch systems introduced and studied in [JMR2].

4.2. A stability result
Assumption 4 is not strong enough to allow the proof of a stability result; that is why we introduce a strong transparency condition, as in [JMR2] and [C]. This strong transparency condition is satisfied by the physical Maxwell-Bloch systems, and we also prove that if it is satisfied then the nonlinearity of the second equation of (**S**) vanishes, so that Assumption 5 can be proved.

ASSUMPTION 7 (Strong transparency condition)
There exists $C > 0$ such that for all η, η', and η'' in $\mathbb{R}^d$, all $1 \le j, k \le p_1$, and $1 \le l \le p_2$, and all $\mathbf{a}, \mathbf{b} \in \mathbb{C}^N$, one has

$$\big\| \pi_M(\beta'') g\big(\pi_L(\beta)\mathbf{a}, \pi_L(\beta')\mathbf{b}\big) \big\| \le C \|\mathbf{a}\| \|\mathbf{b}\| \big| \tau_L^j(\eta) + \tau_L^k(\eta') - \tau_M^l(\eta'') \big|,$$

where $\beta := (\tau_L^j(\eta), \eta)$, $\beta' := (\tau_L^k(\eta'), \eta')$, and $\beta'' := (\tau_M^l(\eta''), \eta'')$.

Remark 5
It is straightforward to see that Assumption 4 can be deduced from Assumption 7.

The following proposition asserts that, under Assumption 7, the nonlinearity of the second equation of (**S**) vanishes and that Assumption 5 can therefore be proved.

PROPOSITION 8
Suppose that Assumption 7 is satisfied; then

(i) *one has*

$$-\pi_M^1(0)g\big(\partial_1\pi_L(\underline{\beta})\mathscr{U}_{11},\overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)+\pi_M^1(0)A_1L(0)^{-1}g\big(\pi(\underline{\beta})\mathscr{U}_{11},\overline{\pi(\underline{\beta})\mathscr{U}_{11}}\big)=0;$$

(ii) *the system* (**S**) *reads*

$$\begin{cases}\partial_T\mathscr{U}_{11}+\dfrac{i}{2}\partial_1^2\tau(\underline{\eta})\partial_\zeta^2\mathscr{U}_{11}=2\pi_L(\underline{\beta})f\big(\pi_L(\underline{\beta})\mathscr{U}_{11},\pi_M^1(0)\mathscr{U}_{20}\big),\\ \partial_T\mathscr{U}_{20}-\pi_M^1(0)A_{II}(\partial_Y)\\ \qquad\qquad\times\displaystyle\sum_{j=2}^{p}\frac{1}{\partial_1\tau_L(\underline{\eta})+v_j}\partial_\zeta^{-1}\pi_M^j(0)A_{II}(\partial_Y)\pi_M^1(0)\mathscr{U}_{20}=0,\end{cases}$$

so that Assumption 5 is satisfied.

Proof

(i) Let α be in a neighborhood of zero in $\mathbb{R}$, and take here $\beta=(\tau_L^1(\underline{\eta}+(\alpha/2,0)),\underline{\eta}+(\alpha/2,0))$, $\beta'=(\tau_L^1(-\underline{\eta}+(\alpha/2,0)),-\underline{\eta}+(\alpha/2,0))$, and $\beta''=(\tau_M^1(\alpha,0),(\alpha,0))$. Expanding $\pi_M(\beta'')g(\pi_L(\beta')\mathbf{a},\pi(\beta)\mathbf{b})$ with respect to α near zero yields, for all $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{C}^N$,

$$\begin{aligned}\pi_M\big(\beta''\big)g\big(\pi_L\big(\beta'\big)\mathbf{a},\pi(\beta)\mathbf{b}\big)&=\pi_M^1(0)g\big(\pi_L(\underline{\beta})\mathbf{a},\pi_L(-\underline{\beta})\mathbf{b}\big)\\&\quad+\alpha\bigg[\frac{1}{2}\pi_M^1(0)g\big(\partial_1\pi_L(\underline{\beta})\mathbf{a},\pi_L(-\underline{\beta})\mathbf{b}\big)\\&\qquad-\frac{1}{2}\pi_M^1(0)g\big(\pi_L(\underline{\beta})\mathbf{a},\partial_1\pi_L(-\underline{\beta})\mathbf{b}\big)\\&\qquad+\big(\pi_M^1\big)'(0)g\big(\pi(\underline{\beta})\mathbf{a},\pi(-\underline{\beta})\mathbf{b}\big)\bigg]+o(\alpha).\end{aligned}$$

The leading term of this expansion vanishes, thanks to Assumption 4. Using the fact that $\pi(-\underline{\beta})=\overline{\pi(\underline{\beta})}$ and taking $b=\overline{a}$ therefore yields

$$\begin{aligned}\pi_M\big(\beta''\big)g\big(\pi_L(\beta')\mathbf{a},\pi(\beta)\overline{\mathbf{a}}\big)&=\alpha\Big[\pi_M^1(0)g\big(\partial_1\pi_L(\underline{\beta})\mathbf{a},\overline{\pi_L(\underline{\beta})\mathbf{a}}\big)\\&\qquad+\big(\pi_M^1\big)'(0)g\big(\pi(\underline{\beta})\mathbf{a},\overline{\pi(\underline{\beta})\mathbf{a}}\big)\Big]+o(\alpha).\end{aligned}\tag{50}$$

Now, introducing $\eta=\underline{\eta}+(0,\alpha/2)$, $\eta'=-\underline{\eta}+(0,\alpha/2)$, and $\eta''=(\alpha,0)$, and expanding $\tau_L^1(\eta)+\tau_L^1(\eta')-\tau_M^1(\eta'')$, yields

$$\tau_L^1(\eta)+\tau_L^1(\eta')-\tau_M^1(\eta'')=0+\alpha\big[\partial_1\tau_L^1(\underline{\eta})-\partial_1\tau_M^1(0)\big]+o(\alpha),$$

but, thanks to Assumption 6, we have $\partial_1\tau_L^1(\underline{\eta})=\partial_1\tau_M^1(0)$, so that

$$\tau_L^1(\eta)+\tau_L^1(\eta')-\tau_M^1(\eta'')=o(\alpha).\tag{51}$$

Thanks to Assumption 7, we know that

$$\frac{\left\|\pi_M(\beta'')g\big(\pi_L(\beta')\mathbf{a},\pi(\beta)\overline{\mathbf{a}}\big)\right\|}{\left|\tau_L^1(\eta)+\tau_L^1(\eta')-\tau_M^1(\eta'')\right|}$$

must remain bounded for all α. Equations (50)–(51) say that this is possible if and only if

$$\pi_M^1(0)g\big(\partial_1\pi_L(\underline{\beta})\mathbf{a},\overline{\pi_L(\underline{\beta})\mathbf{a}}\big)+\big(\pi_M^1\big)'(0)g\big(\pi(\underline{\beta})\mathbf{a},\overline{\pi(\underline{\beta})\mathbf{a}}\big)=0. \tag{52}$$

We now prove that this condition gives the one given in point (i) of the proposition. As in the proof of Proposition 4, we write that, for all α in a neighborhood of zero,

$$\pi_M^1(\alpha)\left(\tau_L''(\alpha)+A_1\alpha+\frac{L_0}{i}\right)=0.$$

Differentiating this equality with respect to α and taking the limit $\alpha\to 0$ yields

$$\big(\pi_M^1\big)'(0)\frac{L_0}{i}+\pi_M^1(0)\big((\tau_L^1)'(0)I+A_1\big)=0,$$

and multiplying on the right by $L(0)^{-1}$ thus gives

$$\big(\pi_M^1\big)'(0)\big(I-\pi_M(0)\big)+\pi_M^1(0)A_1L(0)^{-1}=0.$$

Therefore, one has

$$\big(\pi_M^1\big)'(0)\big(I-\pi_M(0)\big)g\big(\pi(\underline{\beta})\mathbf{a},\overline{\pi(\underline{\beta})\mathbf{a}}\big)=-\pi_M^1(0)A_1L(0)^{-1}g\big(\pi(\underline{\beta})\mathbf{a},\overline{\pi(\underline{\beta})\mathbf{a}}\big);$$

that is, since $\pi_M(0)g(\pi(\underline{\beta})\mathbf{a},\overline{\pi(\underline{\beta})\mathbf{a}})=0$,

$$\big(\pi_M^1\big)'(0)g\big(\pi(\underline{\beta})\mathbf{a},\overline{\pi(\underline{\beta})\mathbf{a}}\big)=-\pi_M^1(0)A_1L(0)^{-1}g\big(\pi(\underline{\beta})\mathbf{a},\overline{\pi(\underline{\beta})\mathbf{a}}\big). \tag{53}$$

Equations (52)–(53) then prove the desired result.

(ii) It is a straightforward consequence of point (i) that system (**S**) takes the form given in the proposition. It is also easy to prove that Assumption 5 can be proved in this case. □

We can now prove a stability result for those Maxwell-Bloch systems.

THEOREM 2

Let $\mathcal{U}_{11}^0=\pi_L(\underline{\beta})\mathcal{U}_{11}^0$ *and* $\mathcal{V}_{20}^0=\pi_M^1(0)\mathcal{V}_{20}^0$ *be in* $H^\infty(\mathbb{R}^d_{Y,\zeta})$*, and suppose Assumptions 5, 6, and 7 are satisfied.*

Then there exists $T_{\max}>0$ *such that*

(i) *for all* $0 < \underline{T} < T_{\max}$, *there exists a unique smooth exact solution* $(\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon)$, *defined on* $[0, \underline{T}/\varepsilon] \times \mathbb{R}^d$, *to problem (48) with initial conditions*

$$\mathbf{u}^\varepsilon|_{t=0}(y) = \varepsilon^{1/2}\Big(\mathscr{U}_{11}^0\big(0, \sqrt{\varepsilon}y_{II}, y_1\big)e^{i\underline{\eta}\cdot y/\varepsilon} + \text{c. c.}\Big)$$

and

$$\mathbf{v}^\varepsilon|_{t=0}(y) = \varepsilon\mathscr{V}_{20}^0\big(0, \sqrt{\varepsilon}y_{II}, y_1\big);$$

(ii) *we can write* $\mathbf{u}^\varepsilon$ *and* $\mathbf{v}^\varepsilon$ *under the form*

$$\mathbf{u}^\varepsilon(x) = \varepsilon^{1/2}\mathbf{U}^\varepsilon\left(\varepsilon t, \sqrt{\varepsilon}y_{II}, t, y_1, \frac{\underline{\beta}\cdot x}{\varepsilon}\right)$$

and

$$\mathbf{v}^\varepsilon(x) = \varepsilon\mathbf{V}^\varepsilon\left(\varepsilon t, \sqrt{\varepsilon}y_{II}, t, y_1, \frac{\underline{\beta}\cdot x}{\varepsilon}\right),$$

with $\mathbf{U}^\varepsilon$ *and* $\mathbf{V}^\varepsilon$ *bounded in* $C([0, \underline{T}]; H^s(\mathbb{R}^d \times \mathbb{T}))$, *and we have*

$$\left\|\mathbf{U}^\varepsilon - \frac{1}{\sqrt{\varepsilon}}\mathscr{U}_a^\varepsilon\right\|_{C([0,\underline{T}]; H^s(\mathbb{R}^d\times\mathbb{T}))} + \left\|\mathbf{V}^\varepsilon - \frac{1}{\varepsilon}\mathscr{V}_a^\varepsilon\right\|_{C([0,\underline{T}]; H^s(\mathbb{R}^d\times\mathbb{T}))} = o(1).$$

In particular,

$$\frac{1}{\sqrt{\varepsilon}}\left\|\mathbf{u}^\varepsilon - u^\varepsilon\right\|_{L^\infty([0,\underline{T}/\varepsilon]\times\mathbb{R}^d\times\mathbb{T})} + \frac{1}{\varepsilon}\left\|\mathbf{v}^\varepsilon - v^\varepsilon\right\|_{L^\infty([0,\underline{T}/\varepsilon]\times\mathbb{R}^d\times\mathbb{T})} = o(1).$$

Proof

Existence on a small time interval (depending on ε) is given by general theorems. It is therefore sufficient to obtain some bounds in H^s for the solution in order to prove the existence part of the theorem. Call $\mathscr{M}^\varepsilon = \mathbf{U}^\varepsilon - (1/\sqrt{\varepsilon})\mathscr{U}_a^\varepsilon$ and $\mathscr{N}^\varepsilon = \mathbf{V}^\varepsilon - (1/\varepsilon)\mathscr{V}_a^\varepsilon$. Then $\mathscr{M}^\varepsilon$ and $\mathscr{N}^\varepsilon$ satisfy

$$\begin{aligned}\left\{\partial_T + \frac{1}{\varepsilon}A_1\partial_\zeta + \frac{1}{\sqrt{\varepsilon}}A_{II}(\partial_Y) + \frac{1}{\varepsilon}\partial_1\tau(\underline{\eta})\partial_\zeta + i\frac{L\big(\underline{\beta}D_\theta\big)}{\varepsilon^2}\right\}\mathscr{M}^\varepsilon \\ = f\big(\mathscr{M}^\varepsilon, \mathscr{N}^\varepsilon\big) + f\left(\mathscr{M}^\varepsilon, \frac{1}{\varepsilon}\mathscr{V}_a^\varepsilon\right) + f\left(\frac{1}{\sqrt{\varepsilon}}\mathscr{U}_a^\varepsilon, \mathscr{N}^\varepsilon\right) + \frac{\mathscr{R}^\varepsilon}{\varepsilon^{3/2}}\end{aligned} \tag{54}$$

and

$$\begin{aligned}\left\{\partial_T + \frac{1}{\varepsilon}B_1\partial_\zeta + \frac{1}{\sqrt{\varepsilon}}B_{II}(\partial_Y) + \frac{1}{\varepsilon}\partial_1\tau(\underline{\eta})\partial_\zeta + i\frac{M\big(\underline{\beta}D_\theta\big)}{\varepsilon^2}\right\}\mathscr{N}^\varepsilon \\ = \frac{1}{\varepsilon}g\big(\mathscr{M}^\varepsilon, \mathscr{M}^\varepsilon\big) + \frac{2}{\varepsilon}g\left(\mathscr{M}^\varepsilon, \frac{1}{\varepsilon}\mathscr{U}_a^\varepsilon\right) + \frac{\mathscr{S}^\varepsilon}{\varepsilon^2},\end{aligned} \tag{55}$$

where, thanks to Proposition 7,

$$|\mathscr{R}^\varepsilon|_{L^\infty([0,\underline{T}];H^s)} = O(\varepsilon^2)$$

and

$$|\mathscr{S}^\varepsilon|_{L^\infty([0,\underline{T}];H^s)} = O(\varepsilon^2).$$

Following [JMR2] and [C], we perform the change of functions

$$\mathscr{P} = e^{-(\varepsilon A_1\partial_\zeta+\varepsilon^{3/2}A_{II}(\partial_Y)+\varepsilon\partial_1\tau(\underline{\eta})\partial_\zeta+iL(\underline{\beta}D_\theta))(T/\varepsilon^2)}\mathscr{M}^\varepsilon := S_1^\varepsilon\left(\frac{T}{\varepsilon^2}\right)\mathscr{M}^\varepsilon$$

and

$$\mathscr{Q} = e^{-(\varepsilon B_1\partial_\zeta+\varepsilon^{3/2}B_{II}(\partial_Y)+\varepsilon\partial_1\tau(\underline{\eta})\partial_\zeta+iM(\underline{\beta}D_\theta))(T/\varepsilon^2)}\mathscr{N}^\varepsilon := S_2^\varepsilon\left(\frac{T}{\varepsilon^2}\right)\mathscr{N}^\varepsilon.$$

