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Weakly transverse Boussinesq systems and the Kadomtsev–Petviashvili approximation

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Abstract
We study here the asymptotic behaviour of weakly transverse water-waves in the long waves regime. It is well-known that the Kadomtsev–Petviashvili (KP) approximation describes formally the dynamics of the exact solutions of the water-waves equations. We provide here a rigorous justification of this approximation, showing that if solutions of the water-waves equations exist over a relevant time scale, then they are well approximated by the KP approximation. A nonphysical zero mass assumption, inherent to the structure of the KP equation, is however needed to obtain this result; this is the reason why we introduce a class of weakly transverse Boussinesq systems. These new systems provide a much more precise approximation than the KP equation and do not require any zero mass assumption.

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1. Introduction

This paper is in some sense a continuation of [4–6] where new 'Boussinesq' systems describing the interaction of long water-waves in two and three spatial dimensions were systematically derived and analysed. In particular, it was proven in [6] that solutions of any of the aforementioned systems yield good approximations to the full Euler equations on the long time scale ε^{-1} where nonlinear and dispersive effects can have an order one relative effect on the velocity field and the wave profile. Our aim here is to start from the complete free-surface Euler equations with a flat bottom (more precisely, from the free-surface Bernouilli equations, which correspond to the Euler equations written in terms of a velocity potential [22]) to derive new Boussinesq systems in the Kadomtsev–Petviashvili (KP) scaling, that is in the regime of weakly transverse long waves. As is well known (see for instance [3, 13] the KP equation has serious drawbacks of having a (unphysical) zero mass constraint [3] and a rather poor convergence rate to the full Euler system. This limits severely its applicability as a realistic model for long water-waves. Adapting the strategy of [4–6] we obtain here a class of Boussinesq systems which have the convergence rate $O(\varepsilon^2 t)$ (where ε is the small dimensionless parameter) and which does not suffer from a zero mass constraint. The only incidence of the uni-directionalization implied by the KP scaling is that the transverse component of the velocity at the surface may grow as $1/\sqrt{\varepsilon}$.

1.1. Formulation of the equations

As said above, we are interested here in the case of a flat bottom; we denote by h the depth of the fluid at rest. The horizontal variables are denoted by x and y, while z stands for the vertical coordinate ($z = 0$ being a parameterization of the surface of the fluid at rest). In this setting, it is presumed that the free surface may be described as the graph of a function ζ defined over the bottom. Since we assume that the fluid is incompressible and irrotational, there exists a velocity potential Φ such that the velocity field is given by $v = \nabla\Phi$ and the equations of motion are given by

$$\begin{array}{ll} \Delta\Phi = 0, & -h \leqslant z \leqslant \zeta(t,x,y), \\ \partial_z\Phi = 0, & z = -h, \\ \partial_t\zeta + \partial_x\Phi\partial_x\zeta + \partial_y\Phi\partial_y\zeta = \partial_z\Phi, & z = \zeta(t,x,y), \\ \partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2 + gz = 0, & z = \zeta(t,x,y), \end{array} \tag{1}$$

where the gradient operator ∇ and the Laplace operator Δ are taken with respect to the three coordinates x, y and z.

The regime of weakly transverse long waves (or KP regime) under study here can be specified in terms of the relevant characteristics of the wave, namely, its typical amplitude a, the mean depth h, the typical wavelength λ along the longitudinal direction (say, the x axis) and μ, the wavelength along the transverse direction (say, the y axis):

$$\frac{a}{h} = \varepsilon, \qquad \frac{\lambda^2}{h^2} = \frac{S_1}{\varepsilon}, \qquad \frac{\mu^2}{h^2} = \frac{S_2}{\varepsilon^2}, \tag{2}$$

where $\varepsilon \ll 1$ is a small dimensionless parameter, while $S_1 \sim 1$ and $S_2 \sim 1$ (so that the Stokes number is S_1 along the longitudinal direction and S_2/ε along the transverse one). For notational simplicity, we set $S_1 = S_2 = 1$ throughout this paper.

1.2. The nondimensionalized water-waves equations

The asymptotic study becomes more transparent when working with variables scaled in such a way that the dependent quantities and the initial data which appear in the initial value problem are all of order one. The relations (2) which set the KP regime studied here are connected with small parameters in the nondimensionalized equations of motion.

Denoting dimensionless variables with a prime, we set

$$x' = \frac{x}{\lambda}, \qquad y' = \frac{y}{\mu}, \qquad z' = \frac{z}{h}, \qquad \zeta' = \frac{\zeta}{a}, \qquad \Phi' = \frac{\Phi}{\Phi_0}, \qquad t' = \frac{\sqrt{gh}}{\lambda}t,$$

where $\Phi_0 = (a/h)\sqrt{gh}\lambda$.

[3] This constraint is due to a singular approximation of the dispersion relation of the linear wave equation in the relevant scaling (see, e.g. [16]) which measures the ratio of the typical wave elevation over the mean depth.

The equations of motion (1) then become

$$\begin{array}{ll} \frac{h^2}{\lambda^2}\partial_{x'}^2\Phi' + \frac{h^2}{\mu^2}\partial_{y'}^2\Phi' + \partial_{z'}^2\Phi' = 0, & -1 \leqslant z' \leqslant \frac{a}{h}\zeta', \\ \partial_{z'}\Phi' = 0, & z' = -1, \\ \partial_{t'}\zeta' + \frac{a}{h}\partial_{x'}\Phi'\partial_{x'}\zeta' + \frac{a\lambda^2}{h\mu^2}\partial_{y'}\Phi'\partial_{y'}\zeta' = \frac{\lambda^2}{h^2}\partial_{z'}\Phi', & z' = \frac{a}{h}\zeta', \\ \partial_{t'}\Phi' + \frac{1}{2}\left(\frac{a}{h}(\partial_{x'}\Phi')^2 + \frac{a\lambda^2}{h\mu^2}(\partial_{y'}\Phi')^2 + \frac{a\lambda^2}{h^3}(\partial_{z'}\Phi')^2\right) + \frac{h}{a}z' = 0, & z' = \frac{a}{h}\zeta'. \end{array}$$

Using the relations (2) and omitting the primes for dimensionless quantities, the above system yields

$$\begin{array}{ll} \varepsilon\partial_x^2\Phi + \varepsilon^2\partial_y^2\Phi + \partial_z^2\Phi = 0, & -1 \leqslant z \leqslant \varepsilon\zeta(t,x,y), \\ \partial_z\Phi = 0, & z = -1, \\ \partial_t\zeta + \varepsilon\partial_x\Phi\partial_x\zeta + \varepsilon^2\partial_y\Phi\partial_y\zeta = \frac{1}{\varepsilon}\partial_z\Phi, & z = \varepsilon\zeta(t,x,y), \\ \partial_t\Phi + \frac{1}{2}\left(\varepsilon(\partial_x\Phi)^2 + \varepsilon^2(\partial_y\Phi)^2 + (\partial_z\Phi)^2\right) + \zeta = 0, & z = \varepsilon\zeta(t,x,y). \end{array} \tag{3}$$

