A numerical study of the long-wave short-wave resonance for 3D water waves

Christophe Besse [a], David Lannes [b*]

[a] *Laboratoire de Mathématiques pour l'Industrie et la Physique (UMR CNRS 5640), Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse cedex 4, France*

[b] *Mathématiques Appliquées de Bordeaux (UMR CNRS 5466), Université Bordeaux 1, 351 Cours de la Libération, 33405 Talence cedex, France*

Abstract – When long-wave short-wave resonance occurs, Davey–Stewartson systems become singular and have to be replaced by another system of equations. This is this system we study here numerically. We use a finite difference scheme for which we prove existence and uniqueness of a solution. We also prove a stability theorem and compute some invariants of the discrete system. We finally give and comment numerical experiments to study the behaviour of the solutions. © 2001 Éditions scientifiques et médicales Elsevier SAS

1. Introduction

The motion of weakly nonlinear capillary-gravity waves on water of finite depth has been studied extensively since the sixties. The variation $\eta(x, y, t)$ of the height of the free surface with respect to the state of rest is traditionnally written under the form

$$\eta(X, Y, \tau) = \varepsilon \left(\frac{i}{\omega} A(x, y, t) \sinh(kh) \, e^{i(kX - \omega\tau)} + c.c \right)\Bigg|_{x=\varepsilon(X-c_g\tau), y=\varepsilon Y, t=\varepsilon^2\tau}$$
$$+ \, \mathrm{O}(\varepsilon^2),$$

where $\omega(k)$ is given by the usual dispersion relation for water-waves

$$\omega^2 = k(g + Tk^2)\tanh(kh),$$

where T denotes the surface tension per unit density of the liquid, and $c_g = \omega'(k)$ is the group velocity.

The parameter ε is small and corresponds to the amplitude of the waves.

It is known (see for instance [1–3] for irrotational flows and [4] for the general case) that this phenomenon is governed by the Davey–Stewartson system:

$$i\,\partial_t A + \frac{1}{2}\omega''(k)\frac{\partial^2 A}{\partial x^2} + \frac{c_g}{2k}\frac{\partial^2 A}{\partial y^2} = \frac{k}{c_g \sinh(2kh)}\left(\frac{c_g\omega}{g} + \sinh(2kh)\right)\zeta A - \nu|A|^2 A,$$

$$(gh - c_g^2)\frac{\partial^2\zeta}{\partial x^2} + gh\frac{\partial^2\zeta}{\partial y^2} = -c_g^2\left(1 + \frac{g\sinh(2kh)}{c_g\omega}\right)\frac{\partial^2|A|^2}{\partial x^2},$$

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* Correspondence and reprints.
E-mail addresses: besse@mip.ups-tlse.fr (C. Besse), lannes@math.u-bordeaux.fr (D. Lannes).

where h is the mean depth of the water, g is the accelaration of gravity and c_g the group velocity. The first of these two partial differential equations is a nonlinear Schrödinger equation with a forcing term, and the second one is a linear equation which is either elliptic or hyperbolic, depending on whether the group velocity c_g of the capillary-gravity wave is (strictly) less than or greater than the velocity $\sqrt{gh}$ of the long gravity waves.

Several instabilities or resonances have been studied for this system (see the book by Mei [5], see also [6] for a study of interaction between long-waves and short-waves). We are concerned here by the long-wave short-wave resonance which occurs when the group velocity c_g of the fundamental mode (capillarity-gravity wave) is exactly equal to the phase velocity $\sqrt{gh}$ of the long-waves (long gravity waves). In 2D, this resonance appears obviously since in this case the Davey–Stewartson equations reduce to the cubic nonlinear Schrödinger equation in which the coefficient of the cubic term is proportional to $1/(c_g^2 - gh)$ which becomes infinite when $c_g^2 = gh$. Djordjevic and Redekopp ([3]) have proposed in this case a new scaling which takes into account this resonance; they have obtained the following set of amplitude equations:

$$i\partial_t A + \frac{1}{2}\omega''(k)\frac{\partial^2 A}{\partial x^2} = BA,$$

$$\partial_t B = -\alpha\partial_x|A|^2,$$

where α can be expressed explicitly (see [3]).

Sulem and Sulem ([7]) have extended this result for the 3D case, and Colin and Lannes ([8]) have proved that such phenomena also occur in a general way in nonlinear optics. Under the long-wave short-wave resonance hypothesis ($c_g^2 = gh$), the equation satisfied by A is coupled with that satisfied by ζ and reads:

$$\partial_t A - \frac{i}{2}\omega''(k)\frac{\partial^2 A}{\partial x^2} = -ik\left(\frac{\omega}{\sinh(2kh)} + \frac{g}{c_g}\right)A\zeta,$$

$$\partial_t \zeta + \frac{c_g}{2}\partial_x^{-1}\frac{\partial^2 \zeta}{\partial y^2} = -\frac{1}{2}\left(\frac{\sinh(2kh)}{\omega} + \frac{c_g}{g}\right)\partial_x|A|^2,$$

where $c_g = \omega'(k)$.

These equations generalize the system obtained by Djordjevic and Redekopp [3]. The only modification is the transverse dispersion on the mean flow ζ in the second equation, i.e. the term $\partial_x^{-1}\partial_y^2\zeta$, which is the same as in the Kadomtsev–Petviashvili (KP) equation.

Remark: Assuming that the resonance condition $c_g^2 = gh$ is exactly satisfied is not very realistic from a physical point of vue. In their book [7], Sulem and Sulem assume that it is approximately satisfied, and more precisely that $\sqrt{gh} - c_g = \varepsilon\nu$, with finite ν. In this case, the second equation of the above system must be replaced by

$$\partial_t \zeta + \nu\zeta_x + \frac{c_g}{2}\partial_x^{-1}\frac{\partial^2 \zeta}{\partial y^2} = -\frac{1}{2}\left(\frac{\sinh(2kh)}{\omega} + \frac{c_g}{g}\right)\partial_x|A|^2.$$

However, since the aim of the present article is to study the dynamics of the long-wave short-wave resonance, we will assume that the resonance condition is exactly satisfied.

From a numerical point of view, the Davey–Stewartson equation have been extensively studied (see [9–12] for instance), but as far as we know the modulation equations obtained in [7,8] under the long-wave short-wave resonance assumption has not been investigated. Moreover, very few numerical studies ([13–15]) deal with the KP equations, which are related to the system considered here since the term $\partial_x^{-1}\partial_y^2\zeta$ is present in both cases.

In fact, to our knowledge, the most related study has been made by Alterman and Rauch ([16–18]), though in a very different physical context. Indeed, they have studied the propagation of short pulses in nonlinear optics

and proved that the classical Schrödinger approximation is then not valid anymore. In the linear case, it must be replaced by the Linear Diffractive Pulse Equation

$$2u_{tx} = \Delta_y u,$$

which is the linear part of the equation of the mean flow in the system derived by Sulem and Sulem. The numerical study carried out by Alterman and deals only with the linear case.

The aim of this paper is to give a numerical study of the nonlinear modulation equations arising under the long-wave short-wave resonance assumption. In the next section, we set up the notation, give the main known properties of the system, and explain the finite differences method we use for the numerical resolution. In section 3, we study the properties of the discrete system, particularly focusing on the conservation of two quantities, and on a stability theorem. Finally, we give in the last section the algorithms we have used, and comment the numerical results obtained.

2. Numerical approach and known results

2.1. Notation

For all sequences of complex numbers $a = (a_{j,k})_{(j,k)\in\mathbb{Z}^2}$ and $b = (b_{j,k})_{(j,k)\in\mathbb{Z}^2}$, we introduce the following quantities

$$(a, b) := \sum_{(j,k)\in\mathbb{Z}^2} a_{j,k}\overline{b_{j,k}}\delta x\delta y,$$

$$(a, b)_k := \sum_{j\in\mathbb{Z}} a_{j,k}\overline{b_{j,k}}\delta x, \quad \forall k \in \mathbb{Z},$$

$$\|a\| := \left(\sum_{(j,k)\in\mathbb{Z}^2} |a_{j,k}|^2\delta x\delta y \right)^{1/2},$$

$$\|a\|_k := \left(\sum_{j\in\mathbb{Z}} |a_{j,k}|^2\delta x \right)^{1/2}, \quad \forall k \in \mathbb{Z},$$

which are finite if a and b are in $l^2(\mathbb{Z}^2)$.

We now define difference operators which, for any sequence $a = (a_{j,k})_{(j,k)\in\mathbb{Z}^2}$, are given by

$$(D_+^x a)_{j,k} = \frac{a_{j+1,k} - a_{j,k}}{\delta x},$$

$$(D_-^x a)_{j,k} = \frac{a_{j,k} - a_{j-1,k}}{\delta x},$$

$$(D_0^x a)_{j,k} = \frac{a_{j+1,k} - a_{j-1,k}}{2\delta x},$$

$$(D_+^y a)_{j,k} = \frac{a_{j,k+1} - a_{j,k}}{\delta y},$$

$$(D_+^y a)_{j,k} = \frac{a_{j,k} - a_{j,k-1}}{\delta y},$$

$$(D_0^y a)_{j,k} = \frac{a_{j,k+1} - a_{j,k-1}}{2\delta y}.$$

2.2. The continuous case

The system studied in this paper is the following

$$i\partial_t u + \partial_x^2 u = u\partial_x v, \tag{1}$$

$$\partial_t v + \partial_x^{-1}\partial_y^2 v = -|u|^2. \tag{2}$$

It is related to the system found in hydrodynamics and described above by identificating $u = A$ and $\partial_x v = \zeta$.

We list here some of the properties of this system. The proof of all the results below can be found in [8]. The first of these results is an existence theorem. The uniqueness of the solution remains open.

Theorem 1: Let u_0 and v_0 be two smooth functions, and let $T > 0$.

Then, if $\|u_0\|_{L_2} + \|\partial_x u_0\|_{L^2} + \|v_0\|_{H^1}$ is small enough there exists (u, v) such that:

  (i) $u \in C([0, T]; L^2(\mathbb{R}^2))$ and $\partial_x u \in L^\infty([0, T]; L^2(\mathbb{R}^2))$;
 (ii) $v \in L^\infty([0, T]; H^1) \cap C([0, T]; L^2_{\text{loc}}(\mathbb{R}^2))$;
(iii) (u, v) solves system (1)–(2) with initial conditions (u_0, v_0).

The following proposition gives three invariants preserved by this system.