Note that this kind of group has also been used in [S], [Gr], [BMN], and [Ga]. The equations satisfied by $\mathscr{P}$ and $\mathscr{Q}$ are written

$$\begin{aligned}\partial_T\mathscr{P} = {} & S_1^\varepsilon\left(\frac{T}{\varepsilon^2}\right) f\left(S_1^\varepsilon\left(-\frac{T}{\varepsilon^2}\right)\mathscr{P}, S_2^\varepsilon\left(-\frac{T}{\varepsilon^2}\right)\mathscr{Q}\right)\\ & + S_1^\varepsilon\left(\frac{T}{\varepsilon^2}\right) f\left(S_1^\varepsilon\left(-\frac{T}{\varepsilon^2}\right)\mathscr{P}, \frac{1}{\varepsilon}\mathscr{V}_a^\varepsilon\right)\\ & + S_1^\varepsilon\left(\frac{T}{\varepsilon^2}\right) f\left(\frac{1}{\sqrt{\varepsilon}}\mathscr{U}_a^\varepsilon, S_2^\varepsilon\left(-\frac{T}{\varepsilon^2}\right)\mathscr{Q}\right) + S_1^\varepsilon\left(\frac{T}{\varepsilon^2}\right)\frac{\mathscr{R}^\varepsilon}{\varepsilon^{3/2}}\end{aligned} \tag{56}$$

and

$$\begin{aligned}\partial_T\mathscr{Q} = {} & \frac{1}{\varepsilon}S_2^\varepsilon\left(\frac{T}{\varepsilon^2}\right) g\left(S_1^\varepsilon\left(-\frac{T}{\varepsilon^2}\right)\mathscr{P}, S_1^\varepsilon\left(-\frac{T}{\varepsilon^2}\right)\mathscr{P}\right)\\ & + \frac{2}{\varepsilon}S_2^\varepsilon\left(\frac{T}{\varepsilon^2}\right) g\left(S_1^\varepsilon\left(-\frac{T}{\varepsilon^2}\right)\mathscr{P}, \frac{1}{\sqrt{\varepsilon}}\mathscr{U}_a^\varepsilon\right) + S_2^\varepsilon\left(\frac{T}{\varepsilon^2}\right)\frac{\mathscr{S}^\varepsilon}{\varepsilon^2}.\end{aligned} \tag{57}$$

As S_1^ε and S_2^ε are unitary groups on all the Sobolev spaces H^s, we just have to find estimates on $\mathscr{P}$ and $\mathscr{Q}$ in $L^\infty_T([0,\underline{T}]; H^s(\mathbb{R}^n_{\zeta,Y}\times\mathbb{T}_\theta))$ for a $\underline{T}>0$. Denoting by $|\cdot|_{\underline{T}}$ the norm associated to this space, (56) yields

$$|\mathscr{P}|_{\underline{T}} \le C\underline{T}\big(|\mathscr{P}|_{\underline{T}}|\mathscr{Q}|_{\underline{T}} + |\mathscr{P}|_{\underline{T}} + |Q|_{\underline{T}} + O(\varepsilon^{1/2})\big) + \big|\mathscr{P}(T=0)\big|_{H^s}, \tag{58}$$

where we have used the fact that

$$\left|\frac{\mathscr{V}_a^\varepsilon}{\varepsilon}\right|_{\underline{T}} \le C, \qquad \left|\frac{\mathscr{U}_a^\varepsilon}{\sqrt{\varepsilon}}\right|_{\underline{T}} \le C, \qquad \text{and} \qquad |\mathscr{R}^\varepsilon|_{\underline{T}} = O(\varepsilon^2).$$

Moreover, one also has

$$\left|\partial_T \mathscr{P}\right|_{\underline{T}} \le C\big(|\mathscr{P}|_{\underline{T}}|\mathscr{Q}|_{\underline{T}} + |\mathscr{P}|_{\underline{T}} + |Q|_{\underline{T}} + O\big(\varepsilon^{1/2}\big)\big). \tag{59}$$

The case of (57) is more delicate. One has

$$\begin{aligned}\mathscr{Q} &= \mathscr{Q}(T=0) + \frac{1}{\varepsilon}\int_0^T S_2^\varepsilon\Big(\frac{s}{\varepsilon^2}\Big) g\Big(S_1^\varepsilon\Big(-\frac{s}{\varepsilon^2}\Big)\mathscr{P}(s), S_1^\varepsilon\Big(-\frac{s}{\varepsilon^2}\Big)\mathscr{P}(s)\Big)\, ds \\ &\quad + \frac{2}{\varepsilon}\int_0^T S_2^\varepsilon\Big(\frac{s}{\varepsilon^2}\Big) g\Big(S_1^\varepsilon\Big(-\frac{s}{\varepsilon^2}\Big)\mathscr{P}, \frac{\mathscr{U}_a^\varepsilon(s)}{\sqrt{\varepsilon}}\Big)\, ds + \int_0^T S_2^\varepsilon\Big(\frac{s}{\varepsilon^2}\Big)\frac{\mathscr{S}^\varepsilon}{\varepsilon^2}\, ds \\ &= \mathscr{Q}(T=0) + I_1 + I_2 + I_3.\end{aligned}$$

We now estimate I_1, I_2, and I_3 separately.

• *Estimate of* I_1

We use the spectralization of the groups S_1^ε and S_2^ε as follows. Denote by m, ξ_1, and ξ_{II} the Fourier dual variables of θ, ζ, and Y, respectively, and introduce $\pi_L^l(\eta) := \pi_L(\tau_L^l(\eta), \eta)$ and $\pi_M^l(\eta) := \pi_M(\tau_M^l(\eta), \eta)$.

We then have

$$\widehat{S_1^\varepsilon\Big(\frac{T}{\varepsilon^2}\Big)} = \sum_l \pi_L^l\big(\underline{\eta}m + \big(\varepsilon\xi_1, \varepsilon^{3/2}\xi_{II}\big)\big)e^{-i[m\underline{\tau}+\varepsilon\partial_1\tau(\underline{\eta})\xi_1 - \tau_L^l(\underline{\eta}m+(\varepsilon\xi_1,\varepsilon^{3/2}\xi_{II}))](T/\varepsilon^2)}$$

and

$$\widehat{S_2^\varepsilon\Big(\frac{T}{\varepsilon^2}\Big)} = \sum_l \pi_M^l\big(\underline{\eta}m + \big(\varepsilon\xi_1, \varepsilon^{3/2}\xi_{II}\big)\big)e^{i[m\underline{\tau}+\varepsilon\partial_1\tau(\underline{\eta})\xi_1 - \tau_M^l(\underline{\eta}m+(\varepsilon\xi_1,\varepsilon^{3/2}\xi_{II}))](T/\varepsilon^2)}.$$

Denoting by $\mathscr{F}(I_1)(m,\xi)$ the Fourier transform of I_1, we have therefore

$$\begin{aligned}&\mathscr{F}(I_1)(m,\xi) \\ &\quad = \frac{1}{\varepsilon}\int_0^T \sum_p\sum_l\sum_{l'}\sum_{l''}\int \pi_M^l\big(\underline{\eta}m + (\varepsilon\xi_1, \varepsilon^{3/2}\xi_{II})\big) \\ &\quad\quad \times e^{-i[m\underline{\tau}+\varepsilon\partial_1\tau(\underline{\eta})\xi_1-\tau_M^l(\underline{\eta}m+(\varepsilon\xi_1,\varepsilon^{3/2}\xi_{II}))](s/\varepsilon^2)} \\ &\quad\quad \times g\Big(\pi_L^{l'}\big(\underline{\eta}p + \big(\varepsilon\eta_1, \varepsilon^{3/2}\eta_{II}\big)\big) \\ &\quad\quad\quad \times e^{i[p\underline{\tau}+\varepsilon\partial_1\tau(\underline{\eta})\eta_1-\tau_L^{l'}(\underline{\eta}p+(\varepsilon\eta_1,\varepsilon^{3/2}\eta_{II}))](s/\varepsilon^2)}\widehat{\mathscr{P}}_p(\eta), \\ &\quad\quad\quad \pi_L^{l''}\big(\underline{\eta}(m-p) + \big(\varepsilon(\xi_1-\eta_1), \varepsilon^{3/2}\big(\xi_{II}-\eta_{II}\big)\big)\big) \\ &\quad\quad\quad \times e^{i[(m-p)\underline{\tau}+\varepsilon\partial_1\tau(\underline{\eta})(\xi_1-\eta_1)-\tau_L^{l''}(\underline{\eta}(m-p)+(\varepsilon(\xi_1-\eta_1),\varepsilon^{3/2}(\xi_{II}-\eta_{II})))](s/\varepsilon^2)} \\ &\quad\quad\quad \times \widehat{\mathscr{P}}_{m-p}(\xi-\eta)\Big)\, d\eta\, ds,\end{aligned}$$

and thus

$$\begin{aligned}
&\mathscr{F}(I_1)(m,\xi)\\
&=\frac{1}{\varepsilon}\int_0^T\sum_P\sum_l\sum_{l'}\sum_{l''}\\
&\quad e^{i[\tau_M^l(\underline{\eta}m+(\varepsilon\xi_1,\varepsilon^{3/2}\xi_{II}))-\tau_L^{l'}(\underline{\eta}p+(\varepsilon\eta_1,\varepsilon^{3/2}\eta_{II}))-\tau_L^{l''}(\underline{\eta}(m-p)+(\varepsilon(\xi_1-\eta_1),\varepsilon^{3/2}(\xi_{II}-\eta_{II}))))](s/\varepsilon^2)}\\
&\quad\times\pi_M^l\big(\underline{\eta}m+\big(\varepsilon\xi_1,\varepsilon^{3/2}\xi_{II}\big)\big)g\Big(\pi_L^{l'}\big(\underline{\eta}p+\big(\varepsilon\eta_1,\varepsilon^{3/2}\eta_{II}\big)\big)\widehat{\mathscr{P}}_p(\eta),\\
&\qquad\qquad\pi_L^{l''}\big(\underline{\eta}(m-p)+\big(\varepsilon\big(\xi_1-\eta_1\big),\\
&\qquad\qquad\quad\varepsilon^{3/2}\big(\xi_{II}-\eta_{II}\big)\big)\big)\widehat{\mathscr{P}}_{m-p}(\xi-\eta)\Big)\,d\eta\,ds.
\end{aligned}$$

Integrating by parts and using Assumption 7 yields

$$\begin{aligned}
\big|\mathscr{F}(I_1)(m,\xi)\big|\le C\varepsilon\int_0^T\sum_p\int\big|\widehat{\mathscr{P}}_p(\eta)\big|\big|\partial_T\widehat{\mathscr{P}}_{m-p}(\xi-\eta)\big|\,d\eta\,ds\\
+C\varepsilon\sum_p\int\big|\widehat{\mathscr{P}}_p(\eta)\big|\big|\widehat{\mathscr{P}}_{m-p}(\xi-\eta)\big|\,d\eta(T)\\
+C\varepsilon\sum_p\int\big|\widehat{\mathscr{P}}_p(\eta)\big|\big|\widehat{\mathscr{P}}_{m-p}(\xi-\eta)\big|\,d\eta(0).
\end{aligned}$$

It follows that

$$|I_1|_{\underline{T}}\le C\varepsilon\underline{T}|\mathscr{P}|_{\underline{T}}|\partial_T\mathscr{P}\big|_{\underline{T}}+C\varepsilon\Big(|\mathscr{P}|_{\underline{T}}^2+\big|\mathscr{P}(T=0)\big|_{H^s}^2\Big).\tag{60}$$

• *Estimate of I_2*

Recall that

$$\begin{aligned}
\mathscr{U}_a^\varepsilon(T,Y,\zeta,\theta)=\sqrt{\varepsilon}\big(\pi_L(\underline{\beta})\mathscr{U}_{11}e^{i\theta}+\text{c.c.}\big)\\
+\varepsilon^{3/2}\big(iL(\underline{\beta})^{-1}A_1\partial_\zeta\pi_L(\underline{\beta})\mathscr{U}_{11}e^{i\theta}+\text{c.c.}\big)\\
+\varepsilon^2\big(iL(\underline{\beta})^{-1}A_{II}(\partial_Y)\pi_L(\underline{\beta})\mathscr{U}_{11}e^{i\theta}+\text{c.c.}\big)+O\big(\varepsilon^{5/2}\big).
\end{aligned}$$

As in [C], introduce

$$\widetilde{\mathscr{U}}_{11}^\varepsilon=\mathbf{1}_{\{|(\partial_\zeta,\partial_Y)|\le 1/\sqrt{\varepsilon}\}}\mathscr{U}_{11}.$$

The following lemma is a direct consequence of the decreasing properties associated to the regularity of $\mathscr{U}_{11}$.

LEMMA 5

The difference between $\mathscr{U}_{11}$ and $\widetilde{\mathscr{U}}_{11}^\varepsilon$ is controlled by

$$\big|\widetilde{\mathscr{U}}_{11}^\varepsilon-\mathscr{U}_{11}\big|_{\underline{T}}\le C\varepsilon^2\big|\mathscr{U}_{11}\big|_{L^\infty(0,\underline{T};H^{s+4})}.$$

As $\underline{\beta}$ is a smooth point of $\mathscr{C}_L$, there exists a local parametrization $\eta \mapsto \tau_L^{l_0}(\eta)$ defined on a neighborhood of $\underline{\eta}$ such that $\tau_L^{l_0}(\underline{\eta}) = \underline{\tau}$. We denote by $\pi_L^{l_0}(\eta)$ the associated spectral projector. Thanks to [C], we know that, for all j,

$$iL(\beta)^{-1}A_j\pi_L^{l_0}(\eta) = -i\partial_j\pi_L^{l_0}(\eta),$$

so that

$$\frac{1}{\sqrt{\varepsilon}}\mathscr{U}_a^\varepsilon = \pi_L^{l_0}\big(\underline{\eta} + \big(\varepsilon\partial_\zeta, \varepsilon^{3/2}\partial_Y\big)\big)\widetilde{\mathscr{U}}_{11}^\varepsilon e^{i\theta} + \text{c. c.} + \varepsilon^2\mathscr{T}_\varepsilon$$

with

$$|\mathscr{T}_\varepsilon|_{\underline{T}} \le C$$

since $\pi_L^{l_0}(\xi)$ is smooth near $\xi = \underline{\eta}$ and since the spectrum of $\widetilde{\mathscr{U}}_{11}^\varepsilon$ is included in $|\xi| \le 1/\sqrt{\varepsilon}$.

We can therefore write

$$\begin{aligned} I_2 &= \frac{2}{\varepsilon}\int_0^T S_2^\varepsilon\Big(\frac{s}{\varepsilon^2}\Big) g\Big(S_1^\varepsilon\Big(-\frac{s}{\varepsilon^2}\Big)\mathscr{P}(s), \pi_L^{l_0}\big(\underline{\eta} + \big(\varepsilon\partial_\zeta, \varepsilon^{3/2}\partial_Y\big)\big)\widetilde{\mathscr{U}}_{11}^\varepsilon + \text{c. c.}\Big)\,ds \\ &\quad + \sqrt{\varepsilon}\int_0^T S_2^\varepsilon\Big(\frac{s}{\varepsilon^2}\Big) g\Big(S_1^\varepsilon\Big(-\frac{s}{\varepsilon^2}\Big)\mathscr{P}(s), \mathscr{T}_\varepsilon\Big)\,ds \\ &= I_{21} + I_{22}. \end{aligned}$$

It is clear that

$$|I_{22}|_{\underline{T}} \le C\underline{T}\sqrt{\varepsilon}|\mathscr{P}|_{\underline{T}}. \tag{61}$$

Now remark that

$$\tau_L^{l_0}\big(\underline{\eta} + \varepsilon\big(\xi_1, \sqrt{\varepsilon}\xi_{II}\big)\big) = \tau_L^{l_0}(\underline{\eta}) + \varepsilon\xi_1\partial_1\tau(\underline{\eta}) + O\big(\varepsilon^2\big)$$

since $\partial_{II}\tau(\underline{\eta}) = 0$. Defining

$$\widehat{\mathscr{W}}_{11}^\varepsilon = e^{i[\tau_L^{l_0}(\underline{\eta}+\varepsilon(\xi_1,\sqrt{\varepsilon}\xi_{II})) - \tau_L^{l_0}(\underline{\eta}) - \varepsilon\xi_1\partial_1\tau(\underline{\eta})](T/\varepsilon^2)}\pi_L^{l_0}\big(\underline{\eta} + \big(\varepsilon\xi_1, \varepsilon^{3/2}\xi_{II}\big)\big)\widehat{\widetilde{\mathscr{U}}_{11}^\varepsilon},$$

we thus obtain

$$\big|\mathscr{W}_{11}^\varepsilon\big|_{\underline{T}} \le C \qquad \text{and} \qquad \big|\partial_T\mathscr{W}_{11}^\varepsilon\big|_{\underline{T}} \le C. \tag{62}$$

We can write

$$\pi_L^{l_0}\big(\underline{\eta} + \big(\varepsilon\partial_\zeta, \varepsilon^{3/2}\partial_Y\big)\big)\widetilde{\mathscr{U}}_{11}^\varepsilon = S_1^\varepsilon\Big(-\frac{T}{\varepsilon^2}\Big)\mathscr{W}_{11}^\varepsilon$$

and

$$I_{21} = \frac{2}{\varepsilon}\int_0^T S_2^\varepsilon\Big(\frac{s}{\varepsilon^2}\Big) g\Big(S_1^\varepsilon\Big(-\frac{s}{\varepsilon^2}\Big)\mathscr{P}(s), S_1^\varepsilon\Big(-\frac{s}{\varepsilon^2}\Big)\mathscr{W}_{11}^\varepsilon\Big)\,ds.$$

It follows that I_{21} has the same form as I_1, and an integration by parts yields, using (62),

$$|I_{21}|_{\underline{T}} \leq C\varepsilon \underline{T}\big(\big|\partial_T \mathscr{P}\big|_{\underline{T}} + |\mathscr{P}|_{\underline{T}}\big) + C\varepsilon\big(|\mathscr{P}|_{\underline{T}} + \big|\mathscr{P}(T=0)\big|_{H^s}\big). \tag{63}$$

It follows from (61) and (63) that

$$|I_2|_{\underline{T}} \leq C\underline{T}\sqrt{\varepsilon}\big(\big|\partial_T \mathscr{P}\big|_{\underline{T}} + |\mathscr{P}|_{\underline{T}}\big) + C\varepsilon\big(|\mathscr{P}|_{\underline{T}} + 1\big). \tag{64}$$

• *Estimate of* I_3

We first recall that, thanks to Proposition 7, we have $\mathscr{S}^\varepsilon/\varepsilon^2 = \mathscr{S}_1^\varepsilon + O(\varepsilon)$ and

$$\mathscr{S}_1^\varepsilon = \mathscr{S}_{10}^\varepsilon(T,\zeta,Y) + \big(\mathscr{S}_{12}^\varepsilon e^{2i\theta} + \text{c.c.}\big),$$

as well as $\pi_M^1(0)\mathscr{S}_{10}^\varepsilon = 0$, thanks to Proposition 7(ii).