1.3. Reduction to a system of two scalar evolution equations

It is well known (see for instance [8, 9, 23] that the water-waves equations reduce to a system of two evolution equations coupling the parameterization of the free surface ζ to the value of the velocity potential at the surface, which we denote ψ. Such a reduction involves usually a Dirichlet–Neumann operator (which maps ψ to the normal derivative of the velocity potential at the surface); as in [6], we rather use the operator $Z^\varepsilon(\zeta)\cdot$ defined for all ζ smooth enough and all $s > 1/2$ as

$$Z^\varepsilon(\zeta): \begin{array}{ccc} H^s(\mathbb{R}^2) & \to & H^{s-1}(\mathbb{R}^2) \\ \psi & \mapsto & \partial_z\Phi_{|z=\varepsilon\zeta} \end{array}, \tag{4}$$

where Φ solves the boundary-value-problem

$$\begin{array}{ll} \varepsilon\partial_x^2\Phi + \varepsilon^2\partial_y^2\Phi + \partial_z^2\Phi = 0, & -1 \leqslant z \leqslant \varepsilon\zeta(x,y), \\ \partial_z\Phi = 0, & z = -1, \\ \Phi = \psi, & z = \varepsilon\zeta(x,y). \end{array}$$

Remark 1. (i) The $2D$-surface waves studied in [6] were assumed to have wavelength of the same order of magnitude in all directions. The elliptic equation determining Φ was therefore different, namely, $\varepsilon\partial_x^2\Phi + \varepsilon\partial_y^2\Phi + \partial_z^2\Phi = 0$.

(ii) The dimensionless version of the Dirichlet–Neumann operator $G^\varepsilon(\zeta)$ used in [8, 9] does not coincide with the operator $Z^\varepsilon(\zeta)$; the link between both operators is given by the relation

$$Z^\varepsilon(\zeta)\psi = \frac{1}{1+\varepsilon^3(\partial_x\zeta)^2 + \varepsilon^4(\partial_y\zeta)^2}(G^\varepsilon(\zeta)\psi + \varepsilon^2\partial_x\zeta\partial_x\psi + \varepsilon^3\partial_y\zeta\partial_y\psi);$$

one can of course work interchangeably with one or the other operator.

Remarking that $\partial_i\Phi_{|z=\varepsilon\zeta} = \partial_i\psi - \varepsilon\partial_i\zeta Z^\varepsilon(\zeta)\psi$, with $i = x, y$ or t, and plugging these relations into (3) yields the following dimensionless version of the equations derived in [8,9,23]:

$$\begin{array}{l} \partial_t\psi + \zeta - \frac{1}{2}|Z^\varepsilon(\zeta)\psi|^2 + \frac{\varepsilon}{2}\left((\partial_x\psi)^2 + \varepsilon(\partial_y\psi)^2\right) - \frac{\varepsilon^3}{2}|Z^\varepsilon(\zeta)\psi|^2\left((\partial_x\zeta)^2 + \varepsilon(\partial_y\zeta)^2\right) = 0, \\ \partial_t\zeta + \varepsilon\left(\partial_x\psi\partial_x\zeta + \varepsilon\partial_y\psi\partial_y\zeta\right) - \varepsilon^2 Z^\varepsilon(\zeta)\psi\left((\partial_x\zeta)^2 + \varepsilon(\partial_y\zeta)^2\right) = \frac{1}{\varepsilon}Z^\varepsilon(\zeta)\psi, \end{array} \tag{5}$$

which is the formulation of the water-waves problem we work on in this paper.

1.4. Description of the results

Proceeding as in [4, 5] we first derive a class of formally equivalent Boussinesq systems for weakly transverse waves, by essentially performing various transformations on the dispersive (linear) part of a reference Boussinesq system in the KP scaling. Next following [6], we perform a nonlinear transform which symmetrizes the nonlinear part, keeping the same order of approximation. We then restrict somehow this (formally equivalent) class of fully symmetric systems by eliminating those which are not linearly well-posed and (using Padé approximants) by choosing those whose dispersions have 'good' approximation properties.

In the next section, we derive rigorously the uncoupled KP approximation from the weakly transverse Boussinesq systems obtained in the previous sections. We also prove a convergence theorem for the aforementioned completely symmetric systems.

The last section is devoted to convergence results for the water-waves system. As advocated above we obtain a better approximation for the weakly transverse Boussinesq systems (convergence rate of order $O(\varepsilon^2(t))$ than for the uncoupled KP system (convergence rate $o(1)$) confirming the advantage of the former system.

Notations

- We use the generic notation C for any numerical constant; when we want to stress out the dependence of a constant on some parameters $\lambda_1, \lambda_2, \ldots$, we write $C(\lambda_1, \lambda_2, \ldots)$.
- We denote by $\mathcal{S}$ the strip $\mathbb{R}^2_{x,y} \times (-1, 0)_z$.
- For all $1 \leqslant p \leqslant \infty$, the usual $L^p(\mathbb{R}^2)$-norm is denoted by $|\cdot|_p$, and the usual $L^p(\mathcal{S})$-norm by $\|\cdot\|_p$.
- For all $f \in L^2(\mathbb{R}^2)$, we denote by $\mathcal{F}f$ or $\widehat{f}$ its Fourier transform and by $\mathcal{F}^{-1}f$ its inverse Fourier transform.
- We use the classical notation $H^s(\mathbb{R}^2)$ for Sobolev spaces over $\mathbb{R}^2$.
- For all $s \in \mathbb{R}$, we define the space $H^{s,0} = H^{s,0}(\mathcal{S})$ by

$$H^s_{tg}(\mathcal{S}) := \{\varphi \in \mathcal{D}'(\mathcal{S}), \|\varphi\|_{H^s_{tg}} := \left(\int_{-1}^0 |\varphi(\cdot,\cdot,z)|^2_{H^s(\mathbb{R}^2)} \mathrm{d}z\right)^{1/2} < \infty\}.$$

- The unit vertical vector is written $\boldsymbol{e}_z$.

2. Boussinesq systems for weakly transverse water-waves

2.1. Derivation of a weakly transverse Boussinesq system

We derive in this section a Boussinesq system which formally describes the asymptotic behaviour of the solutions to (5). This is achieved via the asymptotic expansion of the operator $Z^\varepsilon(\zeta)\cdot$ in terms of ε provided by the following proposition (see also, for instance, [6–9] for rigorous expansions of Dirichlet–Neumann operators).

Proposition 1. *Let $k \in \mathbb{N}$ and $\zeta \in W^{k+4,\infty}(\mathbb{R}^2)$. Then for all ψ such that $\nabla_{x,y}\psi \in H^{k+8}(\mathbb{R}^2)$, one has for all $\varepsilon > 0$ small enough:*

$$\left| Z^\varepsilon(\zeta)\psi - \left(\varepsilon Z_1 + \varepsilon^2 Z_2 + \varepsilon^3 Z_3\right)\right|_{H^{k+1/2}} \leqslant \varepsilon^4 C(|\zeta|_{W^{k+4,\infty}})|\nabla_{x,y}\psi|_{H^{k+8}},$$

with

$$Z_1 := -\partial_x^2\psi,$$
$$Z_2 := -\left(\tfrac{1}{3}\partial_x^4\psi + \zeta\partial_x^2\psi + \partial_y^2\psi\right),$$
$$Z_3 := -\left(\zeta\partial_y^2\psi + \tfrac{2}{3}\partial_x^2\partial_y^2\psi + \zeta\partial_x^4\psi + 2\partial_x\zeta\partial_x^3\psi + \partial_x^2\psi\partial_x^2\zeta + \tfrac{2}{15}\partial_x^6\psi\right).$$

Remark 2. For small amplitude waves, it is possible to make an asymptotic expansion of the operator $Z^\varepsilon(\zeta)\cdot$ in terms of the corresponding operator for unperturbed surfaces $Z^\varepsilon(0)\cdot$ (see [8, 9]); expanding the operator $Z^\varepsilon(0)\cdot$ in terms of ε would lead in the end to the same result as proposition 1. One could also keep the full operator $Z^\varepsilon(0)\cdot$ instead of expanding it: this is the approach used in [1, 15] and which leads to models with full dispersion; this leads to a higher computational cost (the operator $Z^\varepsilon(0)\cdot$ being non-local) and, more specifically, this method is not adapted to our present purpose of justifying rigorously the KP approximation.