Proposition 1: Let (u, v) be given by the above theorem. Then, for all $t \in [0, T]$,

  (i)

$$\int_{\mathbb{R}^2} |u|^2(t) = \int_{\mathbb{R}^2} |u_0|^2; \tag{3}$$

 (ii)

$$\int_{\mathbb{R}^2} \left(|\partial_x u|^2 + |u|^2 \partial_x v + \frac{1}{2}|\partial_y v|^2 \right)(t) = \int_{\mathbb{R}^2} |\partial_x u_0|^2 + |u_0|^2 \partial_x v_0 + \frac{1}{2}|\partial_y v_0|^2; \tag{4}$$

(iii)

$$\int_{\mathbb{R}^2} \left(|\partial_x v|^2 + 2iu\partial_x \overline{u} \right)(t) = \int_{\mathbb{R}^2} |\partial_x v_0|^2 + 2iu_0\partial_x \overline{u_0}. \tag{5}$$

2.3. Numerical method and results

On the one hand, the equation (1) is the classical nonlinear Schrödinger equation but without the transverse dispersion term ∂_y^2. Therefore, all usual methods, such as the Crank–Nicolson [19] or relaxation [20] ones, can be applied to treat (1) numerically. On the other hand, the second equation (2), because of the term $\partial_x^{-1}\partial_y^2$ which occurs also in KP, needs a new approach. The first idea is to attempt to take advantage of the relatively simple form of the $\partial_x^{-1}\partial_y^2$ operator in Fourier variables but unfortunately, this operator is singular on the line $\xi = 0$ where ξ denotes the dual variable of x.

However, the system of equations (1–2) can be written:

$$\begin{aligned} i\partial_t u + \partial_x^2 u &= u\partial_x v, \\ \partial_t \partial_x v + \partial_y^2 v &= -\partial_x |u|^2, \end{aligned} \tag{6}$$

with the following change

$$(2) \rightarrow \partial_x(2).$$

Note that on the line $\xi = 0$ where the above operator is singular, the Fourier transform $\widehat{v}$ of the long-wave amplitude vanishes as one can see on (6). The x-derivative in the source term indeed prevents $\widehat{v}(\xi = 0, \eta \neq 0)$ to be forced.

Let us denote by $u^n(x, y)$ the approximation of $u(t, x, y)$ with $t = n\delta t$, and $v^{n+1/2}(x, y)$ the one of $v(x, y, t)$ with $t = (n + 1/2)\delta t$. Then, the second equation of (6) reads in semi-discrete in time form

$$\partial_x \frac{v^{n+1/2} - v^{n-1/2}}{\delta t} + \partial_y^2 \frac{v^{n+1/2} + v^{n-1/2}}{2} = -\partial_x |u^n|^2. \tag{7}$$

Now, supposing that $v^{n-1/2}$ and u^n are known, equation (7) can be viewed as a non-homogeneous 1D heat equation on $v^{n+1/2}$ with x playing the role of the time variable and y of the spatial one. The problem is that we have to solve this equation on $\mathbb{R}$ and not only on the half-line. This is possible because of the peculiar form of the source term, and requires special care in the discretization of the space variables.

Hence, the numerical method consists in using the relaxation method in order to treat each equation of system (6) at a different time (e.g. in our situation at $t = (n + 1/2)\delta t$ and $t = n\delta t$). Therefore, the semi-discrete in time scheme becomes

$$i \frac{u^{n+1} - u^n}{\delta t} + \partial_x^2 \left(\frac{u^{n+1} + u^n}{2} \right) = \left(\frac{u^{n+1} + u^n}{2} \right) \partial_x v^{n+1/2},$$

$$\partial_x \left(\frac{v^{n+1/2} - v^{n-1/2}}{\delta t} \right) + \partial_y^2 \left(\frac{v^{n+1/2} + v^{n-1/2}}{2} \right) = -\partial_x |u^n|^2.$$

The spatial discretization is done by truncating the real domain $\mathbb{R}^2$ in $\Omega = [x_0, x_1] \times [y_0, y_1]$ and by using a classical Cartesian mesh. Then, finite difference methods are used. Finally, homogeneous Dirichlet boundary conditions are given for u and v on $\Gamma = \partial\Omega$. Let us denote by N_x and N_y the number of points of discretization and set $\delta x = (x_1 - x_0)/N_x$ and $\delta y = (y_1 - y_0)/N_y$. Thereby, we define a point of discretization by $\{x, y\} = \{(j - 1)\delta x, (k - 1)\delta y\}$, with $1 \leqslant j \leqslant N_x$ and $1 \leqslant k \leqslant N_y$.

Then, the complete discrete system is

$$i \frac{u_{j,k}^{n+1} - u_{j,k}^n}{\delta t} + \left(D_+^x D_-^x \left(\frac{u^{n+1} + u^n}{2} \right) \right)_{j,k} = \left(\frac{u_{j,k}^{n+1} + u_{j,k}^n}{2} \right) \left(D_0^x v^{n+1/2} \right)_{j,k}, \tag{8}$$

$$\left(D_0^x \left(\frac{v^{n+1/2} - v^{n-1/2}}{\delta t} \right) \right)_{j,k} + \left(D_+^y D_-^y \left(\frac{v^{n+1/2} + v^{n-1/2}}{2} \right) \right)_{j,k} = -\left(D_0^x |u^n|^2 \right)_{j,k}. \tag{9}$$

We prove in section 3 the existence and uniqueness of $(u^{n+1}, v^{n+1/2})$ solution to (8)–(9) in $l^2(\mathbb{Z}^2)$. In the same time, we perform the computation of two discrete invariants which are the equivalents of the first two continuous ones, namely (3) and (4). Unfortunately, the third invariant (5) is not conserved by the discrete system (8)–(9). Finally, a stability theorem is shown for small enough initial data. Section 4 is then devoted to the algorithms and numerical results of experiments.

3. Properties of the discrete system

3.1. An existence theorem for the discrete system

We prove here that the discrete system described in section 2.3 is well posed in $l^2(\mathbb{Z}^2)$.

Theorem 2: Let u^0 and $v^{-1/2}$ be two sequences of $l^2(\mathbb{Z}^2)$.

There exists one and only one sequence $(u^n, v^{n-1/2})_{n \in \mathbb{N}} \in (l^2(\mathbb{Z}^2) \times l^2(\mathbb{Z}^2))^{\mathbb{N}}$ which solves equations (8)–(9) for all $n \in \mathbb{N}$.

Proof. – Assuming that we know $(u^n, v^{n-1/2}) \in l^2(\mathbb{Z}) \times l^2(\mathbb{Z}^2)$, we prove that there exists a unique $(u^{n+1}, v^{n+1/2}) \in l^2(\mathbb{Z}) \times l^2(\mathbb{Z}^2)$ which solves equations (8)–(9).

First of all, we define the discrete Fourier transform.

DEFINITION 1: *Let $a \in l^2(\mathbb{Z}^2)$. The discrete Fourier transform of a is the function $\widehat{a} \in L^2(\mathbb{T}^2)$ defined as*

$$\widehat{a}(\xi, \eta) = \sum_{(j,k) \in \mathbb{Z}^2} a_{j,k}\, e^{i(j\xi + k\eta)}.$$

We now prove the uniqueness part for $v^{n+1/2}$; we thus assume the existence of a sequence $v^{n+1/2} \in l^2(\mathbb{Z}^2)$ satisfying equation (9) for all $(j, k) \in \mathbb{Z}^2$. Taking the discrete Fourier transform of equation (9) yields

$$i\frac{\sin(\xi)}{\delta x \delta t}\big(\widehat{v^{n-1/2}} - \widehat{v^{n+1/2}}\big)(\xi, \eta) + \frac{2\cos(\eta) - 2}{2\delta y^2}\big(\widehat{v^{n+1/2}} + \widehat{v^{n-1/2}}\big)(\xi, \eta)$$

$$= i\frac{\sin(\xi)}{\delta x}\widehat{|u^n|^2}(\xi, \eta), \quad (\xi, \eta) \in \mathbb{T}^2,$$

which can also read

$$\left(-i\frac{\sin(\xi)}{\delta x \delta t} - \frac{\sin^2(\eta/2)}{\delta y^2}\right)\widehat{v^{n+1/2}}(\xi, \eta) = \left(-i\frac{\sin(\xi)}{\delta x \delta t} + \frac{\sin^2(\eta/2)}{\delta y^2}\right)\widehat{v^{n-1/2}}(\xi, \eta) + i\frac{\sin(\xi)}{\delta x}\widehat{|u^n|^2}(\xi, \eta).$$

Introducing $\lambda := (-i\frac{\sin(\xi)}{\delta x \delta t} - \frac{\sin^2(\eta/2)}{\delta y^2})$, we have therefore, when $(\xi, \eta) \notin \pi\mathbb{Z} \times 2\pi\mathbb{Z}$,

$$\widehat{v^{n+1/2}}(\xi, \eta) = -\frac{\overline{\lambda}}{\lambda}\widehat{v^{n-1/2}}(\xi, \eta) + i\frac{\sin(\xi)}{\lambda \delta x}\widehat{|u^n|^2}(\xi, \eta). \tag{10}$$

Since $|\overline{\lambda}/\lambda| = 1$ and

$$\left|i\frac{\sin(\xi)}{\lambda \delta x}\right| \leqslant \frac{|\sin(\xi)|}{\frac{1}{\delta x \delta t}|\sin(\xi)|\delta x} \leqslant \delta t,$$

these two functions are L^∞ multipliers and the two terms of the right-hand side of equation (10) are in L^2 since we have assumed that u^n and $v^{n-1/2}$ are in l^2.

Hence, if $v^{n+1/2}$ exists and is in $l^2(\mathbb{Z}^2)$ then its discrete Fourier transform is given by equation (10), which proves the uniqueness part.

Taking the inverse Fourier transform of the L^2 function given by the right-hand side of equation (10) proves the existence part for $v^{n+1/2}$.

We now just have to prove existence and uniqueness of a function u^{n+1}, knowing u^n, $v^{n-1/2}$ and $v^{n+1/2}$.

Classical techniques used for the Schrödinger equation give a solution $u^{n+1} \in l^2(\mathbb{Z}^2)$ and the conservation of the l^2 norm we prove below in section 3.2 gives the uniqueness part. $\quad\square$

Remark: The fact that the right-hand side of equation (9) is a derivative in x is crucial in the proof of the above theorem. It it were not the case the L^∞ multiplier $i\frac{\sin(\xi)}{\lambda \delta x}$ would be replaced by $\frac{1}{\lambda \delta x}$, which is not an L^∞ multiplier, so that the theorem would not be valid.

3.2. *Some invariant quantities*

Throughout this section, we will denote by $(u^n, v^{n-1/2})_{n \in \mathbb{N}}$ the unique sequence of $l^2(\mathbb{Z}^2) \times l^2(\mathbb{Z}^2)$ which solves (8)–(9) for all $n \in \mathbb{N}$, and given in section 2.3.

The following proposition says that the first invariant of the continuous system is preserved by the discrete system (8)–(9).