We now introduce the notation

$$I_3^j = \int_0^{\underline{T}} S_2^\varepsilon\left(\frac{s}{\varepsilon^2}\right)\mathscr{S}_{1j}^\varepsilon e^{ij\theta}\,ds.$$

As $\mathscr{S}_{1j}^\varepsilon$ is smooth enough, we have

$$\big|\mathbf{1}_{|(\partial_\zeta,\partial_Y)|\geq 1/\sqrt{\varepsilon}}\mathscr{S}_{1j}^\varepsilon\big|_{\underline{T}} \leq C\varepsilon,$$

and thus

$$\big|I_3^j\big|_{\underline{T}} \leq \left|\int_0^{\underline{T}} S_2^\varepsilon\left(\frac{s}{\varepsilon^2}\right)\mathbf{1}_{|(\partial_\zeta,\partial_Y)|\leq 1/\sqrt{\varepsilon}}\mathscr{S}_{1j}^\varepsilon e^{ij\theta}\,ds\right|_{\underline{T}} + C\varepsilon.$$

For $j = 2$, since $2\underline{\beta}$ is not in the characteristic variety of M, an integration by parts yields

$$\big|I_3^2\big|_{\underline{T}} \leq C\varepsilon^2 + C\varepsilon \leq C\varepsilon. \tag{65}$$

For $j = 0$, one has

$$\begin{aligned}\mathscr{F}\big(I_3^0\big) &= \int_0^T \sum_l e^{i(\varepsilon\partial_1\tau(\underline{\eta})\xi_1 - \tau_M^l(\varepsilon(\xi_1,\sqrt{\varepsilon}\xi_{II})))(s/\varepsilon^2)}\pi_M^l\big(\varepsilon\big(\xi_1,\sqrt{\varepsilon}\xi_{II}\big)\big)\\ &\quad\times \mathbf{1}_{\{|(\partial_\zeta,\partial_Y)|\leq 1/\sqrt{\varepsilon}\}}\widehat{\mathscr{S}}_{10}^\varepsilon(s,\xi)\,ds + O(\varepsilon)\\ &:= \sum_l \mathscr{F}\big(I_3^0(l)\big) + O(\varepsilon).\end{aligned}$$

We may encounter three cases.

(i) We have $\tau_M^l(\varepsilon(\xi_1,\sqrt{\varepsilon}\xi_{II})) \to \tau_M^l(0) \neq 0$ when ε tends towards zero. In this case, an integration by parts yields

$$\left|\sum_{l,\tau_M^l(0)\neq 0} I_3^0(l)\right|_{\underline{T}} \leq C\varepsilon^2.$$

(ii) We have $\tau_M^l(\varepsilon(\xi_1,\sqrt{\varepsilon}\xi_{II})) \sim \varepsilon\partial_1\tau_M^l(0)\xi_1$ and $\partial_1\tau_M^l(0) \neq \partial_1\tau(\underline{\eta})$. In this case, the phase does not vanish except in a neighborhood of zero, and a standard argument yields

$$\left| \sum_{l, \partial_1 \tau_M^l(0) \neq \partial_1 \tau(\underline{\eta})} I_3^0(l) \right|_{\underline{T}} = o(1).$$

(iii) We have $\tau_M^l(\varepsilon(\xi_1, \sqrt{\varepsilon}\xi_{II})) \sim \varepsilon \partial_1 \tau(\underline{\eta})\xi_1$. In this case, we cannot expect anything from the phase; however, we have the following lemma.

LEMMA 6
If $\partial \tau_M^l(0) = \partial_1 \tau(\underline{\eta})$*, then we have*

$$\lim_{\varepsilon \to 0} \pi_M^l\big(\varepsilon\big(\xi_1, \sqrt{\varepsilon}\xi_{II}\big)\big)\big(1 - \pi_M^1(0)\big) = 0.$$

Proof
First recall that $\pi_M^1(0)$ is the spectral projector of $\pi_M(0)B_1\pi_M(0)$ associated to the eigenvalue $-\partial_1 \tau(\underline{\eta})$.

The mapping

$$\varepsilon \longmapsto \pi_M^l\big(\varepsilon\big(\xi_1, \sqrt{\varepsilon}\xi_{II}\big)\big)$$

is analytical and bounded for ε small enough and $\varepsilon \neq 0$. Thanks to [K], we can therefore extend this function analytically to zero. We denote by $\underline{\pi_M^l(0)}$ the value of this extension.

By definition of $\pi_M^l(\varepsilon(\xi_1, \sqrt{\varepsilon}\xi_{II}))$, one has

$$\left(\tau_M^l\big(\varepsilon\big(\xi_1, \xi_{II}\big)\big) + \varepsilon B_1 \xi_1 + \varepsilon^{3/2} B_{II}(\xi_{II}) + \frac{M_0}{i}\right)\pi_M^l\big(\varepsilon\big(\xi_1, \sqrt{\varepsilon}\xi_{II}\big)\big) = 0. \tag{66}$$

Multiplying this expression on the left by $\pi_M(0)$, dividing it by ε, and finally taking the limit when $\varepsilon \to 0$ yields

$$\pi_M(0)\big(\xi_1 \partial_1 \tau_M^l(0) + B_1 \xi_1\big)\underline{\pi_M^l(0)} = 0.$$

As we have $\partial_1 \tau_M^l(0) = \partial_1 \tau(\underline{\eta})$, we can conclude that

$$\pi_M(0) B_1 \underline{\pi_M^l(0)} = -\partial_1 \tau(\underline{\eta}) \pi_M(0) \underline{\pi_M^l(0)}.$$

We just have to prove that the range of $\underline{\pi_M^l(0)}$ is contained in the range of $\pi_M(0)$ to complete the proof.

But taking the limit when $\varepsilon \to 0$ in (66) yields $L_0 \underline{\pi_M^l(0)} = 0$, which proves the desired result. □

It follows that $I_3^0 \to 0$ when $\varepsilon \to 0$, and we thus obtain

$$\lim_{\varepsilon \to 0} I_3 = 0. \tag{67}$$

It follows from (60), (64), and (67) that

$$
\begin{aligned}
|\mathcal{Q}|_{\underline{T}} \leq \big|\mathcal{Q}(T=0)\big|_{H^s} &+ C\varepsilon\underline{T}|\mathcal{P}|_{\underline{T}}\big|\partial_T\mathcal{P}\big|_{\underline{T}} + C\varepsilon\big(|\mathcal{P}|_{\underline{T}}^2 + \big|\mathcal{P}(T=0)\big|_{H^s}\big) \\
&+ C\sqrt{\varepsilon}\underline{T}\big(|\partial_T\mathcal{P}|_{\underline{T}} + |\mathcal{P}|_{\underline{T}}\big) + C\varepsilon\big(|P|_{\underline{T}} + 1\big) + o(1).
\end{aligned} \tag{68}
$$

Thanks to (58), (59), and (68), we can end the proof of the theorem as in [C]. □

5. The 1-dimensional case

We consider in this section 1-dimensional problems that belong to the general class (1). They read

$$\big(\partial_t + A_1\partial_1\big)\mathbf{u}^\varepsilon + \frac{L_0}{\varepsilon}\mathbf{u}^\varepsilon = f\big(\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon\big). \tag{69}$$

As said in the introduction, one seeks in this case approximate solutions to this system under the form

$$u^\varepsilon(x) = \sqrt{\varepsilon}\,\mathcal{U}\left(\varepsilon, \varepsilon t, t, y_1, \frac{\underline{\beta}\cdot x}{\varepsilon}\right),$$

with

$$\mathcal{U}\big(\varepsilon, T, t, y_1, \theta\big) := \big(\mathcal{U}_1 + \sqrt{\varepsilon}\,\mathcal{U}_2 + \varepsilon\mathcal{U}_3 + \varepsilon^{3/2}\mathcal{U}_4 + \varepsilon^2\mathcal{U}_5\big)\big(\varepsilon, T, t, y_1, \theta\big).$$

We have also seen that the long-wave short-wave resonance condition reduces in this case to the usual rectification condition.

The study of this 1-dimensional case can easily be deduced from the multidimensional study made in the previous sections.

The following theorem gives the evolution equations that the leading terms of the ansatz must satisfy in order for u^ε to be a good approximation of the exact solution $\mathbf{u}^\varepsilon$.

THEOREM 3

Suppose that u^ε, given by

$$u^\varepsilon(x) = \sqrt{\varepsilon}\,\mathcal{U}\left(\varepsilon, \varepsilon t, t, y_1, \frac{\underline{\beta}\cdot x}{\varepsilon}\right)$$

with

$$\mathcal{U} := \mathcal{U}_1 + \sqrt{\varepsilon}\,\mathcal{U}_2 + \varepsilon\mathcal{U}_3 + \varepsilon^{3/2}\mathcal{U}_4 + \varepsilon^2\mathcal{U}_5,$$

is the approximate solution to (69) given by geometric optics.

If $\mathcal{U}_1 = \mathcal{U}_{11}e^{i\theta} + \mathrm{c.\,c.}$ and $\mathcal{U}_2 = \mathcal{U}_{20} + \mathcal{U}_{21}e^{i\theta} + \mathrm{c.\,c.}$, then one has

$$\pi(\underline{\beta})\mathcal{U}_{11} = \mathcal{U}_{11}, \qquad \mathcal{U}_{21} = 0, \qquad \textit{and} \qquad \pi(0)\mathcal{U}_{20} = \mathcal{U}_{20}.$$

Moreover, $\pi(\underline{\beta})\mathcal{U}_{11}$ and $\pi^1(0)\mathcal{U}_{20} = \langle\pi(0)\mathcal{U}_{20}\rangle$ are transported at the group velocity $-\partial_1\tau(\underline{\eta})$, that is,

$$\big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi(\underline{\beta})\mathcal{U}_{11} = \big(\partial_t - \partial_1\tau(\underline{\eta})\partial_{y_1}\big)\pi^1(0)\mathcal{U}_{20} = 0,$$

and must also satisfy

$$\partial_T \pi(\underline{\beta})\mathcal{U}_{11} + \frac{i}{2}\partial_1^2 \tau(\underline{\eta})\partial_{y_1}^2 \pi(\underline{\beta})\mathcal{U}_{11} = 2\pi(\underline{\beta}) f\big(\pi(\underline{\beta})\mathcal{U}_{11}, \pi^1(0)\mathcal{U}_{20}\big) \tag{70}$$

and

$$\begin{aligned}\partial_T \pi^1(0)\mathcal{U}_{20} = {} & -2i\,\partial_{y_1}\pi^1(0) f\big(\partial_1 \pi(\underline{\beta})\mathcal{U}_{11}, \overline{\pi(\underline{\beta})\mathcal{U}_{11}}\big) \\ & + 2i\pi^1(0)A_1\partial_{y_1}L(0)^{-1}\Re\Big(f\big(\pi(\underline{\beta})\mathcal{U}_{11}, \overline{\pi(\underline{\beta})\mathcal{U}_{11}}\big)\Big).\end{aligned} \tag{71}$$

The system that $\pi(\underline{\beta})\mathcal{U}_{11}$ and $\pi^1(0)\mathcal{U}_{20}$ must solve is simpler than the system (**S**) found in the multidimensional case since the dispersive term ∂_ξ^{-1} disappears. The system found here can be solved, so that we do not need to do an assumption like Assumption 5.

Since the dependence of $\pi(\underline{\beta})\mathcal{U}_{11}$ and $\pi^1(0)\mathcal{U}_{20}$ on t and y_1 is made through $\zeta := y_1 + t\partial_1\tau(\underline{\eta})$, we can write $\pi(\underline{\beta})\mathcal{U}_{11}(T, t, y_1)$ and $\pi^1(0)\mathcal{U}_{20}(T, t, y_1)$ under the forms $\pi(\underline{\beta})\mathcal{U}_{11}(T, \zeta)$ and $\pi^1(0)\mathcal{U}_{20}(T, \zeta)$. We then have the following theorem.

THEOREM 4
Let $\mathcal{U}_{11}^0 = \pi(\underline{\beta})\mathcal{U}_{11}^0$ and $\mathcal{U}_{20}^0 = \pi^1(0)\mathcal{U}_{20}^0$ be in $H^s(\mathbb{R}_\zeta)$ for $s \geq 0$.

There exists a $\underline{T} > 0$ and a unique couple of profiles $\mathcal{U}_{11}, \mathcal{U}_{20} \in C([0, \underline{T}]; H^s(\mathbb{R}_\zeta))$ satisfying

$$(\mathbf{S_1}) \begin{cases} \partial_T \pi(\underline{\beta})\mathcal{U}_{11} + \frac{i}{2}\partial_1^2\tau(\underline{\eta})\partial_\zeta^2 \pi(\underline{\beta})\mathcal{U}_{11} = 2\pi(\underline{\beta}) f\big(\pi(\underline{\beta})\mathcal{U}_{11}, \pi^1(0)\mathcal{U}_{20}\big), \\ \partial_T \pi^1(0)\mathcal{U}_{20} = -2i\,\partial_\zeta \pi^1(0) f\big(\partial_1\pi(\underline{\beta})\mathcal{U}_{11}, \overline{\pi(\underline{\beta})\mathcal{U}_{11}}\big) \\ \qquad\qquad + 2i\pi^1(0)A_1 L(0)^{-1}\partial_\zeta \Re\Big(f\big(\pi(\underline{\beta})\mathcal{U}_{11}, \overline{\pi(\underline{\beta})\mathcal{U}_{11}}\big)\Big), \end{cases}$$

together with the polarization conditions

$$\mathcal{U}_{11} = \pi(\underline{\beta})\mathcal{U}_{11} \qquad \text{and} \qquad \mathcal{U}_{20} = \pi^1(0)\mathcal{U}_{20},$$

and with the initial conditions

$$\mathcal{U}_{11}\big|_{T=0} = \mathcal{U}_{11}^0 \qquad \text{and} \qquad \mathcal{U}_{20}\big|_{T=0} = \mathcal{U}_{20}^0.$$

Proof
The theorem is proved if we can have an existence/uniqueness result for a general system writing

$$\begin{cases} \partial_T u + i\lambda\partial_\zeta^2 u = f_1(u, v), \\ \partial_T v = \partial_\zeta f_2(u, \overline{u}), \end{cases} \tag{72}$$

where $\lambda \in \mathbb{R}\backslash\{0\}$, f_1 and f_2 are two bilinear mappings and u and v are vector-valued functions defined on $[0, \underline{T}] \times \mathbb{R}_\zeta$. This system is completed with the initial conditions

$$u(T = 0) = u_0 \in H^2(\mathbb{R}) \qquad \text{and} \qquad v(T = 0) = v_0 \in H^1(\mathbb{R}).$$

A direct proof using Picard iterates cannot yield the result for a system like (72) since we must deal with the loss of a derivative because of the term ∂_ζ in front of the second member of the equation. In order to overcome this difficulty, we use a technique introduced in [OT] for the Zakharov equations. We thus introduce the system

$$\begin{cases} \partial_T w + i\lambda\partial_\zeta^2 w = f_1(w,v) + f_1\Big(u_0 + \int_0^T w, \partial_T v\Big), \\ \partial_T v = \partial_\zeta f_2(u,\overline{u}), \\ \big(\partial_\zeta^2 - 1\big)u = \frac{i}{\lambda} w - u_0 - \int_0^T w - \frac{1}{\lambda} f_1\Big(u_0 + \int_0^T w, v\Big), \end{cases} \tag{73}$$

together with the initial conditions

$$v(T=0) = v_0 \qquad \text{and} \qquad w(T=0) = -i\lambda u_0'' + f_1(u_0, v_0) \in L^2(\mathbb{R}).$$

This system is formally obtained by differentiating the first equation in (72) with respect to T and introducing $w = \partial_T u$. The problem due to the loss of derivatives has disappeared from this new formulation.