Proof. First recall that if Φ solves $\varepsilon\partial_x^2\Phi + \varepsilon^2\partial_y^2\Phi + \partial_z^2\Phi = f$ on the fluid domain, then we know by lemma 2.5 of [12] that $\widetilde{\Phi}(x, y, z) = \Phi(x, y, (1+\varepsilon\zeta)z + \varepsilon\zeta)$ solves the following elliptic equation on the flat strip $\mathcal{S} = \mathbb{R}^2 \times (-1, 0)$:

$$\nabla \cdot P^\varepsilon(\zeta)\nabla\widetilde{\Phi} = 0, \tag{6}$$

with

$$P^\varepsilon(\zeta) := \begin{pmatrix} \varepsilon(1+\varepsilon\zeta) & 0 & -\varepsilon^2(1+z)\partial_x\zeta \\ 0 & \varepsilon^2(1+\varepsilon\zeta) & -\varepsilon^3(1+z)\partial_y\zeta \\ -\varepsilon^2(1+z)\partial_x\zeta & -\varepsilon^3(1+z)\partial_y\zeta & \frac{1+\varepsilon^3(1+z)^2(\partial_x\zeta)^2}{1+\varepsilon\zeta} \end{pmatrix}.$$

One checks easily that $P^\varepsilon(\zeta)$ is coercive and that for all $X = (X_1, X_2, X_3)^{\mathrm{T}} \in \mathbb{R}^3$, one has

$$X \cdot P^\varepsilon(\zeta)X \geqslant c_0(|\zeta|_{W^{1,\infty}})\big(\varepsilon X_1^2 + \varepsilon^2 X_2^2 + X_3^2\big), \tag{7}$$

for some $c_0(|\zeta|_{W^{1,\infty}}) > 0$ independent of ε. We have the following elliptic estimates.

Lemma 1. *Let $p \in \mathbb{N}^*$, $k \in \mathbb{N}$ and $\zeta \in W^{k+2,\infty}(\mathbb{R}^2)$. Let $\widetilde{\Phi}_{\rm app}$ satisfy*

$$\begin{aligned} &-\nabla \cdot P^\varepsilon(\zeta)\nabla\widetilde{\Phi}_{\rm app} = \varepsilon^p R^\varepsilon \\ &\widetilde{\Phi}_{{\rm app}\,|z=0} = \psi, \qquad \partial_z\widetilde{\Phi}_{{\rm app}\,|z=-1} = 0, \end{aligned}$$

with $(R^\varepsilon)_\varepsilon$ bounded in $H^{k+1}_{tg}(\mathcal{S})$.
Then one has, for all $\varepsilon > 0$ small enough,

$$\Big|Z^\varepsilon(\zeta)\psi - \frac{1}{1+\varepsilon\zeta}(\partial_z\widetilde{\Phi}_{\rm app})_{|z=0}\Big|_{H^{k+1/2}} \leqslant \varepsilon^p C(|\zeta|_{W^{k+2,\infty}})\|R^\varepsilon\|_{H^{k+1}_{tg}}.$$

Proof. By definition, one has $Z^\varepsilon(\zeta)\psi = (\partial_z\Phi)_{|z=\varepsilon\zeta} = (1/(1+\varepsilon\zeta))\partial_z\widetilde{\Phi}_{|z=0}$. It follows that $Z^\varepsilon(\zeta)\psi - (1/(1+\varepsilon\zeta))(\partial_z\widetilde{\Phi}_{\rm app})_{|z=0} = (1/(1+\varepsilon\zeta)\partial_z(\widetilde{\Phi} - \widetilde{\Phi}_{\rm app})_{|z=0}$. Writing $\Psi := \widetilde{\Phi}_{\rm app} - \widetilde{\Phi}$, it follows from the trace theorem that

$$\Big|Z^\varepsilon(\zeta)\psi - \frac{1}{1+\varepsilon\zeta}(\partial_z\widetilde{\Phi}_{\rm app})_{|z=0}\Big|_{H^{k+1/2}} \leqslant C(|\zeta|_{W^{k+1,\infty}})\big(\|\partial_z\Psi\|_{H^{k+1}_{tg}} + \|\partial_z^2\Psi\|_{H^k_{tg}}\big),$$

and we are thus reduced to bound $\|\partial_z\Psi\|_{H^{k+1}_{tg}}$ and $\|\partial_z^2\Psi\|_{H^k_{tg}}$ from above by $\varepsilon^p C(|\zeta|_{W^{k+2,\infty}})\|R^\varepsilon\|_{H^{k+1}_{tg}}$.
Remark that for all $k \in \mathbb{N}$ and $0 \leqslant l \leqslant k$, the quantity $\partial_x^{k-l}\partial_y^l\Psi$ solves the boundary value problem

$$\begin{aligned} &-\nabla \cdot P^\varepsilon(\zeta)\nabla\partial_x^{k-l}\partial_y^l\Psi = \varepsilon^p\partial_x^{k-l}\partial_y^l R^\varepsilon + \nabla \cdot [\partial_x^{k-l}\partial_y^l, P^\varepsilon(\zeta)]\nabla\Psi, \\ &\partial_x^{k-l}\partial_y^l\Psi_{\,|z=0} = 0, \qquad \partial_z(\partial_x^{k-l}\partial_y^l\Psi)_{\,|z=-1} = 0. \end{aligned}$$

Introducing the notation $\|\varphi\|^2_{\dot{H}^1_\varepsilon} := \varepsilon\|\partial_x\varphi\|_2^2 + \varepsilon^2\|\partial_y\varphi\|_2^2 + \|\partial_z\varphi\|_2^2$, for all functions φ defined over the strip $\mathbb{R}^2 \times (-1, 0)$, we now prove that

$$\forall k \in \mathbb{N}, \qquad \sup_{0\leqslant l\leqslant k} \|\partial_x^{k-l}\partial_y^l\Psi\|_{\dot{H}^1_\varepsilon} \leqslant \varepsilon^p C(|\zeta|_{W^{k+1,\infty}})\|R^\varepsilon\|_{H^k_{tg}}, \tag{8}$$

from which the H^{k+1}_{tg}-estimate on $\partial_z\Psi$ follows directly.