Proposition 2: The l^2 norm of u^n is conserved, that is,

$$\forall n \in \mathbb{N}, \quad \|u^n\| = \|u^0\|.$$

More precisely, the l^2 norm with respect to the first variable is conserved,

$$\forall n \in \mathbb{N}, \ \forall k \in \mathbb{Z}, \quad \|u^n\|_k = \|u^0\|_k.$$

Proof. – Multiplying equation (8) by $2\overline{z_{j,k}^{n+1/2}} = \overline{u_{j,k}^{n+1} + u_{j,k}^n}$ and taking the imaginary part yields

$$\frac{|u_{j,k}^{n+1}|^2 - |u_{j,k}^n|^2}{\delta t} + 2\Im\big((D_+^x D_-^x z^{n+1/2})_{j,k} \, \overline{z_{j,k}^{n+1/2}}\big) = 0,$$

so that summing on j yields

$$\frac{\|u^{n+1}\|_k^2 - \|u^n\|_k^2}{\delta t} + 2\Im(D_-^x D_+^x z^{n+1/2}, z^{n+1/2})_k = 0,$$

and since a discrete integration by parts yields the nullity of the second term of the left-hand side of this equality, we can conclude that $|u^{n+1}|_k^2 = |u^n|_k^2$, for all k in $\mathbb{Z}$. The end of the proof is then straightforward. $\quad\square$

The following proposition asserts that the second invariant of the continuous system has a discrete equivalent.

Proposition 3: The quantity I_n defined as

$$I_n := \|D_+^1 u^n\|^2 + (|u^n|^2, D_0^1 v^{n-1/2}) + \frac{1}{2}\|D_+^2 v^{n-1/2}\|^2$$

is independent of $n \in \mathbb{N}$.

Proof. – Multiplying equation (8) by $\overline{u_{j,k}^{n+1} - u_{j,k}^n}$ and taking the real part yields

$$2\Re\big((D_-^x D_+^x z^{n+1/2})_{j,k} \, (\overline{u_{j,k}^{n+1} - u_{j,k}^n})\big) = (|u_{j,k}^{n+1}|^2 - |u_{j,k}^n|^2)(D_0^x v^{n+1/2})_{j,k}.$$

Summing this expression on $j \in \mathbb{Z}$ yields, for all $k \in \mathbb{Z}$,

$$2\Re(D_-^x D_+^x z^{n+1/2}, u^{n+1} - u^n)_k = (|u^{n+1}|^2, D_0^x v^{n+1/2})_k - (|u^n|^2, D_0^x v^{n+1/2})_k.$$

However, one has

$$\begin{aligned}
2\Re(D_-^x D_+^x z^{n+1/2}, u^{n+1} - u^n)_k &= -2\Re(D_+^x z^{n+1/2}, D_+^x(u^{n+1} - u^n))_k \\
&= -\Re(D_+^x(u^{n+1} + u^n), D_+^x(u^{n+1} - u^n))_k \\
&= \|D_+^x u^n\|_k^2 - \|D_+^x u^{n+1}\|_k^2,
\end{aligned}$$

so that

$$\|D_+^x u^n\|_k^2 - \|D_+^x u^{n+1}\|_k^2 = (|u^{n+1}|^2, D_0^x v^{n+1/2})_k - (|u^n|^2, D_0^x v^{n+1/2})_k. \tag{11}$$

We now study the right-hand side of this equality

$$(|u^{n+1}|^2 - |u^n|^2, D_0^x v^{n+1/2})_k = (|u^{n+1}|^2, D_0^x v^{n+1/2})_k - (|u^n|^2, D_0^x v^{n-1/2})_k$$
$$- (|u^n|^2, D_0^x(v^{n+1/2} - v^{n-1/2}))_k. \tag{12}$$

However, we can deduce from equation (9) that

$$(|u^n|^2, D_0^x(v^{n+1/2} - v^{n-1/2}))_k = -\delta t\,(|u^n|^2, D_-^y D_+^y w^n)_k - \delta t\,(|u^n|^2, D_0^x |u^n|^2)_k$$
$$= -\delta t\,(|u^n|^2, D_-^y D_+^y w^n)_k. \tag{13}$$

We would like to perform a discrete integral by parts with respect to the first variable on this last term, but we first have to prove that the discrete integral of $W := D_-^y D_+^y w^n$ is well defined. This is what asserts the following lemma.

   Lemma 1: For all j and all k in $\mathbb{Z}$, the sum

$$S_{j,k} = 2\delta x \sum_{l=0}^{\infty} W_{j-1-2l,k}$$

converges. Moreover, We have the following properties:
   – for all j and k in $\mathbb{Z}$, $D_0^x S = W$,
   – the sum $S_{j,k}$ is given by

$$S_{j,k} = -\frac{v_{j,k}^{n+1/2} - v_{j,k}^{n-1/2}}{\delta t} - |u_{j,k}^n|^2.$$

   Proof. – If $S_{j,k}$ is well defined, it is easy to check that the last assertion of the lemma is satisfied. We now prove the convergence of $S_{j,k}$.

   Equation (9) writes, at point $(j - 1 - 2l, k)$ and for all $l \in \mathbb{Z}$

$$\frac{1}{\delta t}\big((D_0^x v^{n+1/2})_{j-1-2l,k} - (D_0^x v^{n-1/2})_{j-1-2l,k}\big) + W_{j-1-2l,k} = -(D_0^x |u^n|^2)_{j-1-2l,k}.$$

Summing this expression form $l = 0$ until $l = L$ thus yields

$$\frac{v_{j,k}^{n+1/2} - v_{j-2-2L,k}^{n+1/2} - v_{j,k}^{n-1/2} + v_{j-2-2L,k}^{n-1/2}}{2\delta x \delta t} + \sum_{l=0}^{L} W_{j-1-2l,k} = -\frac{|u_{j,k}^n|^2 - |u_{j-2-2L,k}^n|^2}{2\delta x}.$$

Since $v^{n-1/2}$, $v^{n+1/2}$ and u^n are in l^2, one has, for all $k \in \mathbb{Z}$,

$$\lim_{L\to\infty} (v^{n-1/2}, v^{n+1/2}, |u^n|^2)_{j-2-2L,k} = 0,$$

so that

$$S_{j,k} = -\frac{v_{j,k}^{n+1/2} - v_{j,k}^{n-1/2}}{\delta t} - |u_{j,k}^n|^2,$$

which concludes the proof of the lemma.

We can now perform the desired integration by parts on equation (13)

$$
\begin{aligned}
\left(|u^n|^2, D_0^x(v^{n+1/2} - v^{n-1/2})\right)_k &= -\delta t\left(|u^n|^2, W\right)_k \\
&= -\delta t\left(|u^n|^2, D_0^x S\right)_k \\
&= \delta t\left(D_0^x|u^n|^2, S\right)_k,
\end{aligned}
$$

where the convergence of the last term can easily be deduced from the above lemma. Once again, using equation (9) thus yields

$$
\begin{aligned}
\left(|u^n|^2, D_0^x(v^{n+1/2} - v^{n-1/2})\right)_k &= -\left(D_0^x(v^{n+1/2} - v^{n-1/2}), S\right)_k - \delta t(W, S)_k \\
&= \left(v^{n+1/2} - v^{n-1/2}, D_0^x S\right)_k - \delta t\left(D_0^x S, S\right)_k \\
&= \left(v^{n+1/2} - v^{n-1/2}, W\right)_k.
\end{aligned}
$$

Now, replacing W by its expression $W = D_-^y D_+^y w^n$ in the above equality, and summing over k yields

$$
\begin{aligned}
\left(|u^n|^2, D_0^x(v^{n+1/2} - v^{n-1/2})\right) &= \left(v^{n+1/2} - v^{n-1/2}, D_-^y D_+^y w^n\right) \\
&= -\left(D_+^y(v^{n+1/2} - v^{n-1/2}), D_+^y w^n\right) \\
&= -\frac{1}{2}\left(D_+^y(v^{n+1/2} - v^{n-1/2}), D_+^y(v^{n+1/2} + v^{n-1/2})\right),
\end{aligned}
$$

since we recall that $2w^n = (v^{n+1/2} + v^{n-1/2})$. Since $v^{n+1/2}$ and $v^{n-1/2}$ are sequences of real numbers, we have therefore

$$
\left(|u^n|^2, D_0^x(v^{n+1/2} - v^{n-1/2})\right) = -\frac{1}{2}\left(\|D_+^y v^{n+1/2}\|^2 - \|D_+^y v^{n-1/2}\|^2\right).
$$

We now sum equation (12) over k and replace $(|u^n|^2, D_0^x(v^{n+1/2} - v^{n-1/2}))$ by the above expression to find

$$
\begin{aligned}
\left(|u^{n+1}|^2 - |u^n|^2, D_0^x v^{n+1/2}\right) &= \left(|u^{n+1}|^2, D_0^x v^{n+1/2}\right) - \left(|u^n|^2, D_0^x v^{n-1/2}\right) \\
&\quad + \frac{1}{2}\left(\|D_+^y v^{n+1/2}\|^2 - \|D_+^y v^{n-1/2}\|^2\right),
\end{aligned}
$$

so that equation (11) becomes

$$
\begin{aligned}
\|D_+^x u^n\|_k - \|D_+^x u^{n+1}\|_k^2 &= \left(|u^{n+1}|^2, D_0^x v^{n+1/2}\right) - \left(|u^n|^2, D_0^x v^{n-1/2}\right) \\
&\quad + \frac{1}{2}\left(\|D_+^y v^{n+1/2}\|^2 - \|D_+^y v^{n-1/2}\|^2\right).
\end{aligned}
$$

We can also write this last equality under the form $I_{n+1} = I_n$, and the proposition is thus proved.

Remark: The third invariant of the continuous system is not preserved by the discrete system we consider here. In order to conserve this third invariant we would have to consider a non-linear discrete system, far more difficult to handle.

3.3. A stability theorem

The following theorem asserts that the discrete system (8)–(9) is stable if the initial data are small enough.

Theorem 3: If the initial data u^0 and $v^{-1/2}$ are small enough then there exists $\underline{T} > 0$ such that for all $n = 0, \ldots, N$, with $N = \underline{T}/\delta t$, the l^2 norm of the solution $(u^n, v^{n-1/2})$ of the discrete system (8)–(9) is bounded in $l^2(\mathbb{Z}^2)^2$. More precisely, one has

– $\|u^n\| = \|u^0\|$, for all $n = 0, \ldots, N$;
– $\|v^{n-1/2}\|^2 + \|D_+^x u^n\|^2 + \frac{1}{2}\|D_+^y v^{n-1/2}\|^2 \leqslant 1/6$, for all $n = 0, \ldots, N$.

Remark: At the end of this section, we give explicit conditions for the smallness of the initial data, as well as an explicit condition for the computation of $\underline{T}$.

We first give two lemmas we will often use throughout the proof of the stability theorem.

Lemma 2: Let $a \in l^2(\mathbb{Z}^2)$. Then one has

$$\sum_{k \in \mathbb{Z}} \Big(\sup_{j \in \mathbb{Z}} |a_{j,k}|^2\Big) \delta y \leqslant 2\|a\|\|D_+^x a\|.$$

Proof. – For all $(j, k) \in \mathbb{Z}^2$, one can write

$$|a_{j,k}|^2 = \Re \sum_{l=-\infty}^{k-1} (a_{j+1,k} + a_{j,k})\overline{(a_{j+1,k} - a_{j,k})}$$

$$= \Re \sum_{l=-\infty}^{k-1} \big((a_{j+1,k} + a_{j,k})\sqrt{\delta x}\big)\big((D_+^x a)_{j,k}\sqrt{\delta x}\big)$$

$$\leqslant 2\|a\|_k \|D_+^x a\|_k,$$

the last inequality being a consequence of Cauchy–Schwartz inequality. One has therefore

$$\sup_{j \in \mathbb{Z}} |a_{j,k}|^2 \leqslant 2\|a\|_k \|D_+^x a\|_k, \tag{14}$$

and summing over k and using Cauchy–Schwartz inequality with respect to k then yields the desired result.