The third equation in (73) gives u in terms of v and w, thanks to an elliptical inversion. Using the expression of u thus found, the first two equations of (73) are written in terms of v and w. It is easy to show, using classical Picard iterates, that this system of two equations on v and w admits a unique solution $(v,w) \in C([0,\underline{T}]; H^1(\mathbb{R}) \times L^2(\mathbb{R}))$, for a $\underline{T} > 0$, and satisfying $(v,w)(T=0) = (v_0, w_0)$.

Once v and w are known, we can find u, thanks to the formula

$$u = \big(\partial_\zeta^2 - 1\big)^{-1}\Big(\frac{i}{\lambda} w - u_0 - \int_0^T w - \frac{1}{\lambda} f_1\Big(u_0 + \int_0^T w, v\Big)\Big).$$

The system (73) thus admits a unique solution $(u,v,w) \in C([0,\underline{T}]; H^2(\mathbb{R}) \times H^1(\mathbb{R}) \times L^2(\mathbb{R}))$ such that $(v,w)(T=0) = (v_0, w_0)$. The proof of the theorem will therefore be complete once we have proved that $u \in C^1([0,\underline{T}]; L^2(\mathbb{R}))$ with $\partial_T u = w$, and that $u(T=0) = u_0$.

Differentiating the third equation in (73) with respect to T, one gets

$$\big(\partial_\zeta^2 - 1\big)\partial_T u = \frac{i}{\lambda}\partial_T w - w - \frac{1}{\lambda}\partial_T f_1\Big(u_0 + \int_0^T w, v\Big). \tag{74}$$

But thanks to the first equation of (73), one has

$$\big(\partial_\zeta^2 - 1\big) w = \frac{i}{\lambda}\partial_T w - w - \frac{1}{\lambda}\partial_T f_1\Big(u_0 + \int_0^T w, v\Big),$$

so that we can conclude that $\partial_T u(T) = w(T)$ in $H^{-2}(\mathbb{R})$. But it is easy to see, thanks to (74), that $\partial_T u$ is in $C([0,\underline{T}]; L^2(\mathbb{R}))$, so that $u \in C^1([0,\underline{T}]; L^2(\mathbb{R}))$.

Using the third equation of (73) and the initial conditions associated to this system, one gets $u(T = 0) = u_0$, and the proof of the theorem is thus complete. □

Theorem 4 gives the leading oscillating term and the leading nonoscillating term of the ansatz. As done previously for the multidimensional case, we can determine completely our ansatz thanks to these two profiles. Here again, a stability property for the approximate solution u^ε can be proved, but only in the case of systems of the form (48).

6. About Proposition 1 and Assumption 5

6.1. Proof of Proposition 1

More precisely, we prove the following proposition.

PROPOSITION 9

Suppose that Assumptions 1 and 2 are satisfied, and assume that $\tau'(\underline{\eta}) \cdot \eta^0 \neq 0$.

Then there exists a problem $\widetilde{(1)}$ in one-to-one correspondence with problem (1) and for which the contact direction and the group speed are colinear.

Proof

We can always suppose that β^0 as defined in the introduction is of the form $\beta^0 = (1, \eta_1^0, 0, \dots, 0)$.

Let $P = (p_{jk})$ be an invertible matrix; to any function $u(t, y)$ we associate the function $\widetilde{u}$ defined as

$$\widetilde{u}(t, y) := u\big(t, P^{-1} y\big).$$

Then, if u solves (1), that is, if

$$L^\varepsilon(\partial_x)u + f(u, u) = 0,$$

then $\widetilde{u}$ solves $\widetilde{(1)}$,

$$\widetilde{L}^\varepsilon(\partial_x)\widetilde{u} + f(\widetilde{u}, \widetilde{u}) = 0,$$

where

$$\widetilde{L}^\varepsilon(\partial_x) := \partial_T + \sum_{j=1}^{d} \widetilde{A}_j \partial_j + \frac{L_0}{\varepsilon} \qquad \text{and} \qquad \widetilde{A}_j := \sum_{k=1}^{d} p_{jk} A_k.$$

We also introduce the operators $\widetilde{L}(\beta)$ and $\widetilde{\pi}(\beta)$ which are linked to $\widetilde{(1)}$ and whose definition is straightforward. To $\underline{\beta}$ we also associate $\underline{\widetilde{\beta}}$ defined as $\underline{\widetilde{\beta}} := (\underline{\tau}, (P^{-1})^T \underline{\eta})$.

(a) We prove here that $\widetilde{\pi}(\underline{\widetilde{\beta}}) = \pi(\underline{\beta})$. Indeed, one has

$$\begin{aligned}\widetilde{L}(\underline{\widetilde{\beta}}) &= \underline{\tau} + \sum_{j=1}^{d} \widetilde{A}_j \underline{\widetilde{\eta}}_j + \frac{L_0}{i} \\ &= \underline{\tau} + \sum_{j=1}^{d} \left(\sum_{k=1}^{d} p_{jk} A_k \right) \underline{\widetilde{\eta}}_j + \frac{L_0}{i} \\ &= \underline{\tau} + \sum_{k=1}^{d} \left(\sum_{j=1}^{d} p_{jk} A_k \right) \underline{\widetilde{\eta}}_j + \frac{L_0}{i} \\ &= \underline{\tau} + \sum_{k=1}^{d} \left(P e_k \cdot \underline{\widetilde{\eta}} \right) A_k + \frac{L_0}{i},\end{aligned}$$

where $(e_1, \ldots, e_d)$ denotes the canonical basis of $\mathbb{R}^d$.

Since $\underline{\widetilde{\eta}} = (P^{-1})^T \underline{\eta}$, one has $P e_k \cdot \underline{\widetilde{\eta}} = \underline{\eta}_k$, and therefore

$$\widetilde{L}(\underline{\widetilde{\beta}}) = \underline{\tau} + \sum_{j=1}^{d} A_j \underline{\eta}_j + \frac{L_0}{i};$$

that is, $\widetilde{L}(\underline{\widetilde{\beta}}) = L(\underline{\beta})$. The kernels of these matrices are therefore the same, and thus we have $\widetilde{\pi}(\underline{\widetilde{\beta}}) = \pi(\underline{\beta})$.

(b) Denoting by $\widetilde{\tau}(\eta)$ a parametrization of $\mathscr{C}_{\widetilde{L}}$, we now prove that $\widetilde{\tau}'(\underline{\widetilde{\beta}}) = (\partial_1 \tau(\underline{\beta}), 0, \ldots, 0)$.

We know that

$$\widetilde{\pi}(\underline{\widetilde{\beta}}) \widetilde{A}_j \widetilde{\pi}(\underline{\widetilde{\beta}}) = -\partial_j \widetilde{\tau}(\underline{\widetilde{\beta}}) \widetilde{\pi}(\underline{\widetilde{\beta}}),$$

which, thanks to the result of (a), reads

$$\pi(\underline{\beta}) \widetilde{A}_j \pi(\underline{\beta}) = -\partial_j \widetilde{\tau}(\underline{\widetilde{\beta}}) \pi(\underline{\beta}).$$

We now say which matrix P we take. Denoting by l_j its line vectors, we take $l_1 = e_1$, and for $(l_2, \ldots, l_d)$ we take any basis of the orthogonal hyperplane to $\tau'(\underline{\beta})$. Since we have supposed that $\tau'(\underline{\beta}) \cdot \eta^0 \neq 0$, that is, that $\tau'(\underline{\beta}) \cdot e_1 \neq 0$, P is invertible.

We then have $\pi(\underline{\beta}) \widetilde{A}_1 \pi(\underline{\beta}) = \pi(\underline{\beta}) A_1 \pi(\underline{\beta})$, and since this last quantity is equal to $-\partial_1 \tau(\underline{\beta}) \pi(\underline{\beta})$, we can conclude that $\partial_1 \widetilde{\tau}(\underline{\widetilde{\beta}}) = \partial_1 \tau(\underline{\beta})$. When $j \geq 2$, one has

$$\pi(\underline{\beta}) \widetilde{A}_j \pi(\underline{\beta}) = \sum_{k=1}^{d} p_{jk} \pi(\underline{\beta}) A_k \pi(\underline{\beta}) = -\sum_{k=1}^{d} p_{jk} \partial_k \tau(\underline{\beta}) = -l_j \cdot \tau'(\underline{\beta}) = 0$$

since $(l_j)_{j \geq 2}$ is a basis of the orthogonal hyperplane to $\tau'(\underline{\beta})$.

We therefore have $\partial_j \widetilde{\tau}(\underline{\widetilde{\beta}}) = 0$ for $j \geq 2$, so that

$$\widetilde{\tau}'(\underline{\widetilde{\beta}}) = (\partial_1 \tau(\underline{\beta}), 0, \ldots, 0).$$

(c) Denoting by $\widetilde{\mathscr{C}^0}$ the tangent cone at $\underline{(0, 0)}$ to $\mathscr{C}_{\widetilde{L}}$, we now prove that the tangent plane $\widetilde{\mathscr{P}}$ to $\mathscr{C}_{\widetilde{L}}$ at $\underline{\widetilde{\beta}}$ is also tangent to $\widetilde{\mathscr{C}^0}$ at β^0.

Thanks to the results of (b), we know that the vector $\vec{n} := (1, -\partial_1 \tau(\underline{\beta}), 0, \ldots, 0)$ is normal to $\widetilde{\mathscr{P}}$. We thus have to show that it is also normal to $\widetilde{\mathscr{C}^0}$ at β^0. With arguments similar to those used in (a), we can prove that

$$(\tau, \eta) \in \mathscr{C}^0 \iff \left(\tau, \left(P^{-1}\right)^T \eta\right) \in \widetilde{\mathscr{C}^0},$$

so that if $\tau^0(\eta)$ is a parametrization of $\mathscr{C}^0$, then $\widetilde{\tau^0}(\eta) := \tau^0(P^T \eta)$ is a parametrization of $\widetilde{\mathscr{C}^0}$. We have therefore

$$\widetilde{\tau^{0'}}\left(\eta^0\right) = \tau^{0'}\left(P^T \eta^0\right) P^T = \tau^{0'}(\eta^0) P^T,$$

since $\eta^0 = e_1$. But Assumption 2 says that $\tau^{0'}(\eta^0) = \tau'(\underline{\beta})$, so that one has $\widetilde{\tau^{0'}}(\eta^0) = \tau'(\underline{\beta}) P^T$, and hence $\partial_j \widetilde{\tau^0}(\eta^0) = \tau'(\underline{\beta}) \cdot l_j$ for all j. Thanks to the definition of the l_j, this yields

$$\widetilde{\tau^{0'}}(\eta^0) = \left(\partial_1 \tau(\underline{\beta}), 0, \ldots, 0\right),$$

and therefore $\vec{n} = (1, -\partial_1 \tau(\underline{\beta}), 0, \ldots, 0)$ and is thus normal to $\widetilde{\mathscr{C}^0}$ at β^0, as we wanted to prove.

(d) We have thus proved that for the problem $\widetilde{(1)}$, Assumptions 1 and 2 remain true, and that Proposition 1 is also satisfied. □

6.2. An existence theorem

In Assumption 5 we supposed the existence and uniqueness of a regular solution to the coupled problem (**S**) which gives the leading terms of our approximate solution. We have not proved yet this existence/uniqueness theorem, but we give here an existence theorem for a simplified version of system (**S**) which also appears in the study of water waves (see [L2], [Su]). This system reads

$$(\mathbf{T}) \begin{cases} i\partial_t u + \partial_1^2 u = u \partial_1 v, \\ \partial_t v + \partial_1^{-1} \partial_2^2 v = -|u|^2, \end{cases}$$

where ∂_1 and ∂_2 denote the partial derivative with respect to the first and the second space coordinate, respectively. We want v to be real valued, while u may take complex values.

The second equation does not make sense since the operator $\partial_1^{-1}\partial_2^2$ does not act on distributions. However, the integral equation (used in Theorem 9),

$$v = e^{\partial_1^{-1}\partial_2^2 t} v_0 - \int_0^t e^{\partial_1^{-1}\partial_2^2 (t-s)} |u|^2(s)\, ds,$$

makes sense since the group $e^{\partial_1^{-1}\partial_2^2 t}$ acts on every Sobolev space H^s, and for $u \in L^\infty(0, T; L^2)$, $|u|^2$ lies in $L^\infty(0, T; H^s)$ for some negative s.

This system may be seen as a simplified version of (**S**) in space dimension equal to 2 where u plays the role of $\pi(\underline{\beta})\mathscr{U}_{11}$, and $\partial_1 v$ plays the role of $\pi^1(\underline{0})\mathscr{U}_{20}$.

Throughout this section, the Fourier dual variables of y_1 and y_2 are denoted by ξ_1 and ξ_2, respectively.

6.2.1. The regularized problem

In order to define a regularized problem associated to (**T**), we introduce, for any $\mu > 0$, the operator ∂_μ, whose symbol is given by

$$i\frac{\mu + \xi_1^2}{\xi_1}.$$

The operator ∂_μ^{-1} is therefore given by the symbol

$$-i\frac{\xi_1}{\mu + \xi_1^2},$$

which is also used to regularize the KP equation in $\mathbb{R}^2$ (see [IMS]). In the following lemma, we give some of the properties of these operators.

LEMMA 7

(i) ∂_μ *and* ∂_μ^{-1} *are antiadjoints.*

(ii) *If* φ *is a real-valued function, then* $\partial_\mu\varphi$ *and* $\partial_\mu^{-1}\varphi$ *are also real valued.*

Proof

(i) This is a consequence of the fact that the symbols of ∂_μ and ∂_μ^{-1} are purely imaginary.

(ii) It follows from the fact that these symbols are also odd.

□

We can now define the regularized problem. For $\epsilon > 0$ and $\mu > 0$,

$$(\mathbf{T}_{\epsilon,\mu})\begin{cases} i\partial_t u + \partial_1^2 u = u\partial_1 v, \\ \partial_t\big(1 + \epsilon\Delta^2\big)v + \partial_\mu^{-1}\partial_2^2 v = -|u|^2. \end{cases}$$

The end of this section is devoted to the proof of the following theorem.

THEOREM 5

(i) *Let* $(u_0, v_0) \in L^2 \times H^{5/2}(\mathbb{R}^2)$. *There exists a unique solution* $(u, v) \in C(\mathbb{R}; L^2(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2)) \cap C^1(\mathbb{R}; H^{-2}(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2))$ *of* $(\mathbf{T}_{\epsilon,\mu})$ *with initial values* $(u, v)(t = 0) = (u_0, v_0)$.

(ii) *If* $(u_0, v_0) \in H^2 \times H^5$, *then* $(u, v) \in C(\mathbb{R}; H^2 \times H^5) \cap C^1(\mathbb{R}; L^2 \times H^7)$.

Proof

Solving $(\mathbf{T}_{\epsilon,\mu})$ in the spaces given in the theorem is equivalent to solving the two integral equations

$$u = S_1(t)u_0 - i\int_0^t S_1(t-s)u\partial_1 v(s)\,ds \tag{75}$$

and

$$v = S_2(t)v_0 - \int_0^t S_2(t-s)\big(1+\epsilon\Delta^2\big)^{-1}|u|^2(s)\,ds, \tag{76}$$

where

$$S_1(t) := e^{i\partial_1^2 t} \qquad \text{and} \qquad S_2(t) := e^{-\partial_\mu^{-1}\partial_2^2(1+\epsilon\Delta^2)^{-1}t}$$

are two unitary groups on L^2.

For $(u, v) \in C(\mathbb{R}; L^2 \times H^{5/2}(\mathbb{R}^2))$, let us introduce

$$\mathscr{C}(u, v) = \big(\mathscr{C}_1(u, v), \mathscr{C}_2(u, v)\big)$$

with

$$\mathscr{C}_1(u, v) = S_1(t)u_0 - i\int_0^t S_1(t-s)u\partial_1 v(s)\,ds \tag{77}$$

and

$$\mathscr{C}_2(u, v) = S_2(t)v_0 - \int_0^t S_2(t-s)\big(1+\epsilon\Delta^2\big)^{-1}|u|^2(s)\,ds. \tag{78}$$

We also introduce the space $X_T := C([0, T]; L^2(\mathbb{R}^2) \times H^{5/2}(\mathbb{R}^2))$ and consider its natural norm

$$\|(u, v)\|_{X_T} := \max\Big(|u|_{L^\infty([0,T];L^2)}, |v|_{L^\infty([0,T];H^{5/2})}\Big).$$

For any $R > 0$, we also denote by B_R the ball of X_T with radius R. We can now state the following lemma.