Multiplying the equation by $\partial_x^{k-l}\partial_y^l\Psi$, integrating by part and using (7) and Poincaré's inequality yields classically

$$\begin{aligned} c_0(|\zeta|_{W^{1,\infty}})\|\partial_x^{k-l}\partial_y^l\Psi\|^2_{\dot{H}^1_\varepsilon} &\leqslant \varepsilon^p\|\partial_x^{k-l}\partial_y^l R^\varepsilon\|_2\|\partial_x^{k-l}\partial_y^l\Psi\|_{\dot{H}^1_\varepsilon} \\ &\quad + \left|\left([\partial_x^{k-l}\partial_y^l, P^\varepsilon(\zeta)]\nabla\Psi, \nabla\partial_x^{k-l}\partial_y^l\Psi\right)_{L^2}\right|; \end{aligned} \tag{9}$$

(note that the boundary terms arising in the integration by parts of the commutator term vanish since $\partial_x^{k-l}\partial_y^l\Psi\,\boldsymbol{e}_z \cdot [\partial_x^{k-l}\partial_y^l, P^\varepsilon(\zeta)]\nabla\Psi = 0$ on the upper and lower boundaries of the strip $\mathbb{R}^2 \times (-1, 0)$).

When $k = l = 0$, the second term of the rhs of (9) vanishes and the result follows easily. When $k \geqslant 1$, remark that from the explicit expression of $P^\varepsilon(\zeta)$ and Poincaré's inequality, one also gets

$$\begin{aligned} &\left|\left([\partial_x^{k-l}\partial_y^l, P^\varepsilon(\zeta)]\nabla\Psi, \nabla\partial_x^{k-l}\partial_y^l\Psi\right)_{L^2}\right| \\ &\leqslant \varepsilon C(|\zeta|_{W^{k+1,\infty}}) \sup_{0\leqslant j\leqslant k} \|\partial_x^{k-j}\partial_y^j\Psi\|^2_{\dot{H}^1_\varepsilon} \end{aligned}$$

and one deduces easily from (9) that, for all $0 < \varepsilon < c_0(|\zeta|_{W^{1,\infty}})/C(|\zeta|_{W^{k+1,\infty}})$,

$$\sup_{0\leqslant l\leqslant k} \|\partial_x^{k-l}\partial_y^l\Psi\|_{\dot{H}^1_\varepsilon} \leqslant \varepsilon^p C(|\zeta|_{W^{k+1,\infty}})\|R^\varepsilon\|_{H^k_{tg}},$$

which concludes the proof of (8).

To find the required estimate on $\partial_z^2\Psi$, we use the equation satisfied by Ψ and (8), to get that, for all $k \in \mathbb{N}$, $\|\partial_z^2\Psi\|_{H^k_{tg}} \leqslant \varepsilon^p C(|\zeta|_{W^{k+2,\infty}})\|R^\varepsilon\|_{H^{k+1}_{tg}}$, which concludes the proof of the lemma. □

The end of the proof of the proposition consists of constructing an approximate solution $\widetilde{\Phi}_{\mathrm{app}}$ to (6). This is done via a standard BKW procedure, as in [6]. To avoid too lengthy computations, we only construct an approximation of order $O(\varepsilon^3)$ (that is, $p = 3$ in lemma 1); higher order terms are obtained exactly in the same way. We seek therefore $\widetilde{\Phi}_{\mathrm{app}} = \widetilde{\Phi}_0 + \varepsilon\widetilde{\Phi}_1 + \varepsilon^2\widetilde{\Phi}_2$ such that $\widetilde{\Phi}_{0\,|z=0} = \psi$, $\widetilde{\Phi}_{j\,|z=0} = 0$ $(j = 1, 2)$ and $(\partial_z\widetilde{\Phi}_0)_{|z=-1} = 0$ $(j = 0, 1, 2)$. Plugging this expression into (6), one obtains

$$\nabla \cdot P^\varepsilon(\zeta)\nabla\widetilde{\Phi}_{\mathrm{app}} = R_0 + \varepsilon R_1 + \varepsilon^2 R_2 + \varepsilon^3 R^\varepsilon, \tag{10}$$

with

$$\begin{aligned} R_0 &= \partial_z^2\widetilde{\Phi}_0, \\ R_1 &= \partial_z^2\widetilde{\Phi}_1 + (\partial_x^2 - \zeta\partial_z^2)\widetilde{\Phi}_0, \\ R_2 &= \partial_z^2\widetilde{\Phi}_2 + (\partial_x^2 - \zeta\partial_z^2)\widetilde{\Phi}_1 + \partial_x(\zeta\partial_x\widetilde{\Phi}_0) - \partial_x((1+z)\partial_x\zeta\partial_z\widetilde{\Phi}_0) + \partial_y^2\widetilde{\Phi}_0 \\ &\quad - \partial_z((1+z)\partial_x\zeta\partial_x\widetilde{\Phi}_0) + \zeta^2\partial_z^2\widetilde{\Phi}_0, \end{aligned}$$

so that choosing $\widetilde{\Phi}_0$, $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ in order to cancel R_0, R_1 and R_2 gives an approximate solution $\widetilde{\Phi}_{\mathrm{app}}$ satisfying the assumptions of lemma 1 with $p = 3$. One can easily check that the following values of $\widetilde{\Phi}_0$, $\widetilde{\Phi}_1$ and $\widetilde{\Phi}_2$ cancel R_0, R_1 and R_2 and satisfy the boundary conditions stated above:

$$\widetilde{\Phi}_0 = \psi,$$
$$\widetilde{\Phi}_1 = -\left(\frac{z^2}{2} + z\right)\partial_x^2\psi,$$
$$\widetilde{\Phi}_2 = \left(\frac{z^4}{24} + \frac{z^3}{6} - \frac{z}{3}\right)\partial_x^4\psi - \left(\frac{z^2}{2} + z\right)(2\zeta\partial_x^2\psi + \partial_y^2\psi).$$

With such a choice, the rhs of (10) reduces to $\varepsilon^3 R^\varepsilon$, and it is easy to check that $\|R^\varepsilon\|_{H^{k+1}_{tg}} \leqslant C(|\zeta|_{W^{k+3,\infty}})|\nabla_{x,y}\psi|_{H^{k+6}}$.

Therefore, one just has to compute explicitly $(\partial_z\widetilde{\Phi}_{\mathrm{app}})_{|_{z=0}}$ and to use lemma 1 to obtain the proposition (as said above, we did not give the details of the computations for the $O(\varepsilon^3)$-order term of the expansion of $Z^\varepsilon(\zeta)\psi$; note also that the whole procedure has been implemented and checked with the symbolic computational software MuPad). □

Replacing $Z^\varepsilon(\zeta)\psi$ by its asymptotic expansion provided by proposition 1 in the water-waves equations (5) and setting $v = \partial_x\psi$ and $w = \sqrt{\varepsilon}\partial_y\psi$ yields

$$\begin{aligned}
&\partial_t\psi + \zeta + \frac{\varepsilon}{2}(v^2 + w^2) = O(\varepsilon^2),\\
&\partial_t\zeta + \partial_x v + \sqrt{\varepsilon}\partial_y w + \varepsilon\big(v\partial_x\zeta + \zeta\partial_x v + \tfrac{1}{3}\partial_x^3 v\big)\\
&\quad +\varepsilon^{3/2}(w\partial_y\zeta + \zeta\partial_y w + \tfrac{2}{3}\partial_x^2\partial_y w) = O(\varepsilon^2).
\end{aligned} \tag{11}$$