Lemma 3: Let b be a sequence of real numbers in $l^2(\mathbb{Z}^2)$. Then one has

$$\sup_{k \in \mathbb{Z}} \|b\|_k \leqslant \sqrt{2}\|b\|^{1/2}\|D_+^y b\|^{1/2}.$$

Proof. – Introduce $\psi(k) := \|b\|_k$. We can remark that

$$\frac{\psi(k+1) - \psi(k)}{\delta x} = \frac{1}{\delta x}\frac{\psi(k+1)^2 - \psi(k)^2}{\psi(k+1) + \psi(k)} = \frac{\sum_{j \in \mathbb{Z}}(b_{j,k+1} + b_{j,k})(D_+^y b)_{j,k}\delta x}{\psi(k+1) + \psi(k)},$$

so that Cauchy–Schwartz inequality yields

$$\frac{|\psi(k+1) - \psi(k)|}{\delta x} \leqslant \|D_+^y b\|_k.$$

It is easy to see that $\psi \in l^2(\mathbb{Z})$, and that $(\sum_k |\psi(k)|^2 \delta y)^{1/2} = \|b\|$. Therefore, using the majoration of the derivative of ψ we have just obtained, together with inequality (14), yields

$$\sup_{k \in \mathbb{Z}} \psi(k) \leqslant \sqrt{2}\|b\|^{1/2}\|D_+^y b\|^{1/2},$$

which proves the lemma.

Proof of the theorem. – The first step in the proof of the stability theorem is given by the following lemma. It gives a first control of the growth of $\|D_+^1 u^n\|^2$ and $\|D_+^2 v^{n-1/2}\|^2$.

Lemma 4: If $(u^n, v^{n-1/2})$ is given by the existence Theorem 2, and if I denotes the invariant given by Proposition 3, then one has

$$\|D_+^1 u^n\|^2 + \frac{1}{2}\|D_+^2 v^{n-1/2}\|^2 \leqslant I + 4\|u^0\|^{1/2}\|v^{n-1/2}\|^{1/2}\|D_+^y v^{n-1/2}\|^{1/2}\|D_+^x u^n\|^{3/2}.$$

Proof. – One has

$$\|D_+^1 u^n\|^2 + \frac{1}{2}\|D_+^2 v^{n-1/2}\|^2 = I - \left(|u^n|^2, D_0^1 v^{n-1/2}\right). \tag{15}$$

We now transform the scalar product appearing in the right-hand side as follows

$$\left(|u^n|^2, D_0^1 v^{n-1/2}\right) = -\left(D_0^x |u^n|^2, v^{n-1/2}\right) = -2\Re\left(\overline{z^{n+1/2}} D_0^x u^n, v^{n-1/2}\right),$$

where we recall that $2z^{n+1/2} = u^{n+1} + u^n$. One has therefore

$$\left|\left(|u^n|^2, D_0^1 v^{n-1/2}\right)\right| \leqslant 2\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}}|u_{j,k}^n| \left(D_0^x u^n, v^{n-1/2}\right)_k \delta y$$

$$\leqslant 2\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}}|u_{j,k}^n| \|v^{n-1/2}\|_k \|D_0^x u^n\|_k \delta y$$

$$\leqslant 2\sup_{k\in\mathbb{Z}}\|v^{n-1/2}\|_k \sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}}|u_{j,k}^n| \|D_0^x u^n\|_k \delta y$$

$$\leqslant 2\sup_{k\in\mathbb{Z}}\|v^{n-1/2}\|_k \left(\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}}|u_{j,k}^n|^2 \delta y\right)^{1/2} \|D_0^x u^n\|,$$

which yields, thanks to Lemma 2,

$$\left|\left(|u^n|^2, D_0^1 v^{n-1/2}\right)\right| \leqslant 2\sqrt{2}\sup_{k\in\mathbb{Z}}\|v^{n-1/2}\|_k \|u^n\|^{1/2}\|D_+^x u^n\|^{1/2}\|D_0^x u^n\|.$$

Since $\|D_0^x u^n\| \leqslant \|D_+^x u^n\|$, we thus have

$$\left|\left(|u^n|^2, D_0^1 v^{n-1/2}\right)\right| \leqslant 2\sqrt{2}\sup_{k\in\mathbb{Z}}\|v^{n-1/2}\|_k \|u^n\|^{1/2}\|D_+^x u^n\|^{3/2}.$$

Finally, we use Lemma 3 to obtain

$$\left|\left(|u^n|^2, D_0^1 v^{n-1/2}\right)\right| \leqslant 4\|v^{n-1/2}\|^{1/2}\|D_+^y v^{n-1/2}\|^{1/2}\|u^n\|^{1/2}\|D_+^x u^n\|^{3/2}$$

$$= 4\|u^0\|^{1/2}\|v^{n-1/2}\|^{1/2}\|D_+^y v^{n-1/2}\|^{1/2}\|D_+^x u^n\|^{3/2},$$

the last inequality being a consequence of the conservation of the l^2 norm of u. equation (15) therefore yields

$$\|D_+^1 u^n\|^2 + \frac{1}{2}\|D_+^2 v^{n-1/2}\|^2 \leqslant I + 4\|u^0\|^{1/2}\|v^{n-1/2}\|^{1/2}\|D_+^y v^{n-1/2}\|^{1/2}\|D_+^x u^n\|^{3/2}, \tag{16}$$

which proves the lemma.

In order to exploit the above lemma, we need to control the growth of $\|v^{n-1/2}\|$. Such a control is given by the following lemma.

Lemma 5: If $(u^n, v^{n-1/2})_n$ solves the discrete system (8)–(9), then, with the notations introduced above,

$$\|v\|_T^2 \leqslant \|v^{-1/2}\|^2 + 4T\|u^0\|^{3/2}\|D_+^x u\|_T^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2}.$$

Proof. – As usual, we write $2w^n = v^{n+1/2} + v^{n-1/2}$.

With S being, as in Lemma 1, the discrete integral of $D_-^y D_+^y w^n$ with respect to the first variable, one has, thanks to Lemma 1,

$$(S, w^n) = -2\left(\frac{v^{n+1/2} - v^{n-1/2}}{\delta t}, w^n\right) - (|u^n|^2, w^n),$$

and since integrations by parts show that $(S, w^n) = 0$, one then has

$$\|v^{n+1/2}\|^2 = \|v^{n-1/2}\|^2 - 2\delta t(|u^n|^2, w^n). \tag{17}$$

We now look for an upper bound of $(|u^n|^2, w^n)$. One has

$$(|u^n|^2, w^n) \leqslant \sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}}|u_{j,k}^n|\|u^n\|_k\|w^n\|_k\delta y \leqslant \sup_{k\in\mathbb{Z}}\|w^n\|_k \sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}}|u_{j,k}^n|\|u^n\|_k\delta y$$

$$\leqslant \sup_{k\in\mathbb{Z}}\|w^n\|_k\left(\sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}}|u_{j,k}^n|^2\delta y\right)^{1/2}\|u^n\|,$$

thanks to Cauchy–Schwartz inequality. Now, using Lemmas 2 and 3 yields

$$(|u^n|^2, w^n) \leqslant \sqrt{2}\|w^n\|^{1/2}\|D_+^y w^n\|^{1/2}\sqrt{2}\|u\|^{1/2}\|D_+^x u\|^{1/2}\|u^n\|$$

$$= 2\|u^0\|^{3/2}\|w^n\|^{1/2}\|D_+^y w^n\|^{1/2}\|D_+^x u^n\|^{1/2}, \tag{18}$$

since the l^2 norm of u^n is conserved.

Equations (17)–(18) therefore yield

$$\|v^{n+1/2}\|^2 \leqslant \|v^{n-1/2}\|^2 + 2\delta t\|u^0\|^{3/2}\|v^{n+1/2} + v^{n-1/2}\|^{1/2}\|D_+^y(v^{n+1/2} + v^{n-1/2})\|^{1/2}\|D_+^x u^n\|^{1/2}, \tag{19}$$

and therefore

$$\|v\|_T^2 \leqslant \|v^{-1/2}\|^2 + 4T\|u^0\|^{3/2}\|D_+^x u\|_T^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2},$$

which proves the lemma.

From Lemma 4 we can easily deduce

$$\|D_+^x u\|_T^2 + \frac{1}{2}\|D_+^y v\|_T^2 \leqslant I + 4\|u^0\|^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2}\|D_+^x u\|_T^{3/2}.$$

Summing this inequality with the inequality given by Lemma 5 we obtain

$$\|v\|_T^2 + \|D_+^x u\|_T^2 + \frac{1}{2}\|D_+^y v\|_T^2$$

$$\leqslant I + \|v^{-1/2}\|^2 + 4\|u^0\|^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2}\|D_+^x u\|_T^{3/2} + 4T\|u^0\|^{3/2}\|D_+^x u\|_T^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2}.$$

However, Young's inequality yields

$$4T\|u^0\|^{3/2}\|D_+^x u\|_T^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2} \leqslant \frac{1}{4}(4T\|u^0\|^{3/2})^4 + \frac{1}{4}\|v\|_T^2 + \frac{1}{4}\|D_+^y v\|_T^2 + \frac{1}{4}\|D_+^x u\|_T^2,$$

so that

$$\frac{1}{2}\big(\|v\|_T^2 + \|D_+^x u\|_T^2 + \frac{1}{2}\|D_+^y v\|_T^2\big)$$

$$\leqslant I + \|v^{-1/2}\|^2 + 4\|u^0\|^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2}\|D_+^x u\|_T^{3/2} + \frac{1}{4}\big(4T\|u^0\|^{3/2}\big)^4.$$

We now use another time Young's inequality to obtain

$$\|u^0\|^{1/2}\|v\|_T^{1/2}\|D_+^y v\|_T^{1/2}\|D_+^x u\|_T^{3/2} \leqslant \frac{1}{8}\big(3\|u^0\|^{4/3} + \|v\|_T^4 + \|D_+^y v\|_T^4 + 3\|D_+^x u\|_T^4\big)$$

$$\leqslant \frac{3}{8}\|u^0\|^{4/3} + \frac{3}{8}\big(\|v\|_T^4 + \|D_+^y v\|_T^4 + \|D_+^x u\|_T^4\big),$$

so that

$$\frac{1}{2}\big(\|v\|_T^2 + \|D_+^x u\|_T^2 + \frac{1}{2}\|D_+^y v\|_T^2\big)$$

$$\leqslant I + \|v^{-1/2}\|^2 + \frac{3}{2}\|u^0\|^{4/3} + \frac{1}{4}\big(4T\|u^0\|^{3/2}\big)^4 + \frac{3}{2}\big(\|v\|_T^4 + \|D_+^y v\|_T^4 + \|D_+^x u\|_T^4\big).$$

Introducing $f := \|v\|_T^2 + \|D_+^x u\|_T^2 + \frac{1}{2}\|D_+^y v\|_T^2$, the above inequality reads

$$f \leqslant A + 3f^2,$$

with $A := 2I + 2\|v^{-1/2}\|^2 + 3\|u^0\|^{4/3} + \frac{1}{2}(4T\|u^0\|^{3/2})^4.$

The inequation $3f^2 - f + A \geqslant 0$ is not satisfied for all f if its discriminent $\Delta := 1 - 12A$ is positive. Indeed, if $\Delta > 0$, then either $f \leqslant (1 - \sqrt{\Delta})/6$ or $f \geqslant (1 + \sqrt{\Delta})/6$. In order to have an upper bound for f, we want to be sure that we are in the first case.