LEMMA 8

Let $R := 2\max(|u_0|_{L^2}, |v_0|_{H^{5/2}})$.

There exists $T_1 > 0$ such that, for all $T \leq T_1$, the application $\mathscr{C}$ maps B_R into itself.

Proof
One has

$$\begin{aligned} \left|\mathscr{C}_1(u, v)\right|_{L^\infty([0,T];L^2)} &\leq |u_0|_{L^2} + T\left|u\partial_1 v\right|_{L^\infty([0,T];L^2)} \\ &\leq |u_0|_{L^2} + T|u|_{L^\infty([0,T];L^2)}|\partial_1 v|_{L^\infty([0,T];L^\infty)} \\ &\leq |u_0|_{L^2} + C_1 T|u|_{L^\infty([0,T];L^2)}|v|_{L^\infty([0,T];H^{5/2})}, \end{aligned} \tag{79}$$

since $\partial_1 v \in H^{3/2}(\mathbb{R}^2) \subset L^\infty$.

We also have

$$\begin{aligned} &\left|(1-\Delta)^{5/4}\mathscr{C}_2(u, v)\right|_{L^\infty([0,T];L^2)} \\ &\qquad \leq |v_0|_{H^{5/2}} + T\left|(1-\Delta)^{5/4}\left(1+\epsilon\Delta^2\right)^{-1}|u|^2\right|_{L^\infty([0,T];L^2)}, \end{aligned}$$

but

$$\left|(1-\Delta)^{5/4}\left(1+\epsilon\Delta^2\right)^{-1}|u|^2\right|_{L^\infty([0,T];L^2)} \leq C\left|(1-\Delta)^{3/4}|u|^2\right|_{L^\infty([0,T];L^2)}$$

and

$$\left||u|^2\right|_{H^{-1-\alpha}} \leq C\left||u|^2\right|_{L^1}$$

for any $\alpha > 0$. Taking $\alpha = 1/2$ thus yields

$$\left||u|^2\right|_{H^{-3/2}} \leq C\left||u|^2\right|_{L^1} = C|u|_{L^2}^2.$$

We have therefore

$$\left|(1-\Delta)^{5/4}\mathscr{C}_2(u, v)\right|_{L^\infty([0,T];L^2)} \leq |v_0|_{H^{5/2}} + C_2 T|u|^2_{L^\infty([0,T];L^2)}. \tag{80}$$

With $R = 2\max(|u_0|_{L^2}, |v_0|_{H^{5/2}})$ and $(u, v) \in B_R$, equation (79) yields

$$\left|\mathscr{C}_1(u, v)\right|_{L^\infty([0,T];L^2)} \leq \frac{R}{2} + C_1 T R^2,$$

and (80) yields

$$\left|\mathscr{C}_2(u, v)\right|_{L^\infty([0,T];H^{5/2})} \leq \frac{R}{2} + C_2 T R^2.$$

With $T_1 = \min(1/2C_1 R, 1/2C_2 R)$ and $T \leq T_1$, we have therefore $\|\mathscr{C}(u, v)\|_{X_T} \leq R$, and Lemma 8 is thus proved. □

We now prove another lemma before pursuing the proof of the theorem.

LEMMA 9
There exists $T_2 > 0$ such that, for all $T \leq T_2$, $\mathscr{C}$ is a contraction on the ball B_R of X_T.

Proof
Let (u, v) and $(\tilde{u}, \tilde{v})$ in X_T be such that $(u, v)(t = 0) = (\tilde{u}, \tilde{v})(t = 0) = (u_0, v_0)$. One has

$$\mathscr{C}_1(u, v) - \mathscr{C}_1(\tilde{u}, \tilde{v}) = -i \int_0^t S_1(t - s)\big(u\partial_1 v - \tilde{u}\partial_1\tilde{v}\big)\, ds,$$

so that

$$\begin{aligned}
&\big|\mathscr{C}_1(u, v) - \mathscr{C}_1(\tilde{u}, \tilde{v})\big|_{L^\infty([0,T];L^2)} \\
&\quad \le T\Big(\big|(u - \tilde{u})\partial_1 v\big|_{L^\infty([0,T];L^2)} + |\tilde{u}(\partial_1 v - \partial_1\tilde{v})|_{L^\infty([0,T];L^2)}\Big) \\
&\quad \le T\Big(C_1|u - \tilde{u}|_{L^\infty([0,T];L^2)}|v|_{L^\infty([0,T];H^{5/2})} \\
&\qquad\qquad + C_1|\tilde{u}|_{L^\infty([0,T];L^2)}|v - \tilde{v}|_{L^\infty([0,T];H^{5/2})}\Big).
\end{aligned}$$

If $(u, v) \in B_R$ and $(\tilde{u}, \tilde{v}) \in B_R$, one then has

$$\big|\mathscr{C}_1(u, v) - \mathscr{C}_1(\tilde{u}, \tilde{v})\big|_{L^\infty([0,T];L^2)} \le 2C_1TR\big\|(u, v) - (\tilde{u}, \tilde{v})\big\|_{X_T}, \tag{81}$$

and one can show in the same way that

$$\big|\mathscr{C}_2(u, v) - \mathscr{C}_2(\tilde{u}, \tilde{v})\big|_{L^\infty([0,T];H^{5/2})} \le 2C_2TR\big\|(u, v) - (\tilde{u}, \tilde{v})\big\|_{X_T}, \tag{82}$$

and the lemma is thus proved if we take $T_2 = 1/4C_2R$. □

Thanks to those two lemmas, the proof of the following proposition is straightforward.

PROPOSITION 10
For all $(u_0, v_0) \in L^2 \times H^{5/2}$, there exists a unique maximal solution $(u, v) \in C([0, T_{\max}[; L^2 \times H^{5/2})$ to $(\mathbf{T}_{\epsilon,\mu})$ such that $(u, v)(t = 0) = (u_0, v_0)$.

Moreover, if $T_{\max} < \infty$, then

$$|u|_{L^2}(t) + |v|_{H^{5/2}}(t) \longrightarrow \infty \quad \textit{when } t \longrightarrow T_{\max}.$$

Once the next proposition is shown, the proof of Theorem 5(i) will be complete.

PROPOSITION 11
One has $T_{\max} = +\infty$ (where $T_{\max}$ is defined in Proposition 10), and for all $t \in \mathbb{R}$, one has

$$\Big(\int_{\mathbb{R}^2} |u|^2\Big)(t) = \int_{\mathbb{R}^2} |u_0|^2.$$

Proof
Let (u, v) be as given by Proposition 10. We have

$$i\partial_t u + \partial_1^2 u = u\partial_1 v, \tag{83}$$

and $u \in C([0, T_{\max}), L^2) \cap C^1([0, T_{\max}), H^{-2})$.

Let $\rho_\alpha(y_1, y_2)$ be a regularizing sequence defined on $\mathbb{R}^2_y$. We then take the convolution product of ρ_α and (83). The L^2 scalar product of each term of the equation thus obtained with $\rho_\alpha * \overline{u}$ is well defined. Taking the imaginary part yields

$$\frac{1}{2}\partial_t \int \left|\rho_\alpha * u\right|^2 = \Im\left(\int \rho_\alpha * \left(u\partial_1 v\right)\left(\rho_\alpha * \overline{u}\right)\right).$$

Integrating this equality with respect to the time variable t then yields

$$\int \left|\rho_\alpha * u\right|^2(t) - \int \left|\rho_\alpha * u_0\right|^2 = 2\int_0^t \Im\left(\int \rho_\alpha * \left(u\partial_1 v\right)\left(\rho_\alpha * \overline{u}\right)\right).$$

But since $u \in C([0, T_{\max}), L^2)$, we have $\rho_\alpha * u(t) \to u(t)$ for all t when $\alpha \to 0$.

Moreover, one has $u\partial_1 v \in C([0, T_{\max}), L^2)$, so that $\rho_\alpha * (u\partial_1 v)(t) \to u\partial_1 v(t)$ for all t. We have therefore

$$\int \rho_\alpha * \left(u\partial_1 v\right)\rho_\alpha * \overline{u}(t) \longrightarrow \int |u|^2\partial_1 v(t)$$

when $\alpha \to 0$, and thus

$$\Im\left(\int \rho_\alpha * \left(u\partial_1 v\right)\rho_\alpha * \overline{u}(t)\right) \longrightarrow 0.$$

We now prove a domination property. One has

$$\left|\Im\int \left(\rho_\alpha * \left(u\partial_1 v\right)\rho_\alpha * \overline{u}(t)\right)\right| \le \left|\rho_\alpha * \left(u\partial_1 v\right)\right|_{L^2}\left|\rho_\alpha * \overline{u}\right|_{L^2} \le \left|u\partial_1 v\right|_{L^2}|\overline{u}|_{L^2} \le R^3,$$

with R such that (u, v) is in the ball B_R of X_T.

Thanks to Lebesgue's dominated convergence theorem, we have therefore

$$\int_0^t \Im\left(\int \rho_\alpha * \left(u\partial_1 v\right)\rho_\alpha * \overline{u}\right) ds \longrightarrow 0$$

when $\alpha \to 0$, and we have thus proved that $|u|_{L^2}(t) = |u_0|_{L^2}$ for all t.

Moreover, inequality (80) applied to $\mathscr{C}_2(u, v) = v$ yields, for all $T < T_{\max}$,

$$|v|_{L^\infty([0,T]; H^{5/2})} \le |v_0|_{H^{5/2}} + C_2 T|u|^2_{L^\infty([0,T]; L^2)} = |v_0|_{H^{5/2}} + C_2 T|u_0|^2_{L^2}.$$

Therefore, if $T_{\max} < \infty$, we have

$$|v|_{L^\infty([0,T_{\max}); H^{5/2})} \le |v_0|_{H^{5/2}} + C_2 T_{\max}|u_0|^2_{L^2}$$

and

$$|u|_{L^\infty([0,T_{\max}); L^2)} = |u_0|_{L^2},$$

which is in contradiction with the explosion condition of Proposition 10. We have therefore $T_{\max} = +\infty$, and the proposition is thus proved. □

We now prove Theorem 5(ii), which concerns the regularity of the solutions. Let (u_0, v_0) be in $H^2 \times H^5$. Solving the Cauchy problem in $H^2 \times H^5$ locally in time does not raise any difficulty, and we omit the proof. It remains to show that the result is valid globally in time.

Thanks to Theorem 5(i), we know that we can find a continuous function $C(t)$ such that $|v|_{H^{5/2}}(t) \leq C(t)$ for all t.

From (77) we deduce

$$|u|_{H^1}(t) \leq |u_0|_{H^1} + \int_0^t \left|u\partial_1 v\right|_{H^1}(s)\,ds. \tag{84}$$

But one has $\partial(u\partial_1 v) = \partial u \partial_1 v + u\partial\partial_1 v$ and

$$\left|\partial u\partial_1 v\right|_{L^2} \leq |\partial u|_{L^2}|\partial_1 v|_{L^\infty} \leq |u|_{H^1}C(t); \tag{85}$$

we also have

$$\begin{aligned}\left|u\partial\partial_1 v\right|_{L^2} &\leq |u|_{L^4}\left|\partial\partial_1 v\right|_{L^4}\\ &\leq \text{Cst}\,|u|_{H^{1/2}}\left|\partial\partial_1 v\right|_{H^{1/2}}\\ &\leq \text{Cst}\,|u|_{H^{1/2}}|v|_{H^{5/2}}\\ &\leq \text{Cst}\,C(t)|u|_{H^1}.\end{aligned} \tag{86}$$

Thanks to (84)–(86), we have

$$|u|_{H^1}(t) \leq |u_0|_{H^1} + \text{Cst}\int_0^t C(s)|u|_{H^1}(s)\,ds,$$

so that Gronwall's lemma yields the existence of a continuous function $D(t)$ such that $|u|_{H^1}(t) \leq D(t)$.

From (77) we also deduce

$$|u|_{H^2}(t) \leq |u_0|_{H^2} + \int_0^t \left|u\partial_1 v\right|_{H^2}(s)\,ds. \tag{87}$$

But one has $\partial^2(u\partial_1 v) = \partial^2 u\partial_1 v + 2\partial u\partial\partial_1 v + u\partial^2\partial_1 v$ and

$$\left|\partial^2 u\partial_1 v\right|_{L^2} \leq \left|\partial^2 u\right|_{L^2}|\partial_1 v|_{L^\infty} \leq \text{Cst}\,C(t)|u|_{H^2}; \tag{88}$$

we also have

$$\begin{aligned}\left|\partial u\partial_1\partial v\right|_{L^2} &\leq |\partial u|_{L^4}\left|\partial_1\partial v\right|_{L^4}\\ &\leq \text{Cst}\,|\partial u|_{H^{1/2}}\left|\partial_1\partial v\right|_{H^{1/2}}\\ &\leq \text{Cst}\,|\partial u|_{H^{1/2}}|v|_{H^{5/2}}\\ &\leq \text{Cst}\,C(t)|u|_{H^2}\end{aligned} \tag{89}$$

and

$$\left|u\partial^2\partial_1 v\right|_{L^2} \le |u|_{L^2}\left|\partial^2\partial_1 v\right|_{L^\infty} \le \mathrm{Cst}\,|u|_{L^2}\left|\partial^2\partial_1 v\right|_{H^2} \le \mathrm{Cst}\,|u_0|_{L^2}|u|_{H^5}. \tag{90}$$

Thanks to (87)–(90), we have

$$|u|_{H^2}(t) \le |u_0|_{H^2} + \mathrm{Cst}\int_0^t \left(C(s)|u|_{H^2}(s) + |v(s)|_{H^5}\right) ds. \tag{91}$$

From (78) we deduce

$$|v|_{H^5} \le |v_0|_{H^5} + \int_0^t \left||u|^2\right|_{H^1}(s)\, ds, \tag{92}$$

and we have $\partial|u|^2 = 2\Re(\overline{u}\partial u)$ and

$$|\overline{u}\partial u|_{L^2} \le |u|_{L^4}|\partial u|_{L^4} \le \mathrm{Cst}\,|u|_{H^{1/2}}|\partial u|_{H^{1/2}} \le \mathrm{Cst}\,|u|_{L^2}^{1/2}|u|_{H^1}^{1/2}|u|_{H^{3/2}},$$

so that $|\overline{u}\partial u|_{L^2} \le \mathrm{Cst}\sqrt{D(t)}|u|_{H^2}$. From (92) we then deduce

$$|v|_{H^5} \le |v_0|_{H^5} + \mathrm{Cst}\int_0^t \sqrt{D(s)}\left(|u|_{H^2}(s) + 1\right) ds. \tag{93}$$

Equations (91) and (93), together with Gronwall's lemma, yield that

$$|v|_{H^5} + |u|_{H^2} \le E(t),$$

where $E(t)$ is a continuous function.

It is now easy to conclude the proof of the theorem. □

Remark 6

(i) Since $|v|_{H^5}$ and $|u|_{H^2}$ control $|v|_{W^{1,\infty}}$ and $|u|_\infty$, we can easily obtain results for more regular solutions. One has, for instance, a solution in $H^3 \times H^6$.

(ii) In the above proof, we found two constants $C_1(T)$ and $C_2(T)$ such that

$$|(u,v)|_{L^\infty([0,T];H^2\times H^5)} \le C_1(T) \qquad \text{and} \qquad |(u,v)|_{L^\infty([0,T];L^2\times H^{5/2})} \le C_2(T).$$

These constants $C_1(T)$ and $C_2(T)$ depend on T, ϵ, u_0, and v_0 but not on μ.

We now prove the following theorem, which deals with the continuity of the solutions given by Theorem 5 with respect to the parameter μ.

THEOREM 6

(i) *We take here* $\mu = 0$. *If* $(u_0, v_0) \in L^2 \times H^{5/2}$, *then there exists a unique solution*

$(u, v) \in C(\mathbb{R}; L^2 \times H^{5/2})$ to the integral equations (75)–(76) such that $(u, v)(t = 0) = (u_0, v_0)$.

Moreover, if $(u_0, v_0) \in H^2 \times H^5$, then we also have $(u, v) \in C(\mathbb{R}; H^2 \times H^5)$.
(ii) *Let $(u_0, v_0) \in L^2 \times H^{5/2}$ (resp., $H^2 \times H^5$ or $H^3 \times H^6$), and let (u^μ, v^μ) be the solution of (75)–(76) such that $(u, v)(t = 0) = (u_0, v_0)$, with $\mu \geq 0$. Then the mapping*

$$\mathbb{R}^+ \longrightarrow C\big(\mathbb{R}; L^2 \times H^{5/2}\big) \quad \big(\text{resp., } H^2 \times H^5 \text{ or } H^3 \times H^6\big),$$
$$\mu \longmapsto (u^\mu, v^\mu)$$

is continuous.

Proof

(i) The proof made for Theorem 5 remains valid. The only difference is that we cannot use the partial differential equation satisfied by v because of the operator ∂_1^{-1}, but we do not need it.