Remark now that $\sqrt{\varepsilon}\partial_y v^2 = 2v\partial_x w = 2\sqrt{\varepsilon}v\partial_y v = v\partial_x w + \sqrt{\varepsilon}v\partial_y v$ and that, similarly, $\partial_x w^2 = 2w\partial_x w = 2\sqrt{\varepsilon}w\partial_y v = w\partial_x w + \sqrt{\varepsilon}w\partial_y v$. Differentiating the evolution equation on ψ in the above system with respect to x and y gives therefore the following *weakly transverse Boussinesq system*

$$\begin{aligned}
&\partial_t v + \partial_x\zeta + \varepsilon(v\partial_x v + \frac{1}{2}w\partial_x w) + \varepsilon^{3/2}\tfrac{1}{2}w\partial_y v = O(\varepsilon^2),\\
&\partial_t w + \sqrt{\varepsilon}\partial_y\zeta + \varepsilon(w\partial_y w + \tfrac{1}{2}v\partial_x w) + \varepsilon^{3/2}\tfrac{1}{2}v\partial_y v = O(\varepsilon^2),\\
&\partial_t\zeta + \partial_x v + \sqrt{\varepsilon}\partial_y w + \varepsilon\big(v\partial_x\zeta + \zeta\partial_x v + \tfrac{1}{3}\partial_x^3 v\big)\\
&\quad \varepsilon^{3/2}(w\partial_y\zeta + \zeta\partial_y w + \tfrac{2}{3}\partial_x^2\partial_y w) = O(\varepsilon^2).
\end{aligned} \tag{12}$$

Remark 3. We treat w as a $O(1)$ quantity with respect to ε, which can seem surprising since by definition $w = \sqrt{\varepsilon}\partial_y\psi$, which is formally of order $O(\sqrt{\varepsilon})$. Our choice is motivated by the structure of the equations. For instance, for initial data of the form $\psi_{|_{t=0}} = 0$, $\zeta_{|_{t=0}} = f'(y)$, an explicit solution to the linearization of (11) around the rest state is given by

$$\psi(t,x,y) = \frac{1}{2\sqrt{\varepsilon}}(f(y - \sqrt{\varepsilon}t) - f(y + \sqrt{\varepsilon}t)),$$
$$\zeta(t,x,y) = \tfrac{1}{2}(f'(y - \sqrt{\varepsilon}t) + f'(y + \sqrt{\varepsilon}t)).$$

For this solution, $\sqrt{\varepsilon}\partial_y\psi$ is of size $O(1)$ and not $O(\sqrt{\varepsilon})$.

2.2. *A first class of formally equivalent systems*

In (12), v and w denote approximations of the horizontal components of the velocity field at the free surface. Inspired by [4, 6, 17, 20], we define v_θ and w_σ as

$$v_\theta = \left(1 - \frac{\varepsilon}{2}(1-\theta^2)\partial_x^2\right)^{-1} v, \qquad w_\sigma = \left(1 - \frac{\varepsilon}{2}(1-\sigma^2)\partial_x^2\right)^{-1} w, \tag{13}$$

where $\theta, \sigma \in [0, 1]$. The quantities v_θ and w_σ are therefore approximations of the horizontal components of the velocity field at height $z = -1 + \theta$ (x-axis) and $z = -1 + \sigma$ (y-axis). As shown below, using these quantities will provide formally equivalent weakly transverse Boussinesq systems, with improved dispersion relation for some of them.

Rewriting (12) in terms of v_θ and w_σ yields therefore the following two parameters (namely, θ and σ) family of formally equivalent systems

$$\begin{aligned}
&\partial_t v_\theta + \partial_x \zeta + \varepsilon\big(-\frac{1}{2}(1-\theta^2)\partial_x^2\partial_t v_\theta + v_\theta\partial_x v_\theta + \frac{1}{2}w_\sigma\partial_x w_\sigma\big)\\
&\quad + \varepsilon^{3/2}\frac{1}{2}w_\sigma\partial_y v_\theta = O(\varepsilon^2),\\
&\partial_t w_\sigma + \sqrt{\varepsilon}\partial_y\zeta + \varepsilon\big(-\frac{1}{2}(1-\sigma^2)\partial_x^2\partial_t w_\sigma + w_\sigma\partial_y w_\sigma + \frac{1}{2}v_\theta\partial_x w_\sigma\big)\\
&\quad + \varepsilon^{3/2}\frac{1}{2}v_\theta\partial_y v_\theta = O(\varepsilon^2),\\
&\partial_t\zeta + \partial_x v_\theta + \sqrt{\varepsilon}\partial_y w_\sigma + \varepsilon\big(v_\theta\partial_x\zeta + \zeta\partial_x v_\theta + \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\partial_x^3 v_\theta\big)\\
&\quad + \varepsilon^{3/2}\big(w_\sigma\partial_y\zeta + \zeta\partial_y w_\sigma + \left(\frac{\sigma^2}{2} + \frac{1}{6}\right)\partial_x^2\partial_y w_\sigma\big) = O(\varepsilon^2).
\end{aligned} \tag{14}$$

The same kind of observation made in the derivation of the BBM equation from the KdV equation [2] can be used here to introduce three more parameters. More precisely, one has formally

$$\begin{aligned}
\partial_x^3 v_\theta &= \lambda\partial_x^3 v_\theta - (1-\lambda)\partial_x^2\partial_t\zeta - \sqrt{\varepsilon}(1-\lambda)\partial_x^2\partial_y w_\sigma + O(\varepsilon),\\
\partial_x^2\partial_t v_\theta &= (1-\mu)\partial_x^2\partial_t v_\theta - \mu\partial_x^3\zeta + O(\varepsilon),\\
\partial_x^2\partial_t w_\sigma &= (1-\nu)\partial_x^2\partial_t w_\alpha - \sqrt{\varepsilon}\nu\partial_x^2\partial_y\zeta + O(\varepsilon),
\end{aligned}$$

with $\lambda, \mu, \nu \in \mathbb{R}$ (the introduction of these three parameters gives rise to new Boussinesq systems, some of them having very good dispersive properties as shown in section 2.5).

Plugging these relations into (14) gives therefore

$$\begin{aligned}
&\partial_t v_\theta + \partial_x\zeta + \varepsilon\big(a\partial_x^3\zeta - b\partial_x^2\partial_t v_\theta + v_\theta\partial_x v_\theta + \tfrac{1}{2}w_\sigma\partial_x w_\sigma\big)\\
&\quad + \varepsilon^{3/2}\frac{1}{2}w_\sigma\partial_y v_\theta = O(\varepsilon^2),\\
&\partial_t w_\sigma + \sqrt{\varepsilon}\partial_y\zeta + \varepsilon\big(-e\partial_x^2\partial_t w_\sigma + w_\sigma\partial_y w_\sigma + \tfrac{1}{2}v_\theta\partial_x w_\sigma\big)\\
&\quad + \varepsilon^{3/2}\big(f\partial_x^2\partial_y\zeta + \tfrac{1}{2}v_\theta\partial_y v_\theta\big) = O(\varepsilon^2),\\
&\partial_t\zeta + \partial_x v_\theta + \sqrt{\varepsilon}\partial_y w_\sigma + \varepsilon\left(v_\theta\partial_x\zeta + \zeta\partial_x v_\theta + c\partial_x^3 v_\theta - d\partial_x^2\partial_t\zeta\right)\\
&\quad + \varepsilon^{3/2}\big(w_\sigma\partial_y\zeta + \zeta\partial_y w_\sigma + g\partial_x^2\partial_y w_\sigma\big) = O(\varepsilon^2),
\end{aligned} \tag{15}$$

with

$$a = \frac{1-\theta^2}{2}\mu, \qquad b = \frac{1-\theta^2}{2}(1-\mu), \tag{16}$$

$$c = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)\lambda, \qquad d = \left(\frac{\theta^2}{2} - \frac{1}{6}\right)(1-\lambda), \tag{17}$$

$$e = \frac{1}{2}(1-\sigma^2)(1-\nu), \qquad f = \frac{1}{2}(1-\sigma^2)\nu, \tag{18}$$

$$g = \left(\frac{\sigma^2}{2} + \frac{1}{6}\right) - \left(\frac{\theta^2}{2} - \frac{1}{6}\right)(1-\lambda). \tag{19}$$