From now on, we assume that $\Delta > 0$, which is the case if the initial data are small enough.

Denoting by r^2 the positive number $r^2 = (1 - \sqrt{\Delta})/6$ and introducing $f_n = \|v^{n-1/2}\|^2 + \|D_+^x u^n\|^2 + \frac{1}{2}\|D_+^y v^{n-1/2}\|^2$, we want to prove that if $f_k \leqslant r^2$ for $k = 1, \ldots, n$, then we also have $f_{n+1} \leqslant r^2$.

We thus assume that $f_k \leqslant r^2$ for $k = 1, \ldots, n$.

Since for all $a \in l^2(\mathbb{Z}^2)$, one has $\|D_+^y a\| \leqslant 2/\delta y \|a\|$, we can deduce from inequality (19) that

$$\|v^{n+1/2}\|^2 \leqslant \|v^{n-1/2}\|^2 + 2\sqrt{2}\frac{\delta t}{\sqrt{\delta y}}\|u^0\|^{3/2}\|v^{n+1/2} + v^{n-1/2}\|\|D_+^x u^n\|^{1/2}$$

$$\leqslant \|v^{n-1/2}\|^2 + 2\sqrt{2}\frac{\delta t}{\sqrt{\delta y}}\|u^0\|^{3/2}\|D_+^x u^n\|^{1/2}\big(\|v^{n+1/2}\| + \|v^{n-1/2}\|\big)$$

$$\leqslant r^2 + 2\sqrt{2}\frac{\delta t}{\sqrt{\delta y}}\|u^0\|^{3/2}r^{1/2}\big(\|v^{n+1/2}\| + r\big),$$

thanks to the induction asumption. The inequality obtained is of the form $\|v^{n+1/2}\|^2 - \alpha\|v^{n+1/2}\| - \beta \leqslant 0$, with $\alpha = 2\sqrt{2}\frac{\delta t}{\sqrt{\delta y}}\|u^0\|^{3/2}r^{1/2}$ and $\beta = r^2 + r\alpha$. The study of the function $X^2 - \alpha X - \beta$ yields that one has necessarily

$$\|v^{n+1/2}\| \leqslant r + \alpha. \tag{20}$$

We also know thanks to Lemma 4 that

$$\|D_+^x u^{n+1}\|^2 + \frac{1}{2}\|D_+^y v^{n+1/2}\|^2 \leqslant I + 4\|u^0\|^{1/2}\|v^{n+1/2}\|^{1/2}\|D_+^y v^{n+1/2}\|^{1/2}\|D_+^x u^{n+1}\|^{3/2},$$

so that, thanks to Young's inequality,

$$\left\|D_+^x u^{n+1}\right\|^2 + \frac{1}{2}\left\|D_+^y v^{n+1/2}\right\|^2 \leqslant I + 4\left\|u^0\right\|^{1/2}\left\|v^{n+1/2}\right\|^{1/2}\left(\frac{1}{4}\left\|D_+^y v^{n+1/2}\right\|^2 + \frac{3}{4}\left\|D_+^x u^{n+1}\right\|^2\right).$$

Using inequality (20), we thus obtain

$$\left\|D_+^x u^{n+1}\right\|^2 + \frac{1}{2}\left\|D_+^y v^{n+1/2}\right\|^2 \leqslant I + B\left(\left\|D_+^x u^{n+1}\right\|^2 + \frac{1}{2}\left\|D_+^y v^{n+1/2}\right\|^2\right),$$

with $B = 3\|u^0\|^{1/2}(r + \alpha)^{1/2}$.

From now on, we assume that $B < 1$, which is the case if the initial data are small enough.

In that case, we deduce from the above inequality that

$$\left\|D_+^x u^{n+1}\right\|^2 + \frac{1}{2}\left\|D_+^y v^{n+1/2}\right\|^2 \leqslant \frac{I}{1 - B}. \tag{21}$$

From equations (20) and (21), we obtain therefore

$$f_{n+1} \leqslant \frac{I}{1 - B} + (r + \alpha)^2,$$

and since we have already proved that either $f_{n+1} \leqslant r^2$ or $f_{n+1} \geqslant r^2 + \sqrt{\Delta}/3$, we just have to prove that $f_{n+1} < r^2 + \sqrt{\Delta}/3$ to be sure that it is smaller than r^2. This condition is satisfied if

$$\frac{I}{1 - B} + (r + \alpha)^2 < r^2 + \frac{\sqrt{\Delta}}{3},$$

which is the case if the initial data are small enough (for null initial data, this inequality reads $0 < a/3$).

Since $r^2 < 1/6$, the theorem is thus proved, since the conservation of the l^2 norm of u^n has already been proved.

We now list the explicit conditions under which the theorem is valid.

3.4. Explicit conditions of validity for Theorem 3

3.4.1. Condition 1

$$A := 2I + 2\left\|v^{-1/2}\right\|^2 + 3\left\|u^0\right\|^{4/3} + \frac{1}{2}\left(4\underline{T}\|u^0\|^{3/2}\right)^4 < 12,$$

where I is the invariant given by Proposition 3.

3.4.2. Condition 2

$$B := 3\|u^0\|^{1/2}\left(r + 2\sqrt{2}\frac{\delta t}{\sqrt{\delta y}}\|u^0\|^{3/2}r^{1/2}\right)^{1/2} < 1,$$

where $r^2 = (1 - \sqrt{1 - 12A})/6$.

3.4.3. Condition 3

$$\frac{I}{1-B} + \left(r + 2\sqrt{2}\frac{\delta t}{\sqrt{\delta y}}\|u^0\|^{3/2}r^{1/2}\right)^2 < r^2 + \frac{\sqrt{1-12A}}{3}.$$

4. Numerical algorithms and results

4.1. Algorithms

We recall that the numerical semi-discrete in time scheme is defined by

$$i\frac{u^{n+1}-u^n}{\delta t} + \partial_x^2\left(\frac{u^{n+1}+u^n}{2}\right) = \left(\frac{u^{n+1}+u^n}{2}\right)\partial_x v^{n+1/2},$$

$$\partial_x\left(\frac{v^{n+1/2}-v^{n-1/2}}{\delta t}\right) + \partial_y^2\left(\frac{v^{n+1/2}+v^{n-1/2}}{2}\right) = -\partial_x|u^n|^2.$$

Let us explain how we resolve each equations. In order to reduce the CPU time, let us denote by $v^{n+1/2} = \frac{u^{n+1}+u^n}{2}$ and $w^n = \frac{v^{n+1/2}+v^{n-1/2}}{2}$. Thus, the scheme becomes

$$2i\frac{v^{n+1/2}-u^n}{\delta t} + \partial_x^2 v^{n+1/2} = \partial_x v^{n+1/2}v^{n+1/2},$$

$$\partial_x\left(\frac{2w^n - 2v^{n-1/2}}{\delta t}\right) + \partial_y^2 w^n = -\partial_x|u^n|^2.$$

Thus, we have

$$\left(\frac{2i}{\delta t}I + \partial_x^2 - \partial_x v^{n+1/2}\right)v^{n+1/2} = \frac{2i}{\delta t}u^n,$$

$$\left(\frac{2}{\delta t}\partial_x + \partial_y^2\right)w^n = -\partial_x|u^n|^2 + \frac{2}{\delta t}\partial_x v^{n-1/2}.$$

So, in totally discrete form, it reads

$$\left(\frac{2i}{\delta t}I + D_+^x D_-^x - \left(D_0^x v^{n+1/2}\right)_{j,k}\right)v_{j,k}^{n+1/2} = \frac{2i}{\delta t}u_{j,k}^n,$$

$$\left(\frac{2}{\delta t}D_0^x + D_+^y D_-^y\right)w_{j,k}^n = -\left(D_0^x|u^n|^2\right)_{j,k} + \frac{2}{\delta t}\left(D_0^x v^{n-1/2}\right)_{j,k},$$

for $2 \leqslant j \leqslant N_x - 1$ and $1 \leqslant k \leqslant N_y$. For a point on the boundary of Ω, we have

$$\left(\frac{2i}{\delta t}I + D_+^x D_-^x - \left(D_+^x v^{n+1/2}\right)_{j,k}\right)v_{j,k}^{n+1/2} = \frac{2i}{\delta t}u_{j,k}^n,$$

$$\left(\frac{2}{\delta t}D_+^x + D_+^y D_-^y\right)w_{j,k}^n = -\left(D_+^x|u^n|^2\right)_{j,k} + \frac{2}{\delta t}\left(D_+^x v^{n-1/2}\right)_{j,k},$$

for $j = 1$, $1 \leqslant k \leqslant N_y$, and

$$\left(\frac{2i}{\delta t}I + D_+^x D_-^x - \left(D_-^x v^{n+1/2}\right)_{j,k}\right)v_{j,k}^{n+1/2} = \frac{2i}{\delta t}u_{j,k}^n,$$