(ii) We consider here the case $L^2 \times H^{5/2}$.

We write the integral equations (75)–(76) for μ and $\mu_0 \geq 0$:

$$\begin{cases} u^\mu = S_1(t)u_0 - i \displaystyle\int_0^t S_1(t-s)u^\mu \partial_1 v^\mu(s)\, ds, \\ v^\mu = S_2^\mu(t)v_0 - \displaystyle\int_0^t S_2^\mu(t-s)\big(1+\epsilon\Delta^2\big)^{-1}|u^\mu|^2(s)\, ds \end{cases}$$

and

$$\begin{cases} u^{\mu_0} = S_1(t)u_0 - i \displaystyle\int_0^t S_1(t-s)u^{\mu_0} \partial_1 v^{\mu_0}(s)\, ds, \\ v^{\mu_0} = S_2^{\mu_0}(t)v_0 - \displaystyle\int_0^t S_2^{\mu_0}(t-s)\big(1+\epsilon\Delta^2\big)^{-1}|u^{\mu_0}|^2(s)\, ds. \end{cases}$$

Subtracting those two systems yields on the one hand

$$\begin{aligned} \big|u^\mu - u^{\mu_0}\big|_{L^2} &\leq \int_0^t \big|u^\mu \partial_1 v^\mu - u^{\mu_0}\partial_1 v^{\mu_0}\big|_{L^2}(s)\, ds \\ &\leq \int_0^t |u^\mu|_{L^2}\big|\partial_1 v^\mu - \partial_1 v^{\mu_0}\big|_{L^\infty} + |\partial_1 v^{\mu_0}|_{L^\infty}|u^\mu - u^{\mu_0}|_{L^2}(s)\, ds. \end{aligned}$$

We have seen that $|u^\mu|_{L^2} = |u_0|_{L^2}$ and $|v^{\mu_0}|_{H^{5/2}}(t) \leq C(t)$, where $C(t)$ is a continuous function of t which does not depend on μ. We have therefore

$$\big|u^\mu - u^{\mu_0}\big|_{L^2} \leq \text{Cst} \int_0^t \Big(\big|v^\mu - v^{\mu_0}\big|_{H^{5/2}}(s) + C(s)\big|u^\mu - u^{\mu_0}\big|_{L^2}(s)\Big)\, ds. \tag{94}$$

One has on the other hand

$$\begin{aligned}\left|v^{\mu}-v^{\mu_0}\right|_{H^{5/2}} \le{}& \left|\left(S_2^{\mu}(t)-S_2^{\mu_0}(t)\right)v_0\right|_{H^{5/2}}\\ &+\int_0^t \left|\left(S_2^{\mu}(t-s)-S_2^{\mu_0}(t-s)\right)\left(1+\epsilon\Delta^2\right)^{-1}\left|u^{\mu_0}\right|^2(s)\right|_{H^{5/2}}ds\\ &+\int_0^t \left|\left(1+\epsilon\Delta^2\right)^{-1}\left(|u^{\mu_0}|^2-|u^{\mu}|^2\right)\right|_{H^{5/2}}ds\\ \le{}& \left|\left(S_2^{\mu}(t)-S_2^{\mu_0}(t)\right)v_0\right|_{H^{5/2}}\\ &+\int_0^t \left|\left(S_2^{\mu}(t-s)-S_2^{\mu_0}(t-s)\right)|u^{\mu_0}|^2(s)\right|_{H^{-3/2}}ds\\ &+\int_0^t \left|\left(|u^{\mu_0}|^2-|u^{\mu}|^2\right)\right|_{H^{-3/2}}(s)\,ds.\end{aligned}$$

But we know that

$$\begin{aligned}\left||u^{\mu_0}|^2-|u^{\mu}|^2\right|_{H^{-3/2}} &\le \mathrm{Cst}\left||u^{\mu_0}|^2-|u^{\mu}|^2\right|_{L^1}\\ &\le \mathrm{Cst}\left(|u^{\mu_0}|_{L^2}+|u^{\mu}|_{L^2}\right)\left|u^{\mu_0}-u^{\mu}\right|_{L^2}\\ &\le \mathrm{Cst}\left|u^{\mu_0}-u^{\mu}\right|_{L^2},\end{aligned}$$

so that

$$\begin{aligned}\left|v^{\mu}-v^{\mu_0}\right|_{H^{5/2}} \le{}& \left|\left(S_2^{\mu}(t)-S_2^{\mu_0}(t)\right)v_0\right|_{H^{5/2}}+\mathrm{Cst}\int_0^t\left|u^{\mu_0}-u^{\mu}\right|_{L^2}\\ &+\int_0^t\left|\left(S_2^{\mu}(t-s)-S_2^{\mu_0}(t-s)\right)|u^{\mu_0}|^2\right|_{H^{-3/2}}ds.\end{aligned} \tag{95}$$

We introduce

$$m_1(t) := \int_0^t \left|\left(S_2^{\mu}(t-s)-S_2^{\mu_0}(t-s)\right)|u^{\mu_0}|^2(s)\right|_{H^{-3/2}}ds,$$

and we want to prove that $m_1(t)\to 0$ when $\mu\to\mu_0$. Denoting by iP_μ the symbol of ∂_μ, we have

$$\begin{aligned}&\mathscr{F}\Big(\left(S_2^{\mu}(t-s)-S_2^{\mu_0}(t-s)\right)\left|u^{\mu_0}\right|^2(s)\Big)(\xi_1,\xi_2)\\ &\quad=\Big(e^{-iP_\mu(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2(t-s)}-e^{-iP_{\mu_0}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2(t-s)}\Big)\mathscr{F}\left(|u^{\mu_0}|^2\right)(\xi),\end{aligned}$$

and it is clear that the second member tends toward zero for almost every ξ_1, ξ_2, and t when $\mu\to\mu_0$.

Since the integrand that appears in the definition of $m_1(t)$ is dominated by $||u^{\mu_0}|^2|_{H^{-3/2}}(s)\in L^1_{\mathrm{loc}}(\mathbb{R})$, we can conclude, thanks to Lebesgue's dominated convergence theorem, that

$$m_1(t)\longrightarrow 0 \quad \text{in } L^\infty_{\mathrm{loc}}(\mathbb{R}) \quad \text{when } \mu\longrightarrow\mu_0.$$

We now introduce

$$m_2(t) := \left|\left(S_2^{\mu}(t) - S_2^{\mu_0}(t)\right)v_0\right|_{H^{5/2}},$$

and we want to prove that it also tends towards zero when $\mu \to \mu_0$. We have

$$\begin{aligned}
&\mathscr{F}\left(\left(S_2^{\mu}(t) - S_2^{\mu_0}(t)\right)v_0\right) \\
&\quad = \left(e^{-iP_{\mu}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t} - e^{-iP_{\mu_0}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t}\right)\widehat{v_0}(\xi) \\
&\quad = \left(e^{-iP_{\mu}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t} - e^{-iP_{\mu_0}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t}\right)\mathbb{1}_{\{|\xi|\le\alpha\}}\mathbb{1}_{\{|\xi_1|\ge\beta\}}\widehat{v_0}(\xi) \\
&\qquad + (\cdots)(1 - \mathbb{1}_{\{|\xi|\le\alpha\}}\mathbb{1}_{\{|\xi_1|\ge\beta\}})\widehat{v_0}(\xi) \\
&\quad = f_{\mu}(t,\xi) + g_{\mu}(t,\xi).
\end{aligned}$$

Let $\gamma > 0$, and choose $\alpha > 0$ sufficiently big and $\beta > 0$ sufficiently small to have

$$\left|\left(1 + |\xi|^2\right)^{5/4} g_{\mu}(t,\xi)\right|_{L^2} \le \gamma. \tag{96}$$

With the same α and β, one has

$$\begin{aligned}
\left|\left(1 + |\xi|^2\right)^{5/4} f_{\mu}(t,\xi)\right|_{L^2}^2 = \int \Big[&\left(1 + |\xi|^2\right)^{5/2}\left|\widehat{v_0}(\xi)\right|^2 \\
&\times \left|e^{-iP_{\mu_0}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t} - e^{-iP_{\mu}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t}\right|^2 \\
&\times \mathbb{1}_{\{|\xi|\le\alpha\}}\mathbb{1}_{\{|\xi_1|\ge\beta\}}\Big]\, d\xi.
\end{aligned}$$

For $|\xi| \le \alpha$ and $|\xi_1| \ge \beta$, one has, for $t \le T$,

$$\left|e^{-iP_{\mu_0}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t} - e^{-iP_{\mu}(\xi_1)(1+\epsilon|\xi|^4)^{-1}\xi_2^2 t}\right|^2 \le C(\alpha,\beta)T^2|\mu - \mu_0|^2,$$

so that it is easy to see that

$$\left|\left(1 + |\xi|^2\right)^{5/4} f_{\mu}(t,\xi)\right|_{L^2}^2 \le \gamma^2$$

if μ and μ_0 are close enough. Together with (96), this yields

$$m_2(t) \longrightarrow 0 \quad \text{in } L^{\infty}_{\text{loc}}(\mathbb{R}) \quad \text{when } \mu \longrightarrow \mu_0.$$

Equation (95) thus is written

$$\left|v^{\mu} - v^{\mu_0}\right|_{H^{5/2}}(t) \le m_1(t) + m_2(t) + \text{Cst}\int_0^t \left|u^{\mu} - u^{\mu_0}\right|_{L^2}(s)\, ds,$$

with $m_1(t) + m_2(t) \to 0$ as $\mu \to \mu_0$ in $L^{\infty}_{\text{loc}}(\mathbb{R})$. Using (94) and Gronwall's lemma yields

$$\left|u^{\mu} - u^{\mu_0}\right|_{L^2} + \left|v^{\mu} - v^{\mu_0}\right|_{H^{5/2}} \longrightarrow 0 \quad \text{in } L^{\infty}_{\text{loc}}(\mathbb{R}) \text{ as } \mu \longrightarrow \mu_0,$$

and the proof is thus complete. □

6.2.2. Energy estimates

We first prove a few energy estimates linked to the regularized problem ($\mathbf{T}_{\epsilon,\mu}$), for $\mu > 0$. These estimates are very similar to those obtained by Ph. Laurençot [La] for the 1-dimensional problem.

THEOREM 7

Let $(u_0, v_0) \in H^3 \times H^6$.

Then the solution $(u, v) \in C(\mathbb{R}; H^3 \times H^6)$ *given by Theorem 5 for* $\mu > 0$ *satisfies*

(i)

$$\int_{\mathbb{R}^2} |u|^2(t) = \int_{\mathbb{R}^2} |u_0|^2;$$

(ii)

$$\int_{\mathbb{R}^2} |\partial_1 u|^2 + |u|^2 \partial_1 v + \frac{1}{2}\left|\partial_2\left(\partial_\mu^{-1}\partial_1\right)^{1/2} v\right|^2 = \int_{\mathbb{R}^2} |\partial_1 u_0|^2 + |u_0|^2 \partial_1 v_0 + \frac{1}{2}\left|\partial_2\left(\partial_\mu^{-1}\partial_1\right)^{1/2} v_0\right|^2;$$

(iii)

$$\int_{\mathbb{R}^2} \left|\left(1+\epsilon\Delta^2\right)^{1/2}\partial_1 v\right|^2 + 2iu\partial_1\overline{u} = \int_{\mathbb{R}^2} \left|\left(1+\epsilon\Delta^2\right)^{1/2}\partial_1 v_0\right|^2 + 2iu_0\partial_1\overline{u_0};$$

(iv)

$$\int_{\mathbb{R}^2} \left|(1+\epsilon\Delta)^{1/2} v\right|^2 = \int_{\mathbb{R}^2} \left|(1+\epsilon\Delta)^{1/2} v_0\right|^2 - 2\int_0^t \int_{\mathbb{R}^2} |u|^2 v(s)\, ds.$$

Proof

(i) Taking the imaginary part of the L^2 product of the first equation of ($\mathbf{T}_{\epsilon,\mu}$) with $\overline{u}$ yields

$$\partial_t \int |u|^2 = 0,$$

and the result follows.

(ii) Taking the real part of the L^2 product of the first equation of ($\mathbf{T}_{\epsilon,\mu}$) with $\partial_t\overline{u}$ yields

$$\begin{aligned} -\frac{1}{2}\partial_t \int |\partial_1 u|^2 &= \Re\left(\int \partial_t\overline{u}u\partial_1 v\right) \\ &= \frac{1}{2}\int \partial_t|u|^2\partial_1 v \\ &= \frac{1}{2}\partial_t\int |u|^2\partial_1 v - \frac{1}{2}\int |u|^2\partial_t\partial_1 v, \end{aligned}$$

and therefore

$$\partial_t \int |\partial_1 u|^2 + \partial_t \int |u|^2 \partial_1 v = \int |u|^2 \partial_t \partial_1 v. \tag{97}$$

The second equation of $(\mathbf{T}_{\epsilon,\mu})$ may be written under the form

$$\partial_t v + \partial_\mu^{-1} \partial_2^2 \big(1 + \epsilon \Delta^2\big)^{-1} v = -\big(1 + \epsilon \Delta^2\big)^{-1} |u|^2, \tag{98}$$

so that $\int \partial_1 (98) |u|^2$ reads

$$\int \partial_1 \partial_t v |u|^2 + \int \partial_\mu^{-1} \partial_1 \partial_2^2 \big(1 + \epsilon \Delta^2\big)^{-1} v |u|^2 = 0, \tag{99}$$

since $(1 + \epsilon \Delta^2)^{-1} \partial_1$ is antiadjoint.

We now compute $\int (98) \partial_2^2 \partial_\mu^{-1} \partial_1 v$ and find

$$-\frac{1}{2} \partial_t \int \big|\partial_2 \big(\partial_\mu^{-1} \partial_1\big)^{1/2} v\big|^2 = -\int \big(1 + \epsilon \Delta^2\big)^{-1} |u|^2 \partial_2^2 \partial_\mu^{-1} \partial_1 v. \tag{100}$$

Since $(1 + \epsilon \Delta^2)^{-1}$ is self-adjoint, (99)–(100) yield

$$\int \partial_1 \partial_t v |u|^2 = -\frac{1}{2} \partial_t \int \big|\partial_2 \big(\partial_\mu^{-1} \partial_1\big)^{1/2} v\big|^2,$$

so that plugging this equation in (97) yields

$$\partial_t \int_{\mathbb{R}^2} |\partial_1 u|^2 + |u|^2 \partial_1 v + \frac{1}{2} \big|\partial_2 \big(\partial_\mu^{-1} \partial_1\big)^{1/2} v\big|^2 = 0,$$

and the result follows.

(iii) Taking the L^2 product of the second equation of $(\mathbf{T}_{\epsilon,\mu})$ with $\partial_1^2 v$ yields

$$-\frac{1}{2} \partial_t \int \big|\big(1 + \epsilon \Delta^2\big)^{1/2} \partial_1 v\big|^2 = -\int |u|^2 \partial_1^2 v = \int \overline{u} \partial_1 u \partial_1 v + u \partial_1 \overline{u} \partial_1 v.$$

One then takes the expressions on $\overline{u} \partial_1 v$ and $u \partial_1 v$ given by the first equation of $(\mathbf{T}_{\epsilon,\mu})$ and plugs them into the above equation, and thus one obtains

$$\begin{aligned} -\frac{1}{2} \partial_t \int \big|\big(1 + \epsilon \Delta^2\big)^{1/2} \partial_1 v\big|^2 &= \int \partial_1 u \big(-i \partial_t \overline{u} + \partial_1^2 \overline{u}\big) + \partial_1 \overline{u} \big(i \partial_t u + \partial_1^2 u\big) \\ &= i \int \partial_1 \overline{u} \partial_t u - \partial_1 u \partial_t \overline{u} + 0 \\ &= i \int \partial_1 \overline{u} \partial_t u + u \partial_t \partial_1 \overline{u} \\ &= i \partial_t \int u \partial_1 \overline{u}, \end{aligned}$$

so that

$$\partial_t\left(\int \left|\left(1+\epsilon\Delta^2\right)^{1/2}\partial_1 v\right|^2 + 2i\int u\partial_1\overline{u}\right) = 0,$$

and the result follows.

(iv) Taking the L^2 product of the second equation of $(\mathbf{T}_{\epsilon,\mu})$ with v reads

$$\partial_t \int \left|\left(1+\epsilon\Delta^2\right)^{1/2} v\right|^2 = -2\int |u|^2 v,$$

which yields the result. □

The following corollary gives energy estimates associated to the solutions of $(\mathbf{T}_{\epsilon,0})$.