Notation. Writing $\boldsymbol{p} := (\theta, \sigma, \lambda, \mu, \nu) \in [0, 1]^2 \times \mathbb{R}^3$, we denote by $S_{\boldsymbol{p}}(\partial)$ the operator formed by the lhs of (15) and corresponding to this set of parameters. We denote by $\mathfrak{S}$ the class of all the operators $S_{\boldsymbol{p}}(\partial)$ obtained for all $\boldsymbol{p} \in [0, 1]^2 \times \mathbb{R}^3$.

As in [6], we introduce the following notion of consistency.

Definition 1. *Let $T > 0$, $\varepsilon_0 > 0$, $s > 2$ and $\boldsymbol{p} := (\theta, \sigma, \lambda, \mu, \nu) \in [0, 1]^2 \times \mathbb{R}^3$.*

A family $(U^\varepsilon)_{0<\varepsilon<\varepsilon_0} := (v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$, bounded in $L^\infty([0, T/\varepsilon]; H^{s+3}(\mathbb{R}^2))^3 \cap W^{1,\infty}([0, T/\varepsilon]; H^s(\mathbb{R}^2))^3$, is consistent (of order s) with $S_{\boldsymbol{p}}(\partial)$ if $S_{\boldsymbol{p}}(\partial)U^\varepsilon = \varepsilon^2 R^\varepsilon$, with $(R^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ bounded in $L^\infty([0, T/\varepsilon]; H^s(\mathbb{R}^2))^3$.

Example 1. *From the previous section, we know that if $(\psi^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is a family of solutions of (5) such that $(v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is bounded in $L^\infty([0, T/\varepsilon], H^{s'}(\mathbb{R}^2)^3)$, with $T > 0$, $v^\varepsilon := \partial_x \psi^\varepsilon$, $w^\varepsilon := \sqrt{\varepsilon}\partial_y \psi^\varepsilon$ and s' big enough then $(v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is consistent with $S_{\boldsymbol{p}_0}(\partial)$, where $\boldsymbol{p}_0 := (1, 1, 1, 0, 0)$.*

The following proposition (similar to proposition 1 of [6]) provides a rigorous statement of the formal observations made in the derivation of (15) from (12). Its proof is a direct consequence of relations (13).

Proposition 2. *Let $\varepsilon_0 > 0$, $T > 0$ and $\boldsymbol{p} := (\theta, \sigma, \lambda, \mu, \nu) \in [0, 1]^2 \times \mathbb{R}^3$. Let also $(U^\varepsilon)_{0<\varepsilon<\varepsilon_0} := (v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be consistent with $S_{\boldsymbol{p}}(\partial)$.*

For all $\theta_1, \sigma_1 \in [0, 1]$, define v_1^ε and w_1^ε as

$$v_1^\varepsilon := \left(1 - \frac{\varepsilon}{2}(1 - {\theta_1}^2)\partial_x^2\right)^{-1}\left(1 - \frac{\varepsilon}{2}(1 - \theta^2)\partial_x^2\right)v^\varepsilon,$$
$$w_1^\varepsilon := \left(1 - \frac{\varepsilon}{2}(1 - {\sigma_1}^2)\partial_x^2\right)^{-1}\left(1 - \frac{\varepsilon}{2}(1 - \sigma^2)\partial_x^2\right)w^\varepsilon.$$

If $\boldsymbol{p}_1 = (\theta_1, \sigma_1, \lambda_1, \mu_1, \nu_1)$, with $\lambda_1, \mu_1, \nu_1 \in \mathbb{R}$, then $(v_1^\varepsilon, w_1^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is consistent with $S_{\boldsymbol{p}_1}(\partial)$.

Example 2. *Taking $\lambda = \mu = 1/2$, $\theta^2 = 2/3$, $\sigma^2 = 1/3$ and $\nu = 3/4$, one gets $a = b = c = d = e = 1/12$ and $f = g = 1/4$. The corresponding system has therefore a symmetric dispersive part.*

2.3. A second class of formally equivalent systems

The transformations performed in the previous section do not affect the nonlinear part of the equations; this nonlinear part is not symmetric but can be symmetrized as in [6] using nonlinear changes in variables, namely:

$$\widetilde{v}_\theta = v_\theta\left(1 + \frac{\varepsilon}{2}\zeta\right), \qquad \widetilde{w}_\sigma = w_\sigma\left(1 + \frac{\varepsilon}{2}\zeta\right) \tag{20}$$

(and thus $v_\theta = \widetilde{v}_\theta(1 - \frac{\varepsilon}{2}\zeta) + O(\varepsilon^2)$ and $w_\sigma = \widetilde{w}_\sigma(1 - \frac{\varepsilon}{2}\zeta) + O(\varepsilon^2)$).

Replacing v_θ by $\widetilde{v}_\theta$ and w_σ by $\widetilde{w}_\sigma$ in (15) yields

$$\begin{aligned}
&\partial_t\widetilde{v}_\theta + \partial_x\zeta + \varepsilon\big(a\partial_x^3\zeta - b\partial_x^2\partial_t\widetilde{v}_\theta + \frac{3}{2}\widetilde{v}_\theta\partial_x\widetilde{v}_\theta + \tfrac{1}{2}\zeta\partial_x\zeta + \tfrac{1}{2}\widetilde{w}_\sigma\partial_x\widetilde{w}_\sigma\big)\\
&\quad+\varepsilon^{3/2}\big(\tfrac{1}{2}\widetilde{v}_\theta\partial_y\widetilde{w}_\sigma + \tfrac{1}{2}\widetilde{w}_\sigma\partial_y\widetilde{v}_\theta\big) = O(\varepsilon^2),\\
&\partial_t\widetilde{w}_\sigma + \sqrt{\varepsilon}\partial_y\zeta + \varepsilon\big(-e\partial_x^2\partial_t\widetilde{w}_\sigma + \tfrac{1}{2}v_\theta\partial_x\widetilde{w}_\sigma + \tfrac{1}{2}\widetilde{w}_\sigma\partial_x\widetilde{v}_\theta + \widetilde{w}_\sigma\partial_y\widetilde{w}_\sigma\big)\\
&\quad+\varepsilon^{3/2}\big(f\partial_x^2\partial_y\zeta + \tfrac{1}{2}\widetilde{v}_\theta\partial_y\widetilde{v}_\theta + \tfrac{1}{2}\widetilde{w}_\sigma\partial_y\widetilde{w}_\sigma + \tfrac{1}{2}\zeta\partial_y\zeta\big) = O(\varepsilon^2),\\
&\partial_t\zeta + \partial_x\widetilde{v}_\theta + \sqrt{\varepsilon}\partial_y\widetilde{w}_\sigma + \varepsilon\big(\tfrac{1}{2}\widetilde{v}_\theta\partial_x\zeta + \tfrac{1}{2}\zeta\partial_x\widetilde{v}_\theta + c\partial_x^3\widetilde{v}_\theta - d\partial_x^2\partial_t\zeta\big)\\
&\quad+\varepsilon^{3/2}\big(g\partial_x^2\partial_y\widetilde{w}_\sigma + \tfrac{1}{2}\widetilde{w}_\sigma\partial_y\zeta + \tfrac{1}{2}\zeta\partial_y\widetilde{w}_\sigma\big) = O(\varepsilon^2),
\end{aligned} \tag{21}$$

whose linear part is the same as in (15), but whose nonlinear part is *symmetric*.