$$\left(\frac{2}{\delta t}D_-^x + D_+^y D_-^y\right)w_{j,k}^n = -\left(D_-^x|u^n|^2\right)_{j,k} + \frac{2}{\delta t}\left(D_-^x v^{n-1/2}\right)_{j,k},$$
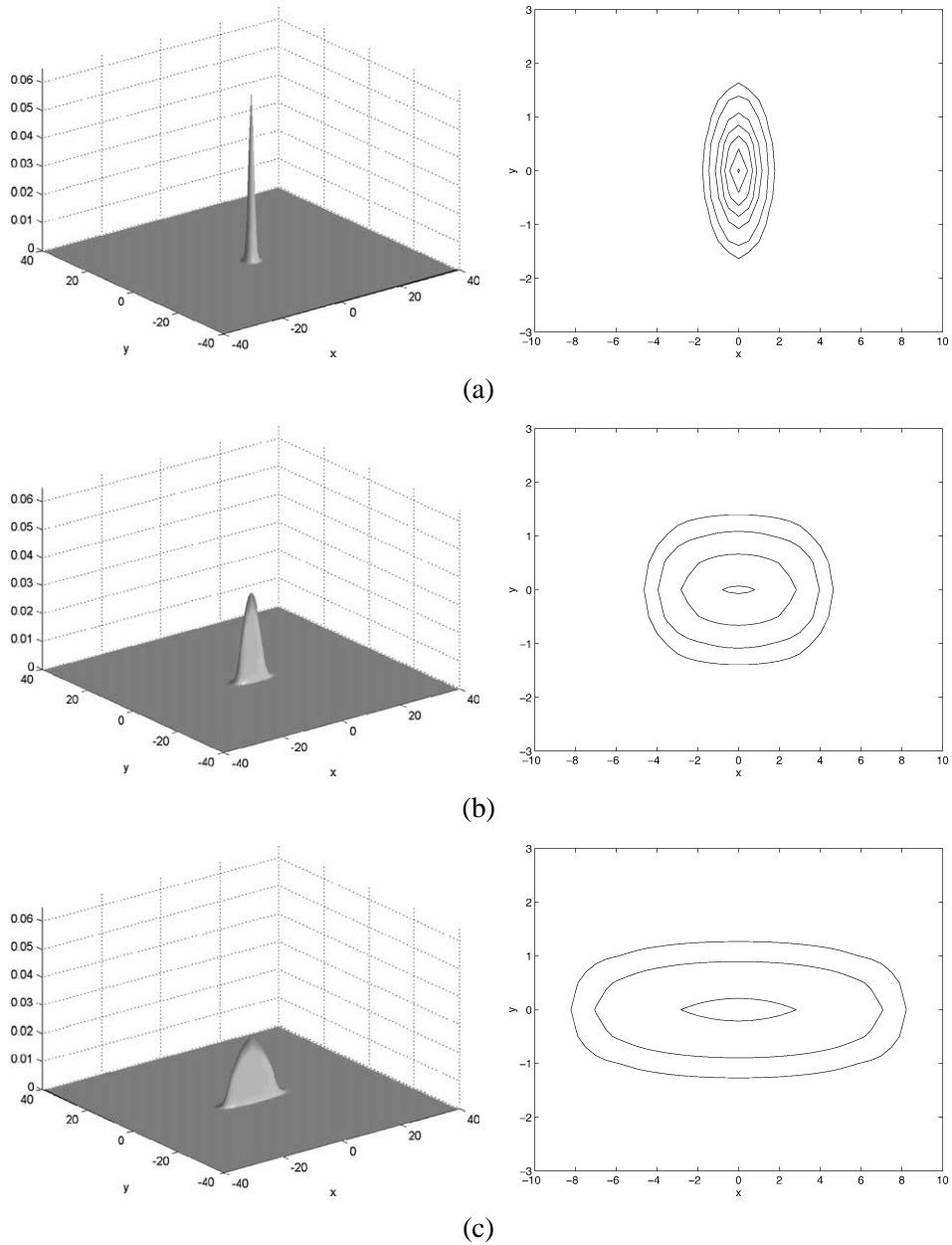
(a)

(b)

(c)

Figure 1. u and contour of $|u|$ at: (a) $t = 0.1$; (b) $t = 1.0$; (c) $t = 2.0$.

for $j = N_x$, $1 \leqslant k \leqslant N_y$.

Therefore, we have for each equation a linear system to solve which gives $v^{n+1/2}$ and w^n. Finally, we get u^{n+1} by $u^{n+1} = 2v^{n+1/2} - u^n$ and $v^{n+1/2}$ by $v^{n+1/2} = 2w^n - v^{n-1/2}$.
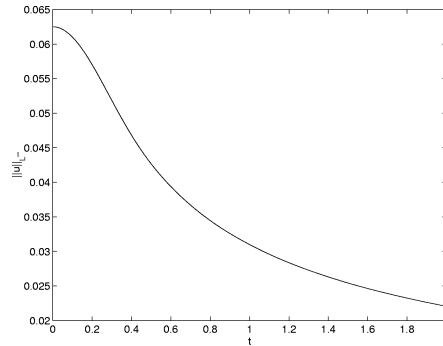
Figure 2. $\|u\|_\infty$.

4.2. Experiments

As for classical nonlinear Schrödinger equation, we take a Gaussian function as initial datum. In order to verify the hypothesis of smallness of initial datum, we take $u_0(x, y) = \frac{\exp-(x^2+y^2)}{16}$. Unfortunately, we do not know an a priori initial datum for v. However, it represents physically the mean flow, that must be null a time 0. The experiment is done on a box $[-160, 40] \times [-40, 40]$. The discretization steps are set to $\delta t = 0.001$, $\delta x = \delta y = 1/2$. Actually, the constant N_x and N_y are $N_x = 480$ and $N_y = 160$. We plot hereafter the modulus of u and v for different times. Moreover, we show the decay of L^∞ norm of u, the relative L^2 norm error, i.e., $\frac{\|u^n\|_2-\|u^0\|_2}{\|u_0\|_2}$, and the relative error made on second invariant I_n, i.e., $\frac{I^n-I^0}{I_0}$.

Remark: In order to have small boundary conditions, we have to take big boxes for v because this function decays very slowly. The mean flow v has only slow decay because if it were strongly decreasing, its Fourier transform would be regular, which is not the case due to the presence of the symbol $\exp(-it\eta^2/\xi)$ in the associated semi-group.

Figure 1 describes the evolution of $|u|$ with t. Likewise, we plot on the same plot lines the contour of $|u|$. One has to notice the remarkable behaviour of the modulus of u. As expected by the analysis of equation (1), the dispersion is only in the x-direction which we see on the contour figures. At this stage, nonlinear dynamics cannot be observed on u. In order to see them, one must go beyond the validity conditions of Theorem 3, considering for instance large initial data, as we do in section 4.3.

We expect a decay of L^∞ norm of u because the system is dispersive. We get this behaviour on *figure 2*. *Figure 3* shows how the solution v behaves. It is striking that influence propagates to the left and that the trail left by the nonlinearity which acts as a source term in the second equation of system (6) is parabolic. This is due to the fact that the linear part of this equation is simply the wave equation in coordinates rotated 45 degrees. The domain of determination at a point $(\underline{t}, \underline{x}, \underline{y})$ is therefore a backward light cone whose intersection with the plane $\{t = 0\}$ is a parabolic region. This has been discussed in [18].

Figure 3(e) presents reflexions on boundary $x = -160$. This is due to the homogeneous Dirichlet conditions. Finally, we show the good result of relative error for L^2 norm and I^n invariant (*figure 4*).

4.3. Large data

In this section, we consider 'large' initial data, where the factor $1/16$ has been removed from the expression of u_0. That is, we consider here $u_0(x, y) = \exp(-x^2 - y^2)$. The results of such an experiment are interesting
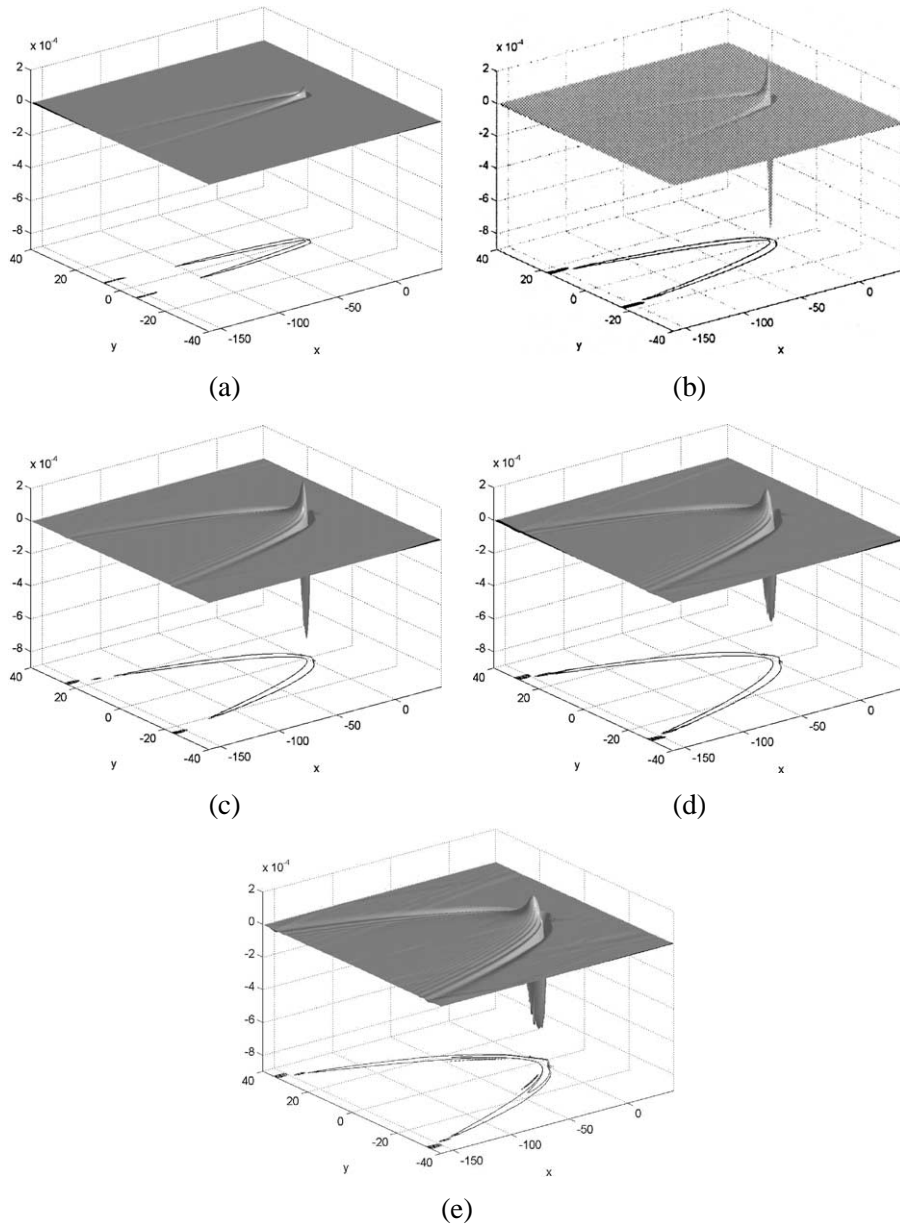
(a)   (b)

(c)   (d)

(e)

Figure 3. v at: (a) $t = 0.1$; (b) $t = 0.5$; (c) $t = 1.0$; (d) $t = 1.5$; (e) $t = 2.0$.

for two reasons: the numerical scheme we have proposed appears to be stable beyond the conditions stated in Theorem 3, and the dynamics we observe on u is not purely dispersive as previously, but nonlinear as *figure 5* shows.

4.4. Stability with respect to perturbations

In this section, we observe numerically that perturbing the initial conditions in frequency or in space does not affect significantly the behaviour of the solution.