COROLLARY 1

We take here $\mu = 0$. The solution (u, v) given in this case by Theorem 6 satisfies

(i)

$$\int_{\mathbb{R}^2} |u|^2(t) = \int_{\mathbb{R}^2} |u_0|^2;$$

(ii)

$$\int_{\mathbb{R}^2} |\partial_1 u|^2 + |u|^2\partial_1 v + \frac{1}{2}|\partial_2 v|^2 = \int_{\mathbb{R}^2} |\partial_1 u_0|^2 + |u_0|^2\partial_1 v_0 + \frac{1}{2}|\partial_2 v_0|^2;$$

(iii)

$$\int_{\mathbb{R}^2} \left|\left(1+\epsilon\Delta^2\right)^{1/2}\partial_1 v\right|^2 + 2iu\partial_1\overline{u} = \int_{\mathbb{R}^2} \left|\left(1+\epsilon\Delta^2\right)^{1/2}\partial_1 v_0\right|^2 + 2iu_0\partial_1\overline{u_0};$$

(iv)

$$\int_{\mathbb{R}^2} \left|(1+\epsilon\Delta)^{1/2} v\right|^2 = \int_{\mathbb{R}^2} \left|(1+\epsilon\Delta)^{1/2} v_0\right|^2 - 2\int_0^t \int_{\mathbb{R}^2} |u|^2 v(s)\,ds.$$

Proof

This corollary is a consequence of Theorem 7 and of the continuity of the flow with respect to the parameter μ. □

Remark 7

The results of the corollary cannot be obtained directly without treating the case $\mu > 0$. Indeed, the estimates cannot be done directly on $(\mathbf{T}_{\epsilon,0})$ since $\partial_1^{-1}\partial_2^2 v$ and $\partial_t v$ are not distributions.

6.2.3. Finding bounds independent of ϵ

Useful inequalities

We first give two useful inequalities that we use throughout this section.

LEMMA 10

If u and $\partial_1 u$ are in $L^2(\mathbb{R}^2)$, then

$$\int \left(\sup_{y_1 \in \mathbb{R}} |u|^2 \right)(y_2)\, dy_2 \leq 2|u|_2 |\partial_1 u|_2.$$

Proof

Since for any function $f \in H^1(\mathbb{R})$ one has $|f|_\infty \leq \sqrt{2}|f|_2|f'|_2$, we can write

$$\left(\sup_{y_1 \in \mathbb{R}} |u|^2 \right)(y_2) \leq 2|u|_{2,y_1}(y_2)|\partial_1 u|_{2,y_1}(y_2).$$

Integrating this inequality with respect to y_2 and using the Cauchy-Schwarz inequality then yields the result. □

LEMMA 11

If v and $\partial_2 v$ are in $L^2(\mathbb{R}^2)$, then

$$\sup_{y_2} \left(\int v^2(y_1, y_2)\, dy_1 \right)^{1/2} \leq \sqrt{2}|v|_2^{1/2}|\partial_2 v|_2^{1/2}.$$

Proof

Let ψ be defined as

$$\psi : y_2 \longmapsto \left(\int_{\mathbb{R}} v^2(y_1, y_2)\, dy_1 \right)^{1/2}.$$

One has $\psi \in L^2(\mathbb{R})$ and $|\psi|_2 = |v|_2$. Moreover, we have

$$\psi'(y_2) = \frac{\int v \partial_2 v\, dy_1}{\left(\int v^2(y_1, y_2)\, dy_1 \right)^{1/2}},$$

so that $|\psi'(y)| \leq (\int |\partial_2 v|^2\, dy_1)^{1/2}$ by Cauchy-Schwarz.

One has therefore $\psi \in H^1(\mathbb{R})$ and $|\psi'|_2 \leq |\partial_2 v|_2$, and thus

$$|\psi|_{\infty, y_2} \leq \sqrt{2}|v|_2^{1/2}|\partial_2 v|_2^{1/2}.$$ □

Local bounds in time, for small initial data

The following theorem gives useful bounds independent of ϵ.

THEOREM 8

We take here $\mu = 0$, and we let $T > 0$.

There exists $\epsilon_0 > 0$, and there exist $\lambda > 0$ and $C > 0$ independent of ϵ, such that *if* $(u_0, v_0) \in H^3 \times H^6$ *is such that*

$$|u_0|_2^2 + \left|\partial_1 u_0\right|_2^2 + |v_0|_{H^1}^2 \le \lambda,$$

then the solution (u, v) *of* $(\mathbf{T}_{\epsilon,0})$ *such that* $(u, v)(t = 0) = (u_0, v_0)$*, given by Theorem 6, satisfies*

$$|u|_{L^\infty([0,T];L^2)} + |\partial_1 u|_{L^\infty([0,T];L^2)} + |v|_{L^\infty([0,T];H^1)} \le C.$$

Proof

We first introduce the quantity N_1 defined as

$$N_1 := \int \left|\partial_1 u_0\right|^2 + |u_0|^2 \partial_1 v_0 + \frac{1}{2}|\partial_2 v_0|^2.$$

Thanks to Corollary 1(ii), one has

$$|\partial_1 u|_2^2 + \frac{1}{2}|\partial_2 v|_2^2 \le N_1 + \left| \int |u|^2 \partial_1 v \right| \le N_1 + 2\int |u|\,|\partial_1 u|\,|v|, \tag{101}$$

but one also has

$$\begin{aligned}
2\int |u||\partial_1 u||v| &= 2\int \left(\int |u||\partial_1 u||v|\, dy_1 \right) dy_2 \\
&\le 2\int |u|_{\infty,y_1}(y_2) \left(\int |\partial_1 u||v|\, dy_1 \right) dy_2 \\
&\le 2\int |u|_{\infty,y_1}(y_2) \left(\int |\partial_1 u|^2 dy_1 \right)^{1/2} \left(\int |v|^2 dy_1 \right)^{1/2} dy_2 \\
&\le 2\sup_{y_2} \left(\int |v|^2\, dy_1 \right)^{1/2} \int |u|_{\infty,y_1} \left(\int |\partial_1 u|^2\, dy_1 \right)^{1/2} dy_2 \\
&\le 2\sup_{y_2} \left(\int |v|^2\, dy_1 \right)^{1/2} \left(\int |u|_{\infty,y_1}\, dy_2 \right)^{1/2} |\partial_1 u|_2 \\
&\le 4|v|_2^{1/2} |\partial_2 v|_2^{1/2} |u|_2^{1/2} |\partial_1 u|_2^{3/2},
\end{aligned} \tag{102}$$

the last inequality being a consequence of Lemmas 10 and11.

Thanks to (101), we have therefore

$$|\partial_1 u|_2^2 + \frac{1}{2}|\partial_2 v|_2^2 \le N_1 + 4|v|_2^{1/2}|\partial_2 v|_2^{1/2}|u_0|_2^{1/2}|\partial_1 u|_2^{3/2}, \tag{103}$$

since for all t, $|u|_2(t) = |u_0|_2$.

It is also a consequence of (102) that

$$N_1 \le |u_0|_2^2 + \frac{1}{2}\left|\partial_2 v_0\right|_2^2 + 4|v_0|_2^{1/2}\left|\partial_2 v_0\right|_2^{1/2}|u_0|_2^{1/2}\left|\partial_1 u_0\right|_2^{3/2}. \tag{104}$$

Thanks to Corollary 1(iv), one also has

$$\left|\left(1+\epsilon\Delta^2\right)^{1/2}v\right|_2^2 = \left|\left(1+\epsilon\Delta^2\right)^{1/2}v_0\right|^2 - 2\int_0^t \int |u|^2 v,$$

where one can write

$$\int |u|^2 v = \int \left(\int u\overline{u}v\, dy_1\right) dy_2,$$

so that

$$\begin{aligned}
\left|\int |u|^2 v\right| &\le \int |u|_{\infty,y_1}\left(\int |u||v|\, dy_1\right) dy_2 \\
&\le \int |u|_{\infty,y_1}\left(\int |u|^2\, dy_1\right)^{1/2}\left(\int |v|^2\, dy_1\right)^{1/2} dy_2 \\
&\le \sup_{y_2}\left(\int |v|^2\, dy_1\right)^{1/2}\int |u|_{\infty,y_1}\left(\int |u|^2\, dy_1\right)^{1/2} dy_2 \\
&\le \sup_{y_2}\left(\int |v|^2\, dy_1\right)^{1/2}\left(\int |u|^2_{\infty,y_1}\, dy_2\right)^{1/2}|u|_2 \\
&\le 2|v|_2^{1/2}|\partial_2 v|_2^{1/2}|u_0|_2^{3/2}|\partial_1 u|_2^{1/2},
\end{aligned}$$

where the last inequality is a consequence of Lemmas 10 and 11 and of the conservation of the L^2 norm of u.

We have therefore

$$\left|\left(1+\epsilon\Delta^2\right)^{1/2}v\right|_2^2 \le \left|\left(1+\epsilon\Delta^2\right)^{1/2}v_0\right|_2^2 + 4\int_0^t |v|_2^{1/2}|\partial_2 v|_2^{1/2}|u_0|_2^{3/2}|\partial_1 u|_2^{1/2}, \quad (105)$$

so that, for $T > 0$ and $t \le T$, we have

$$\left|\left(1+\epsilon\Delta^2\right)^{1/2}v\right|^2 \le \left|\left(1+\epsilon\Delta^2\right)^{1/2}v_0\right|_2^2 + 4T|v|^{1/2}|\partial_2 v|^{1/2}|u_0|_2^{3/2}|\partial_1 y|^{1/2},$$

where the norm $|\cdot|_{L^\infty([0,T];L^2)}$ is denoted by $|\cdot|$.

Since $|v|_2 \le |(1+\epsilon\Delta^2)^{1/2}v|_2$, and since for ϵ small enough we have $|(1+\epsilon\Delta^2)^{1/2}v_0|_2^2 \le 2|v_0|_2^2$, we have

$$|v|^2 \le 2|v_0|_2^2 + 4T|v|^{1/2}|\partial_2 v|^{1/2}|u_0|_2^{3/2}|\partial_1 y|^{1/2}. \quad (106)$$

Taking the sup in time in (103) and summing with (106) then yields

$$\begin{aligned}
|\partial_1 u|^2 + \frac{1}{2}|\partial_2 v|^2 + |v|^2 \le N_1 + 2|v_0|_2^2 &+ 4|v|^{1/2}|\partial_2 v|^{1/2}|u_0|_2^{1/2}|\partial_1 u|^{3/2} \\
&+ 4T|v|^{1/2}|\partial_2 v|^{1/2}|u_0|_2^{3/2}|\partial_1 u|^{1/2}.
\end{aligned} \quad (107)$$

We now use the Young inequality $abcd \leq (1/4)(a^4+b^4+c^4+d^4)$ with $a = 4T|u_0|_2^{3/2}$, $b = |v|^{1/2}$, $c = |\partial_2 v|^{1/2}$, and $d = |\partial_1 u|^{1/2}$ to obtain

$$\begin{aligned}\frac{3}{4}\Big(|\partial_1 u|^2 + \frac{1}{3}|\partial_2 v|^2 + |v|^2\Big) \\ \leq N_1 + 2|v_0|_2^2 + 4|v|^{1/2}|\partial_2 v|^{1/2}|u_0|_2^{1/2}|\partial_1 u|^{3/2} + \frac{1}{4}\big(4T|u_0|_2^{3/2}\big)^4.\end{aligned} \tag{108}$$

We now use another Young inequality, $abcd \leq (1/8)(a^8+b^8+3c^{8/3}+3d^{8/3})$ with $a = |v|^{1/2}$, $b = |\partial_2 v|^{1/2}$, $c = |u_0|_2^{1/2}$, and $d = |\partial_1 u|^{3/2}$ to obtain

$$\begin{aligned}|\partial_1 u|^2 + |\partial_2 v|^2 + |v|^2 &\leq \mathrm{Cst}\,\big(N_1 + |v_0|_2^2 + T^4|u_0|_2^6\big) \\ &\quad + \mathrm{Cst}\,\Big(|u_0|_2^{4/3} + |v|^4 + |\partial_2 v|^4 + |\partial_1 u|^4\Big).\end{aligned}$$

Introducing $f := |v|^2 + |\partial_1 u|^2 + |\partial_2 v|^2$, we obtain from the above equation

$$f \leq \mathrm{Cst}\,\Big(N_1 + T^4|u_0|_2^6 + |v_0|_2^2 + |u_0|_2^{4/3}\Big) + \mathrm{Cst}\, f^2,$$

which is of the form $\alpha X^2 - X + \beta \geq 0$. We want to choose α and β such that the trinomial $\alpha X^2 - X + \beta$ has two distinct real roots. We want therefore $1 - 4\alpha\beta > 0$, which reads

$$\mathrm{Cst}\,\Big(N_1 + T^4|u_0|^6 + |v_0|_2^2 + |u_0|_2^{4/3}\Big) < \frac{1}{4}. \tag{109}$$

For $\lambda > 0$ small enough, it is a consequence of (104) that if

$$|u_0|^2 + \big|\partial_1 u_0\big|^2 + |v_0|_{H^1}^2 \leq \lambda,$$

then condition (109) is satisfied, and we denote by $X_0 < X_1$ the two roots. Since for all t such that $0 \leq t \leq T$ one has $\alpha f(t)^2 - f(t) + \beta \geq 0$, one has either $f(t) < X_0$ or $f(t) > X_1$ for all $t \leq T$. We are in the first case if $f(0) < X_0$ and in the second otherwise. In order to have an upper bound for $f(t)$, we therefore want to have $f(0) < X_0$, which is the case if $2\alpha f(0) - 1 < 0$, that is, if

$$\mathrm{Cst}\,\big(|v_0|^2 + \big|\partial_1 u_0\big|^2 + \big|\partial_2 v_0\big|^2\big) < 1,$$

which is satisfied if the λ defined above is small enough. One then has for all $t \leq T$,

$$\big(|v|^2 + |\partial_1 u|^2 + |\partial_2 v|^2\big)(t) \leq X_0 = \frac{1 - \sqrt{1 - \alpha\beta}}{2} \leq \frac{1}{2}. \tag{110}$$

We now want a bound for $|\partial_1 v|_2$; one has

$$\begin{aligned}\int |\partial_1 v|^2 &\leq \int \big|\big(1 + \epsilon\Delta^2\big)^{1/2}\partial_1 v\big|^2 \\ &\leq \Big|\int \big|\big(1 + \epsilon\Delta^2\big)^{1/2}\partial_1 v\big|^2 + 2iu\partial_1\overline{u}\Big| + 2\Big|\int u\partial_1\overline{u}\Big|,\end{aligned}$$

and using Corollary 1(iii) yields

$$\int |\partial_1 v|^2 \leq \left| \int \left|\left(1+\epsilon\Delta^2\right)^{1/2}\partial_1 v_0\right|^2 + 2iu_0\partial_1\overline{u_0}\right| + 2|u|_{L^2}|\partial_1 u|_{L^2}.$$

For ϵ small enough, one has therefore

$$\int |\partial_1 v|^2 \leq 2|\partial_1 v_0|_{L^2}^2 + 2\int \left|u_0\partial_1\overline{u_0}\right| + 2|u|_{L^2}|\partial_1 u|_{L^2}.$$

Since $f(t) \leq 1/2$, for all $t \leq T$, we can conclude that

$$\int |\partial_1 v|^2 \leq \text{Cst}\,.$$

This inequality, together with (110), proves the theorem. □

6.2.4. Conclusion

Throughout this section, we denote by $(u^\varepsilon, v^\varepsilon)$ the solution to $(\mathbf{T}_{\epsilon,0})$ given by Theorem 6. Thanks to Theorem 8, we can consider a subsequence, still denoted by $(u^\varepsilon, v^\varepsilon)$, such that

$$\begin{aligned} u^\varepsilon &\rightharpoonup u \quad \text{in } L^\infty\big([0,T];L^2\big) \text{ weak } *, \\ \partial_1 u^\varepsilon &\rightharpoonup \partial_1 u \quad \text{in } L^\infty\big([0,T];L^2\big) \text{ weak } *, \\ v^\varepsilon &\rightharpoonup v \quad \text{in } L^\infty\big([0,T];H^1\big) \text{ weak } *. \end{aligned}$$

We want to prove that (u, v) solves $(\mathbf{T})$.

We first give a compactness result for v^ε.

LEMMA 12

If $|u_0|^2 \in H^1$, then one has $v^\varepsilon \to v$ strongly in $L^\infty([0,T];L^2_{\text{loc}})$.