Notation. Writing $\boldsymbol{p} := (\theta, \sigma, \lambda, \mu, \nu) \in [0,1]^2 \times \mathbb{R}^3$, we denote by $S'_{\boldsymbol{p}}(\partial)$ the operator formed by the lhs of (21) and corresponding to this set of parameters. We denote by $\mathfrak{S}'$ the class of all the operators $S'_{\boldsymbol{p}}(\partial)$ obtained for all $\boldsymbol{p} \in [0,1]^2 \times \mathbb{R}^3$.

As a direct consequence, one gets a second consistency result.

Proposition 3. *Let* $\varepsilon_0 > 0$, $T > 0$ *and* $\boldsymbol{p} := (\theta, \sigma, \lambda, \mu, \nu) \in [0,1]^2 \times \mathbb{R}^3$. *Let also* $(U^\varepsilon)_{0<\varepsilon<\varepsilon_0} := (v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *be consistent with* $S_{\boldsymbol{p}}(\partial)$.

Define also $\widetilde{v}^\varepsilon := (1 + \frac{\varepsilon}{2}\zeta)v^\varepsilon$ *and* $\widetilde{w}^\varepsilon := (1 + \frac{\varepsilon}{2}\zeta)w^\varepsilon$.

Then $(\widetilde{v}^\varepsilon, \widetilde{w}^\varepsilon, \zeta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ *is consistent with* $S'_{\boldsymbol{p}}(\partial)$.

2.4. The completely symmetric weakly transverse Boussinesq systems

An interesting subclass of $\mathfrak{S}'$ consists of all the systems $S'_{\boldsymbol{p}}(\partial)$ which are completely symmetric (that is, whose linear *and* nonlinear part are symmetric). We denote by Σ this subclass.

For the systems of Σ, it is easy to prove local well-posedness over times of order $O(1/\varepsilon)$.

Proposition 4. *Let* $\boldsymbol{p} := (\theta, \sigma, \lambda, \mu, \nu)$ *be such that* $S'_{\boldsymbol{p}}(\partial) \in \Sigma$ *and* $U_0 := (v_0, w_0, \zeta_0) \in H^s(\mathbb{R}^2)^3$ *with* $s > 2$. *Then*

(i) For all $\varepsilon > 0$, *there exist* $\underline{T} > 0$ *and a unique family of solutions* $U^\varepsilon := (v^\varepsilon, w^\varepsilon, \zeta^\varepsilon) \in C([0, \underline{T}/\varepsilon], H^s(\mathbb{R}^2)^3) \cap C^1([0, \underline{T}/\varepsilon], H^{s-3}(\mathbb{R}^2)^3)$ *to* $S'_{\boldsymbol{p}}(\partial)$ *such that* $U^\varepsilon\,_{|_{t=0}} = U_0$.

(ii) If on the time interval $[0, T/\varepsilon]$ *(with* $T > 0$*),* $(U^\varepsilon_{cons})_\varepsilon$ *is consistent of order s with* $S'_{\boldsymbol{p}}(\partial) \in \Sigma$, *then* $\underline{T} \geqslant T$ *and one has for all* $\varepsilon > 0$ *small enough,*

$$\forall 0 \leqslant t \leqslant \frac{T}{\varepsilon}, \qquad |U^\varepsilon - U^\varepsilon_{\mathrm{cons}}|_{L^\infty([0,t],H^s} \leqslant C\varepsilon^2 t.$$

Proof. Local existence follows easily from the classical results on quasilinear hyperbolic equations. To prove the fact that the existence time is of order $O(1/\varepsilon)$ one can change the time, introducing $\tau = \varepsilon t$. One is therefore led to prove a local well-posedness result over times $\tau = O(1)$ for a new evolution system with respect to τ. Recalling that the energy estimates for such quasilinear systems depend only on the gradient of the coefficients, the singular terms (i.e. the terms of size $O(1/\varepsilon)$) in this new system are harmless because they have constant coefficients. The proof of the first point of the proposition is thus straightforward. The second point is classical and we omit the proof. □

2.5. How to choose a 'good' weakly transverse Boussinesq system

2.5.1. Linear well-posedness It is worth checking the well-posedness of the Cauchy problem associated to the linearization of the systems (15). For a given set of parameters $\boldsymbol{p} := (\theta, \sigma, \lambda, \mu, \nu)$, the linearization of $S_{\boldsymbol{p}}(\partial)$ around the rest state (which coincides of course with the linearization of $S'_{\boldsymbol{p}}(\partial)$) reads

$$\begin{aligned}
&\partial_t v_\theta + \partial_x \zeta + \varepsilon\big(a\partial_x^3\zeta - b\partial_x^2\partial_t v_\theta\big) = 0,\\
&\partial_t w_\sigma + \sqrt{\varepsilon}\partial_y\zeta + \varepsilon\big(-e\partial_x^2\partial_t w_\sigma\big) + \varepsilon^{3/2} f\partial_x^2\partial_y\zeta = 0,\\
&\partial_t\zeta + \partial_x v_\theta + \sqrt{\varepsilon}\partial_y w_\sigma + \varepsilon\left(c\partial_x^3 v_\theta - d\partial_x^2\partial_t\zeta\right) + \varepsilon^{3/2} g\partial_x^2\partial_y w_\sigma = 0,
\end{aligned} \tag{22}$$

and one readily checks that the Cauchy problem associated to (22), where the coefficients (a, b, c, d, e, f, g) are given by (16)–(19), is L^2-well posed provided one of the following conditions is satisfied:

- (i) $e \geqslant 0, b \geqslant 0, d \geqslant 0, f \leqslant 0, g \leqslant 0, a \leqslant 0, c \leqslant 0$,
- (ii) $e \geqslant 0, b \geqslant 0, d \geqslant 0, f = g, a \leqslant 0, c \leqslant 0$,
- (iii) $e \geqslant 0, b \geqslant 0, d \geqslant 0, f \leqslant 0, g \leqslant 0, a = c$,
- (iv) $e \geqslant 0, b \geqslant 0, d \geqslant 0, f = g, a = c$.

It does not seem to be obvious to prove the local well-posedness of the Cauchy problem associated to $S_p(\partial)$ in the range of parameters given by the above conditions, due to the lack of control of y-derivatives. Long time existence (i.e. $O(1/\varepsilon)$) would be of course harder and this is one of the reasons why the new symmetric class of systems $\mathfrak{S}'$ introduced in section 2.3 proves very useful.