(a)

(b)
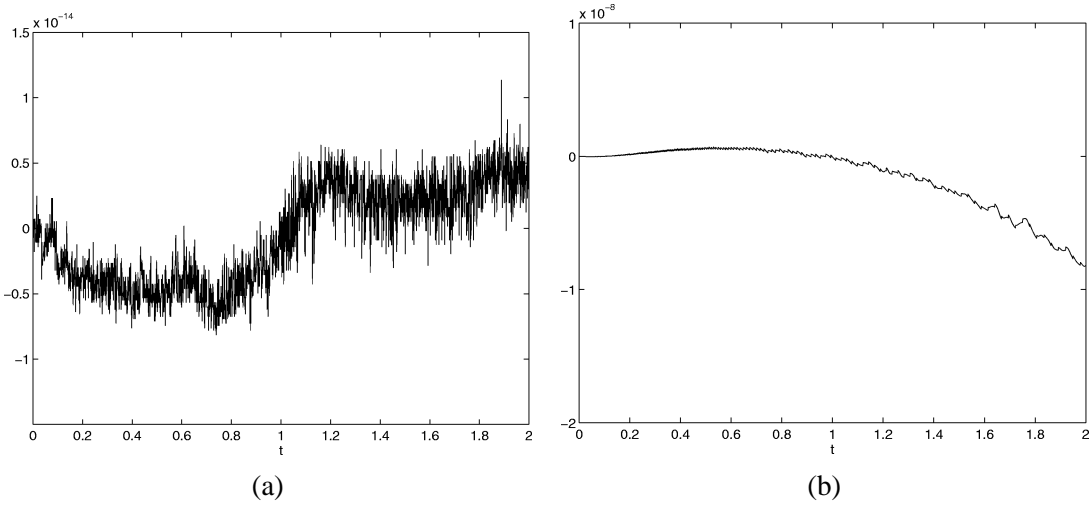
Figure 4. Relative error: (a) of L^2 norm; (b) on I^n.

(a)

(b)

(c)

(d)
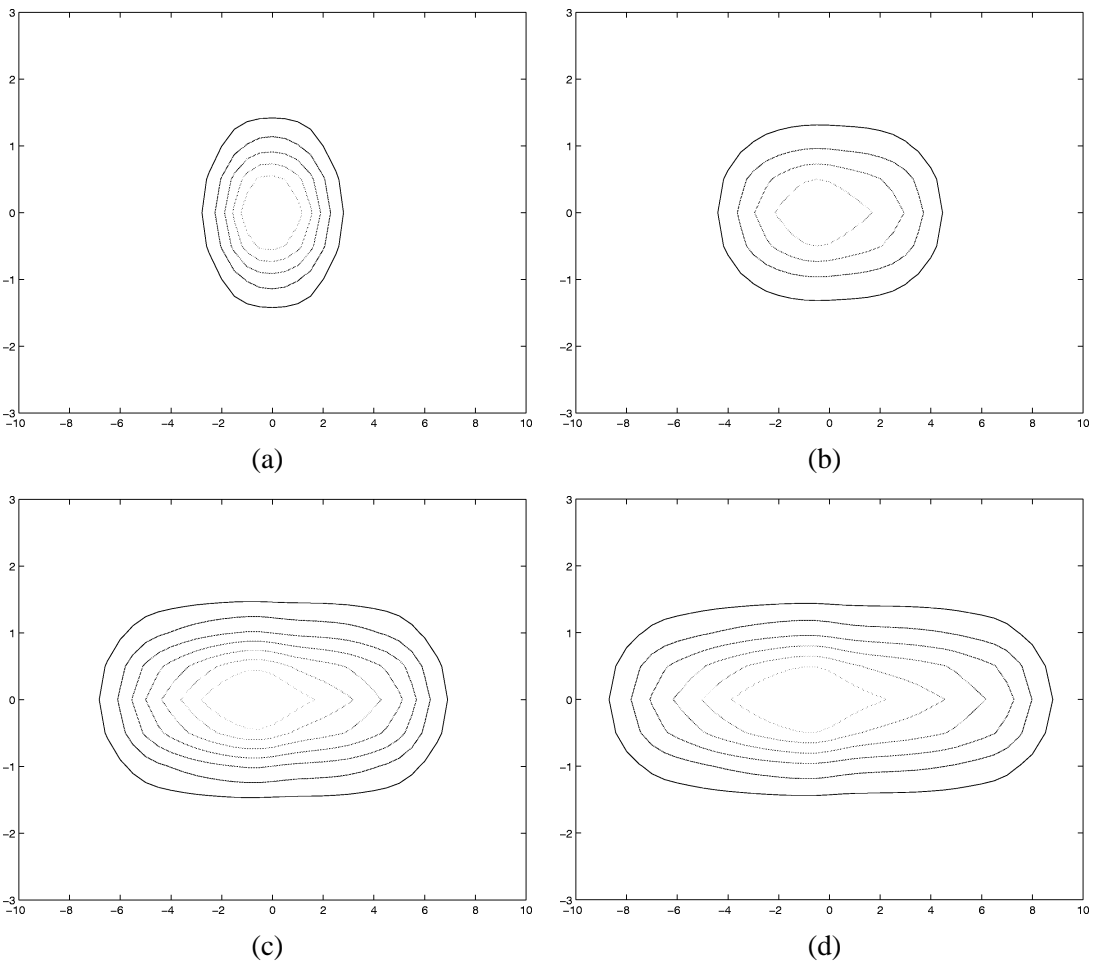
Figure 5. Contour of $|u|$ at (a) $t = 0.5$; (b) $t = 1.0$; (c) $t = 1.5$; (d) $t = 2.0$.
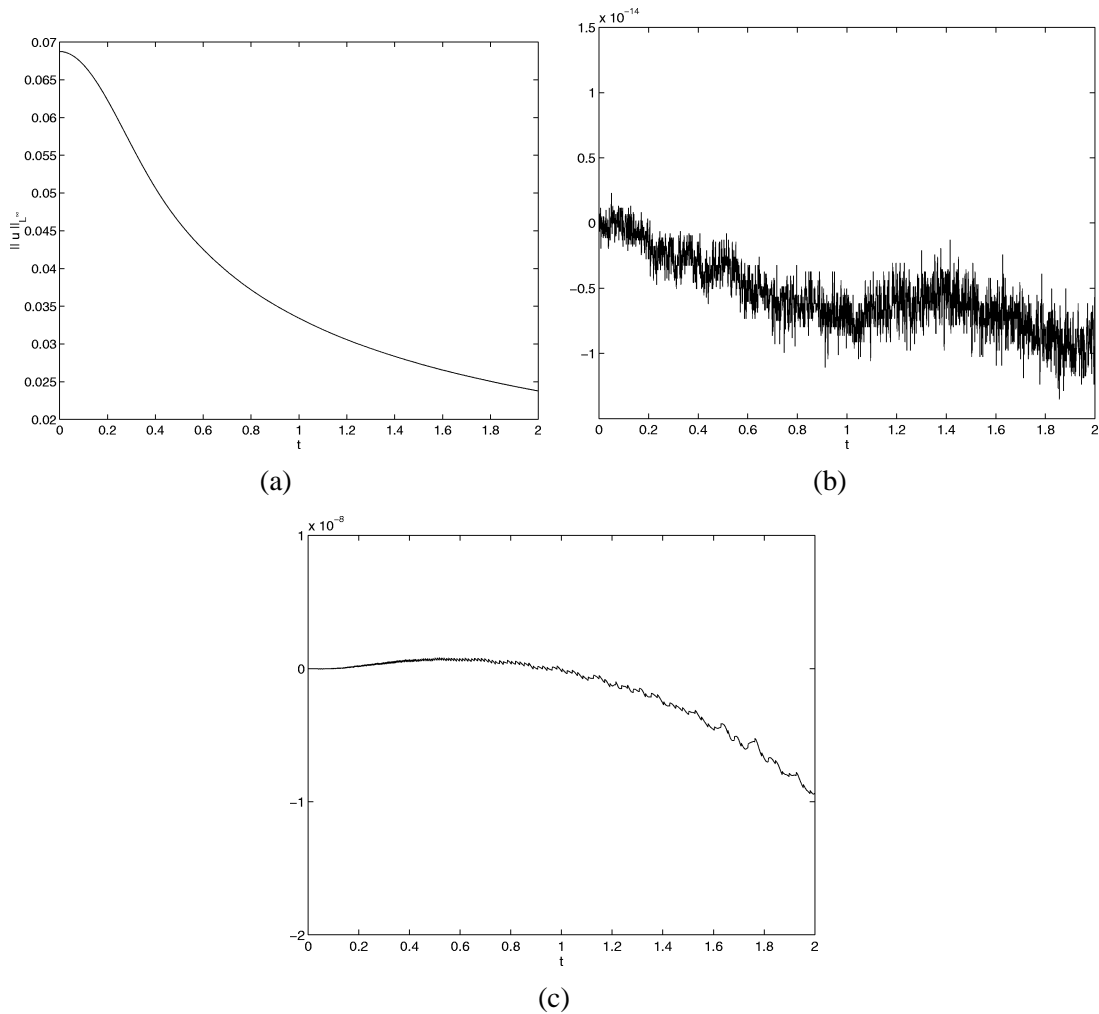
(a)

(b)

(c)

Figure 6. Perturbation in frequency: (a) $\|u\|_\infty$; (b) relative error of L^2 norm; (c) relative error on I^n.

4.4.1. Perturbations in frequency

We perturb here the initial condition in frequency (i.e., we perturb its Fourier transform). More precisely, the initial condition we consider here is $u_0(x, y) = \exp(-x^2 - y^2)/16 \cdot (1 + \cos(x)/10)$.

Figure 6(a) gives the evolution in time of the L^∞ norm. It is similar to the one observed in the nonperturbed case. *Figures 6(b)* and *(c)* show that the relative errors made on the L^2 norm and on the invariant I^n remain very small.

We now give the contour of $|u|$ and the behaviour of v at times $t = 0.1$ and $t = 2.0$ (*figure 7*). Here again, one can observe that the perturbation does not affect the dynamics of the solution.