Proof

Multiplying the first equation of $(\mathbf{T}_{\epsilon,0})$ by $\overline{u}^\varepsilon$ and taking the imaginary part yields

$$\frac{1}{2}\partial_t|u^\varepsilon|^2 + 2\Im\big(\partial_1^2 u^\varepsilon\overline{u}^\varepsilon\big) = 0,$$

and therefore

$$|u^\varepsilon|^2 = |u_0|^2 - 4\partial_1\int_0^t \Im\big(\partial_1 u^\varepsilon\overline{u}^\varepsilon\big)(s)\,ds,$$

since $\partial_1 u^\varepsilon\partial_1\overline{u}^\varepsilon$ is real. Introduce now

$$U^\varepsilon := \int_0^t \Im\big(\partial_1 u^\varepsilon\overline{u}^\varepsilon\big)(s)\,ds,$$

so that

$$v^\varepsilon = e^{-\partial_1^{-1}\partial_2^2(1+\epsilon\Delta)^{-1}t}v_0 - \int_0^t e^{-\partial_1^{-1}\partial_2^2(1+\epsilon\Delta)^{-1}(t-s)}(1+\epsilon\Delta)^{-1}\big[|u_0|^2 - 4\partial_1 U^\varepsilon\big]\,ds.$$

We also introduce

$$V^\varepsilon := e^{-\partial_1^{-1}\partial_2^2(1+\epsilon\Delta)^{-1}t}v_0 - \int_0^t e^{-\partial_1^{-1}\partial_2^2(1+\epsilon\Delta)^{-1}(t-s)}(1+\epsilon\Delta)^{-1}|u_0|^2\,ds$$

and

$$W^\varepsilon := 4\int_0^t e^{-\partial_1^{-1}\partial_2^2(1+\epsilon\Delta)^{-1}(t-s)}(1+\epsilon\Delta)^{-1}\partial_1 U^\varepsilon(s)\,ds,$$

so that $v^\varepsilon = V^\varepsilon + W^\varepsilon$.

As soon as $v_0 \in H^1$ and $|u_0|^2 \in H^1$, we have V^ε bounded in $L^\infty([0,T]; H^1)$ and $V^\varepsilon \to V$ in $L^\infty([0,T]; H^1)$ when $\epsilon \to 0$, where

$$V := e^{-\partial_1^{-1}\partial_2^2 t}v_0 - \int_0^t e^{-\partial_1^{-1}\partial_2^2(t-s)}|u_0|^2\,ds.$$

Since v^ε and V^ε are bounded in $L^\infty([0,T]; H^1)$, then so is $W^\varepsilon = v^\varepsilon - V^\varepsilon$. Moreover, one has

$$\partial_t W^\varepsilon = 4\big(1+\varepsilon\Delta^2\big)^{-1}\partial_1 U^\varepsilon(t) - 4\int_0^t e^{-\partial_1^{-1}\partial_2^2(1+\epsilon\Delta^2)^{-1}(t-s)}\big(1+\epsilon\Delta^2\big)^{-2}\partial_2^2 U^\varepsilon\,ds.$$

But the sequence U^ε, as defined above, is bounded in $L^\infty([0,T]; L^1)$ and therefore in $L^\infty([0,T]; H^{-3/2})$, so that $\partial_t W^\varepsilon$ is bounded in $L^\infty([0,T]; H^{-7/2})$.

It follows that W^ε is strongly compact in $L^\infty([0,T]; L^2_{\mathrm{loc}})$, and the lemma is thus proved. □

The following lemma says that (u, v) solves the first equation of (**T**).

LEMMA 13

The functions u and v solve

$$i\partial_t u + \partial_1^2 u = u\partial_1 v.$$

Proof

We know that

$$i\partial_t u^\varepsilon + \partial_1^2 u^\varepsilon = u\partial_1 v^\varepsilon,$$

which is equivalent to

$$i\partial_t u^\varepsilon + \partial_1^2 u^\varepsilon = \partial_1\big(u^\varepsilon v^\varepsilon\big) - \partial_1 u^\varepsilon v^\varepsilon.$$

But since $v^\varepsilon \to v$ strongly in $L^\infty([0,T]; L^2_{\mathrm{loc}})$ and u^ε and $\partial_1 u^\varepsilon$ converge weakly in $L^\infty([0,T]; L^2)$, we can take the limit in the above equation; that is,

$$i\partial_t u + \partial_1^2 u = \partial_1(uv) - \partial_1 u v,$$

which yields the result of the lemma. □

In order to prove a strong compactness result for u^ε, we need the following lemma.

LEMMA 14
(i) *One has* $u\partial_1 v \in L^\infty([0,T]; L^1_{y_2}(L^2_{y_1}))$.
(ii) *Let* $u_0 \in L^1_{y_2}(L^2_{y_1})$ *and* $f \in L^\infty([0,T]; L^1_{y_2}(L^2_{y_1}))$.
Then the solution w *of*

$$\begin{cases} i\partial_t w + \partial_1^2 w = f, \\ w(0,y) = u_0(y), \end{cases}$$

is in $C([0,T]; L^1_{y_2}(L^2_{y_1}))$.

Proof
(i) One has

$$\int \left|u\partial_1 v\right|^2 dy_1 \le |u|^2_{\infty,y_1}\left(\int |\partial_1 v|^2\, dy_1\right),$$

and thus

$$\left(\int \left|u\partial_1 v\right|^2 dy_1\right)^{1/2} \le |u|_{\infty,y_1}\left(\int |\partial_1 v|^2\, dy_1\right)^{1/2},$$

so that

$$\begin{aligned}\int\left(\int \left|u\partial_1 v\right|^2 dy_1\right)^{1/2} dy_2 &\le \left(\int |u|^2_{\infty,y_1}\, dy_2\right)^{1/2}|\partial_1 v|_2 \\ &\le \sqrt{2}|u|_2^{1/2}|\partial_1 u|_2^{1/2}|\partial_1 v|_2,\end{aligned}$$

thanks to Lemma 10, and the proof is thus complete.

(ii) The function w is written

$$w = e^{i\partial_1^2 t}u_0 - i\int_0^t e^{i\partial_1^2(t-s)} f(s)\, ds,$$

but since

$$\int \left|e^{i\partial_1^2 t}u_0\right|^2 dy_1 = \int |u_0|^2\, dy_1,$$

the function $t \mapsto e^{i\partial_1^2 t}u_0$ is in $C([0,T]; L^1_{y_2}(L^2_{y_1}))$.

The proof does not differ for the component of w concerning the second member f. □

We can now state a compactness result for u^ε.

PROPOSITION 12
Let $u_0 \in L^1_{y_2}(L^2_{y_1})$.
Then $u^\varepsilon \to u$ *strongly in* $L^2([0,T]\times\mathbb{R}^2)$.

Proof

Thanks to Lemma 14(i) and (ii), we know that the weak limit u of u^ε is in $C([0,T];L^1_{y_2}(L^2_{y_1}))$.

We now introduce a regularizing sequence $\rho_\alpha(y_1)$ of $\mathbb{R}_{y_1}$, and we consider

$$\partial_t \int_{\mathbb{R}} \big(\rho_\alpha * u\big)^2\,dy_1 = 2\Re\bigg(\int \big(\rho_\alpha * \overline{u}\big)\big(\rho_\alpha * \partial_t u\big)\,dy_1\bigg).$$

We know, thanks to Lemma 13, that

$$\partial_t u - i\partial_1^2 u = -iu\partial_1 v,$$

so that

$$\begin{aligned}\partial_t \int_{\mathbb{R}} \big(\rho_\alpha * u\big)^2\,dy_1 &= 2\Re\bigg(\int \big(\rho_\alpha * \overline{u}\big)\big[i\partial_1^2\big(\rho_\alpha * u\big) - i\rho_\alpha * \big(u\partial_1 v\big)\big]\,dy_1\bigg)\\ &= 2\Im\bigg(\int \rho_\alpha * \big(u\partial_1 v\big)\big(\rho_\alpha * \overline{u}\big)\,dy_1\bigg).\end{aligned}$$

But for almost every y_2 and t we have $u\partial_1 v \in L^2_{y_1}$ (because $u \in H^1_{y_1} \subset L^\infty_{y_1}$). We therefore have $\rho_\alpha * (u\partial_1 v) \to u\partial_1 v$ in $L^2_{y_1}$ when $\alpha \to 0$.

Moreover, for almost all y_2, $u(\cdot, y_2) \in L^2_{y_1}$, and therefore $\rho_\alpha * \overline{u} \to \overline{u}$ in $L^2_{y_1}$. We have therefore

$$g_\alpha(y_2,t) := 2\Im\bigg(\int \rho_\alpha * \big(u\partial_1 v\big)\big(\rho_\alpha * \overline{u}\big)\,dy_1\bigg) \longrightarrow 0$$

almost everywhere in y_2 and t.

But we also have

$$g_\alpha(y_2,t) = -2\Im\bigg(\int \rho_\alpha * \big(\partial_1 uv\big)\big(\rho_\alpha * \overline{u}\big)\,dy_1\bigg) - 2\Im\bigg(\int \rho_\alpha * (uv)\big(\rho_\alpha * \partial_1\overline{u}\big)\,dy_1\bigg),$$

so that

$$\big|g_\alpha(y_2,t)\big| \le 4|\partial_1 u|_{2,y_1}|v|_{2,y_1}|u|_{\infty,y_1} := g(y_2,t).$$

We have

$$\begin{aligned}\int_0^t \int_{\mathbb{R}} g(y_2,t)\,dy_2\,dt &\le 4\int_0^T |\partial_1 u|_2\bigg(\int |u|^2_{\infty,y_1}\,dy_2\bigg)^{1/2} \sup_{y_2}|v|_{2,y_1}\,dt\\ &\le 8\int_0^T |\partial_1 u|_2|u|_2^{1/2}|\partial_1 u|_2^{1/2}|v|_2^{1/2}|\partial_2 v|_2^{1/2}\,dt,\end{aligned}$$

thanks to Lemmas 10 and 11, and thus, by Theorem 8,

$$\int_0^t \int_{\mathbb{R}} g(y_2,t)\,dy_2\,dt \le \mathrm{Cst}\,T.$$

We have therefore a domination condition on g_α. Since we have also seen that $g_\alpha \to 0$ almost everywhere in y_2 and t, we can conclude, thanks to Lebesgue's dominated convergence theorem, that $g_\alpha \to 0$ in $L^1([0,T]\times\mathbb{R})$.

We have therefore

$$\partial_t \int_{\mathbb{R}} |u|^2\,dy_1 = 0,$$

and therefore

$$\int_{\mathbb{R}} |u|^2\,dy_1 = \mathrm{Cst}\,.$$

We now prove that this constant is equal to $\int_{\mathbb{R}} |u_0|^2\,dy_1$. As we have $u \in C([0,T]; L^1_{y_2}(L^2_{y_1}))$, we have

$$\int \left(\int |u-u_0|^2\,dy_1 \right)^{1/2} dy_2 \longrightarrow 0 \quad \text{as } t \longrightarrow 0,$$

and therefore

$$\int |u-u_0|^2\,dy_1 \longrightarrow 0$$

when $t \to 0$ almost everywhere in y_2.

Hence, we have $\int_{\mathbb{R}} |u|^2\,dy_1 \to \int_{\mathbb{R}} |u_0|^2\,dy_1$ almost everywhere in y_2. The constant $\int_{\mathbb{R}} |u|^2\,dy_1$ is therefore equal to $\int_{\mathbb{R}} |u_0|^2\,dy_1$.

Integrating this relation with respect to y_2 yields

$$|u|_2 = |u_0|_2.$$

We recall that we also have $|u^\varepsilon|_2 = |u_0|_2$, so that u^ε converges weakly towards u, and it converges also in L^2 norm. We can therefore conclude that $u^\varepsilon \to u$ strongly in $L^2([0,T]\times\mathbb{R}^2)$, and the proposition is thus proved. □

Remark 8

Thanks to the compactness properties of $(u^\varepsilon, v^\varepsilon)$, given by Lemma 12 and Proposition 12, Theorem 8 remains valid with initial values $(u_0, v_0) \in L^2 \times H^1$ instead of $H^3 \times H^6$. One just has to consider regularizations of these initial values and then take the limit.

Thanks to Proposition 12, we can now take the limit in the expression which gives v^ε,

$$v^\varepsilon = e^{\partial_1^{-1}\partial_2^2(1+\epsilon\Delta^2)^{-1}t} v_0 - \int_0^t e^{\partial_1^{-1}\partial_2^2(1+\epsilon\Delta^2)^{-1}(t-s)} \left(1+\epsilon\Delta^2\right)^{-1} |u^\varepsilon|^2(s)\,ds,$$

and we state the following theorem.

THEOREM 9

Let (u_0, v_0) be two functions such that

- *u_0 and $\partial_1 u_0$ are in L^2, $|u_0|^2 \in H^1$, and $u_0 \in L^1_{y_2}(L^2_{y_1})$;*
- *$v_0 \in H^1$.*

Let $T > 0$. *If* $|u_0|_2 + |\partial_1 u_0|_2 + |v_0|_{H^1}$ *is small enough, then there exists* (u, v) *such that*

$$\begin{cases} i\partial_t u + \partial_1^2 u = u\partial_1 v, \\ v = e^{\partial_1^{-1}\partial_2^2 t} v_0 - \int_0^t e^{\partial_1^{-1}\partial_2^2 (t-s)} |u|^2(s)\, ds \end{cases}$$

and

$$u \in C\big([0, T]; L^2\big), \quad \partial_1 u \in L^\infty\big([0, T]; L^2\big),$$
$$u(0, y_1, y_2) = u_0(y_1, y_2),$$
$$v \in L^\infty\big([0, T]; H^1\big) \cap C\big([0, T]; L^2_{\mathrm{loc}}\big).$$

Recall that the integral equation for v used in this result makes sense since the group $e^{\partial_1^{-1}\partial_2^2 t}$ acts on every Sobolev space H^s, and, for $u \in L^\infty(0, T; L^2)$, $|u|^2$ lies in $L^\infty(0, T; H^s)$ for some negative s.

Acknowledgments. The authors want to thank J.-L. Joly and G. Métivier for fruitful discussions about this work.

References

[BMN] A. BABIN, A. MAHALOV, and B. NICOLAENKO, *Global splitting, integrability and regularity of 3D Euler and Navier-Stokes equations for uniformly rotating fluids*, European J. Mech. B Fluids **15** (1996), 291–300.

[C] T. COLIN, *Rigorous derivation of the nonlinear Schrödinger equation and Davey-Stewartson systems from quadratic hyperbolic systems*, preprint, Université Bordeaux I, 1999.

[D] P. DONNAT, *Equations hyperboliques semi-linéaires dispersives*, cours à l'Ecole Polytechnique, 1995.

[DJMR] P. DONNAT, J.-L. JOLY, G. MÉTIVIER, and J. RAUCH, "Diffractive nonlinear geometric optics" in *Séminaire sur les équations aux dérivées partielles (1995–1996)*, Sémin. Équ. Dériv. Partielles, École Polytech., Palaiseau, 1996, exp. no. 17.

[Ga] I. GALLAGHER, *Applications of Schochet's methods to parabolic equations*, J. Math. Pures Appl. (9) **77** (1998), 989–1054.

[Gr] E. GRENIER, *Oscillatory perturbations of the Navier-Stokes equations*, J. Math. Pures Appl. (9) **76** (1997), 477–498.

[IMS] P. ISAZA, J. MEJIA, and V. STALLBOHM, *Local solution for the Kadomtsev-Petviashvili equation in* $\mathbb{R}^2$, J. Math. Anal. Appl. **196** (1995), 566–587.

[JMR1] J.-L. JOLY, G. MÉTIVIER, and J. RAUCH, *Diffractive nonlinear geometric optics with rectification*, Indiana Univ. Math. J. **47** (1998), 1167–1241.

[JMR2] ———, *Transparent nonlinear geometric optics and Maxwell-Bloch equations*, J. Differential Equations **166** (2000), 175–250.

[K] T. KATO, *Perturbation Theory for Linear Operators*, Grundlehren Math. Wiss. **132**, Springer, New York, 1966.

[L1] D. LANNES, *Dispersive effects for nonlinear geometrical optics with rectification*, Asymptot. Anal. **18** (1998), 111–146.

[L2] ———, *Quelques phénomènes d'interaction d'ondes en optique non linéaire*, thèse, Université Bordeaux I, 1999.

[La] Ph. Laurençot, *On a nonlinear Schrödinger equation arising in the theory of water waves*, Nonlinear Anal. **24** (1995), 509–527.

[OT] T. OZAWA and Y. TSUTSUMI, *Existence and smoothing effect of solutions for the Zakharov equations*, Publ. Res. Inst. Math. Sci. **28** (1992), 329–361.

[S] S. SCHOCHET, *Fast singular limits of hyperbolic PDEs*, J. Differential Equations **114** (1994), 476–512.

[Su] C. SULEM and P.-L. SULEM, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse*, Appl. Math. Sci. **139**, Springer, New York, 1999.

Mathématiques Appliquées de Bordeaux, Université Bordeaux 1 et Centre National de la Recherche Scientifique Unité Mixte de Recherche 5466, 351 Cours de la Libération, 33405 Talence Cedex, France; colin@math.u-bordeaux.fr, lannes@math.u-bordeaux.fr