4.4.2. Perturbation in space

We perturb here the initial condition in space. More precisely, we consider $u_0(x, y) = \exp(-x^2 - y^2)/16 + \exp(-(x - 1)^2 - (y - 1)^2)/160$.
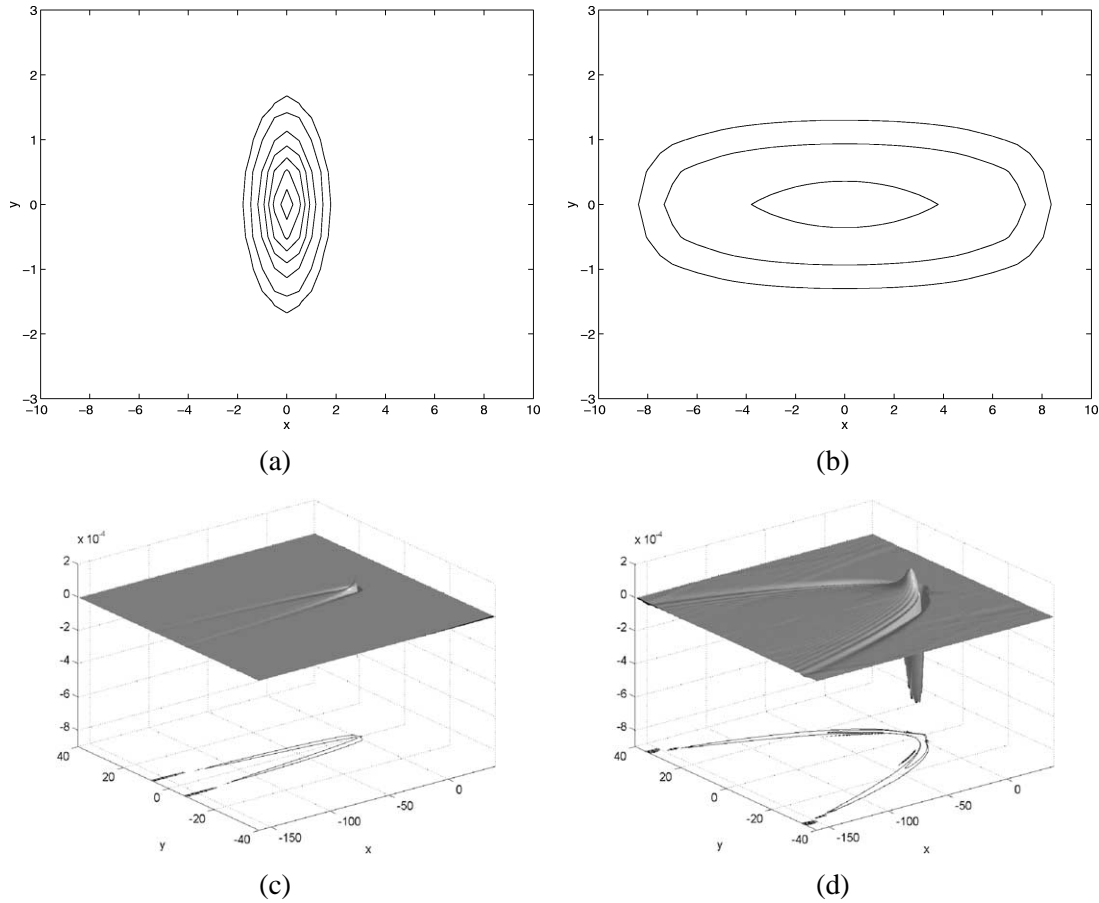
Figure 7. Perturbation in frequency: contour of $|u|$ at (a) $t = 0.1$ and (b) $t = 2.0$; v at (c) $t = 0.1$; (d) $t = 2.0$.

Here again the L^∞ norm keeps the same behaviour as in the unperturbed case, and the relative errors made on the L^2 norm and on I^n remain very small (*figure 8*).

Figure 9 shows that, as for a perturbation in frequency, the dynamics are not affected by a perturbation in space of the initial conditions.

5. Conclusion

The scheme we have studied manages to handle the main difficulty, which is the resolution of the heat equation on the whole real line. As we have shown in the existence theorem, this is due to the peculiar form of the right-hand side in equation (2), which allows us to find $v^{n-1/2}$ in $l^2(\mathbb{Z}^2)$. This function is therefore decaying, but the expression of its Fourier transform given in the proof of the existence Theorem, shows that this decay is at most algebraic. This fact is confirmed by the above numerical experiments, and appears to be the principal difficulty in the computations. Indeed, in order to neglect the boundary terms, we have to work with big boxes, and the calculus is therefore heavy.
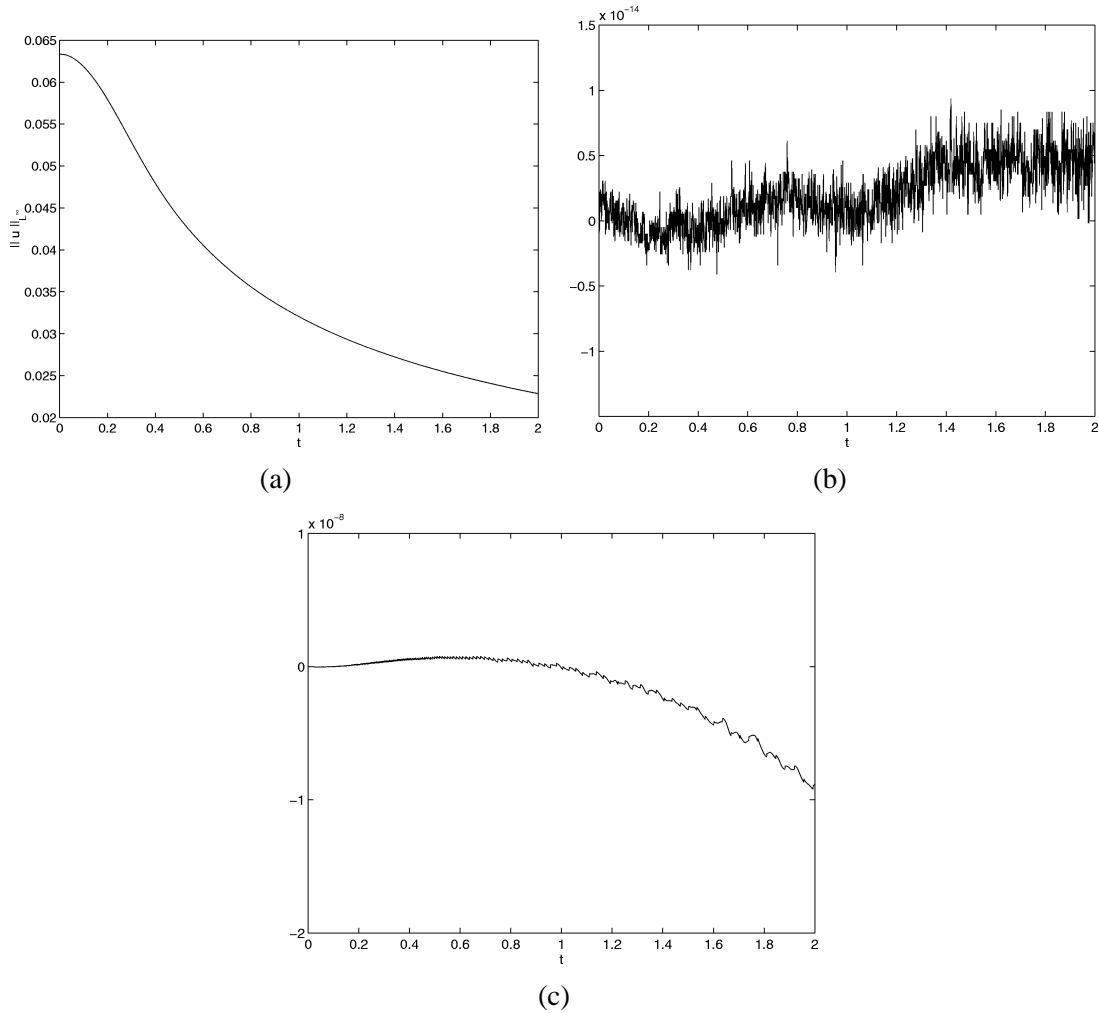
Figure 8. Perturbation in space: (a) $\|u\|_\infty$; (b) relative error of L^2 norm; (c) relative error on I^n.

As we do not have an existence/uniqueness theorem for the continuous case, we cannot expect for the moment proving the convergence of our scheme. However, we have proved its stability, and the first two invariants of the continuous system are well conserved numerically.

Another kind of approach would consist in following the proof of the existence theorem of the continuous case. In this case, the term $\partial_x^{-1}\partial_y^2$ is approximated by a regularized operator whose symbol T_μ is given by

$$T_\mu := i\frac{\xi}{\mu + \xi}\eta^2,$$

where ξ and μ denote the dual variables of x and y respectively.

The idea would be to perform a Fourier analysis of the regularized version of the second equation of system (6), and to treat the first equation as here.
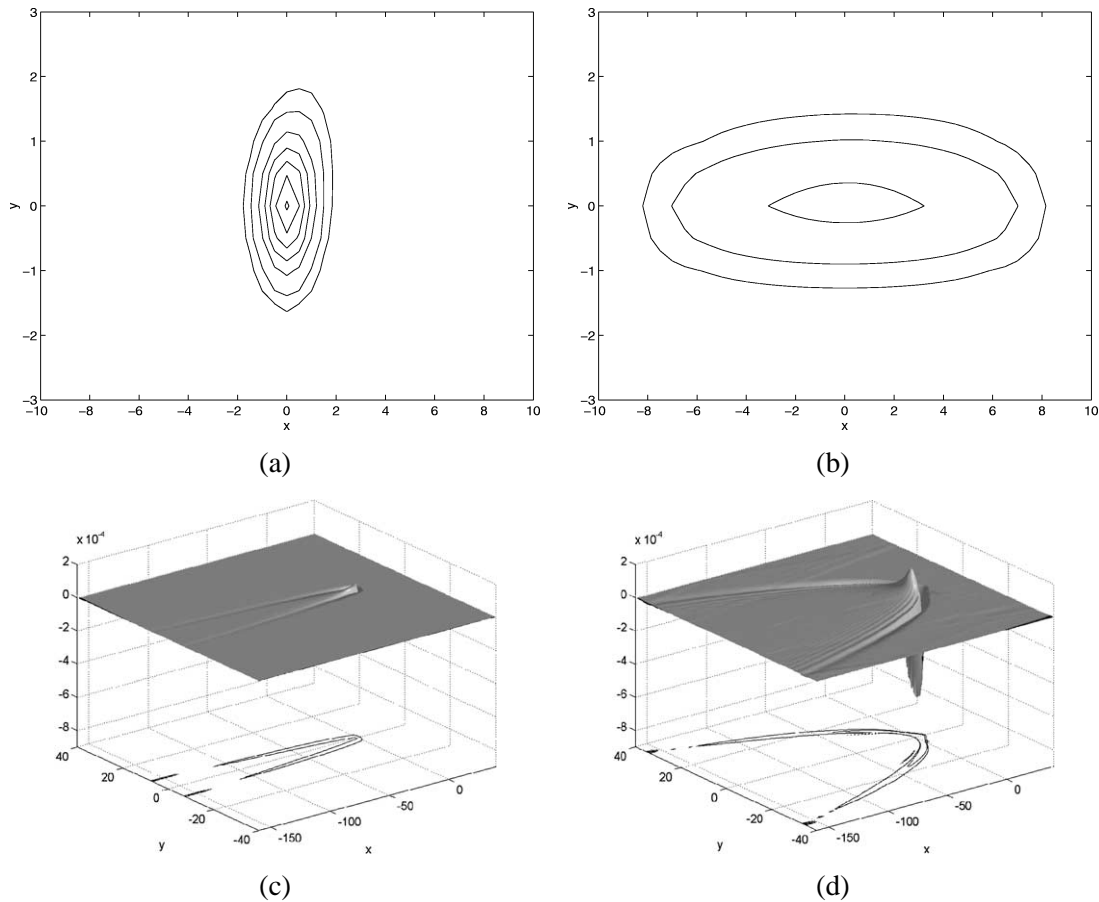
Figure 9. Perturbation in space: contour of $|u|$ at (a) $t = 0.1$ and (b) $t = 2.0$; v at (c) $t = 0.1$ and (d) $t = 2.0$.

Acknowledgements

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