2.5.2. Matching the dispersion relation A common criterion used in oceanography to evaluate the relevance of a formal asymptotic model is that its dispersion relation must be as close as possible to the dispersion relation of the linearized water-waves equations around the rest state. The better the matching, the larger the validity of the asymptotic model.

For unperturbed surfaces (i.e. when $\zeta = 0$), the operator $Z^\varepsilon(0)\cdot$ introduced in (4) can be explicitly computed:

$$Z^\varepsilon(0)\cdot = g^\varepsilon(D_x, D_y), \qquad \text{with } g^\varepsilon(\xi, \eta) := \sqrt{\varepsilon|\xi|^2 + \varepsilon^2|\eta|^2}\tanh\left(\sqrt{\varepsilon|\xi|^2 + \varepsilon^2|\eta|^2}\right),$$

where we used the classical notation for Fourier multipliers, $g^\varepsilon(D_x, D_y)f := \mathcal{F}^{-1}(g^\varepsilon(\xi, \eta)\widehat{f}(\xi, \eta))$.

It follows therefore from (5) that the linearized water-waves equations around the equilibrium state ($\underline{\zeta} = 0$, $\underline{\psi} = 0$) are given by

$$\begin{aligned} &\partial_t \psi + \zeta = 0, \\ &\partial_t \zeta - \frac{1}{\varepsilon} g_\varepsilon(D)\psi = 0. \end{aligned} \tag{23}$$

System (23) admits plane-wave solutions of the form $(\psi_0, \zeta_0)^{\mathrm{T}} \exp(\mathrm{i}(kx + ly - \omega t))$ provided that the following *dispersion relation is satisfied*,

$$\omega_{\mathrm{eul}}^2 = \frac{1}{\varepsilon}\sqrt{\varepsilon k^2 + \varepsilon^2 l^2}\tanh\left(\sqrt{\varepsilon k^2 + \varepsilon^2 l^2}\right). \tag{24}$$

We now turn to seek the dispersion relation associated to the systems of the class $\mathcal{S}$. Obviously, $(v_0, w_0, \zeta_0)^{\mathrm{T}} \exp(\mathrm{i}(kx + ly - \omega t))$ is a plane-wave solution to the linearization of (15) around the equilibrium state ($\underline{v}^\varepsilon = 0$, $\underline{w}^\varepsilon = 0$, $\underline{\zeta} = 0$) if and only if

$$\begin{aligned} &-\mathrm{i}\omega v_0 + \mathrm{i}k\zeta_0 + \varepsilon a(\mathrm{i}k)^3\zeta_0 - \varepsilon b(\mathrm{i}k)^2(-\mathrm{i}\omega)v_0 = 0, \\ &-\mathrm{i}\omega w_0 + \mathrm{i}l\zeta_0 - \varepsilon e(\mathrm{i}k)^2(-\mathrm{i}\omega)w_0 + \varepsilon f(\mathrm{i}k)^2(\mathrm{i}l)\zeta_0 = 0, \\ &-\mathrm{i}\omega\zeta_0 + \mathrm{i}kv_0 + \mathrm{i}\sqrt{\varepsilon}lw_0 + \varepsilon c(\mathrm{i}k)^3 v_0 - \varepsilon d(\mathrm{i}k)^2(-\mathrm{i}\omega)\zeta_0 + \varepsilon^{3/2} g(\mathrm{i}k)^2(\mathrm{i}l)w_0 = 0, \end{aligned}$$

so that nontrivial solutions are possible if and only if

$$\det\begin{pmatrix} -\omega(1 + \varepsilon bk^2) & 0 & k(1 - \varepsilon ak^2) \\ 0 & -\omega(1 + \varepsilon ek^2) & \sqrt{\varepsilon}l(1 - \varepsilon fk^2) \\ k(1 - \varepsilon ck^2) & \sqrt{\varepsilon}l(1 - \varepsilon gk^2) & -\omega(1 + \varepsilon dk^2) \end{pmatrix} = 0,$$

that is, if (ω, k, l) satisfies the *dispersion relation*

$$\omega_{\mathrm{bouss}}^2 = k^2\frac{(1 - \varepsilon ak^2)(1 - \varepsilon ck^2)}{(1 + \varepsilon bk^2)(1 + \varepsilon dk^2)} + \varepsilon l^2\frac{(1 - \varepsilon gk^2)(1 - \varepsilon fk^2)}{(1 + \varepsilon ek^2)(1 + \varepsilon dk^2)}. \tag{25}$$

A criterion to choose a 'good' system among all the systems derived in the previous section is therefore to match (24) and (25) the best way possible.

- Good degeneracy to a $1D$ model. One wants the weakly transverse Boussinesq model to give a good $1D$ Boussinesq model in the limiting case of surface waves independent of y. Setting $l = 0$ in (24) and (25) gives the corresponding $1D$ dispersion relations, namely,

$$\frac{\omega^2}{k^2} = \frac{\tanh(\sqrt{\varepsilon}k)}{\sqrt{\varepsilon}k} \qquad \text{and} \qquad \frac{\omega^2}{k^2} = \frac{(1-\varepsilon ak^2)(1-\varepsilon ck^2)}{(1+\varepsilon bk^2)(1+\varepsilon dk^2)}.$$

A good matching is obtained by choosing a, b, c, d such that the second expression is the [4 : 4]-Padé approximant of the first one. Since

$$\left(\frac{\tanh(\sqrt{\varepsilon}k)}{\sqrt{\varepsilon}k}\right)_{[4:4]} = \frac{1+\frac{\varepsilon}{9}k^2+\frac{\varepsilon^2}{945}k^4}{1+\varepsilon\frac{4}{9}k^2+\frac{\varepsilon^2}{63}k^4},$$

one can check after easy computations that this is achieved provided that

$$\{a, c\} = \left\{-(1+\sqrt{23/35})/18, -(1-\sqrt{23/35})/18\right\}, \tag{26}$$

$$\{b, d\} = \left\{(2-\sqrt{19/7})/9, (2+\sqrt{19/7})/9\right\}. \tag{27}$$

- Good transverse properties. Performing a Taylor expansion of (24) around $l = 0$ (that is around the limit $1D$ case), one gets, with $F(x) := \sqrt{x}\tanh(\sqrt{x})$,

$$\omega^2 = \frac{1}{\varepsilon}F(\varepsilon k^2) + \varepsilon l^2 F'(\varepsilon k^2) + O(l^4).$$

Comparing this expression with (25), one can see that the model will have good transverse dispersive properties if

$$\frac{(1-\varepsilon gk^2)(1-\varepsilon fk^2)}{(1+\varepsilon ek^2)(1+\varepsilon dk^2)}$$

is a good approximation of $F'(\varepsilon k)$. It turns out that it is not possible to match this expression with the [4 : 4]-Padé approximant of $F'(\varepsilon k)$ but we can choose e, f, g in order to match the Taylor expansion of $F'(\varepsilon k)$ the best way possible. Choosing

$$e = \frac{6-10d}{10-15d}, \qquad f = 0 \qquad \text{and} \qquad g = -\frac{6-10d}{10-15d} + \frac{2}{3} - d \tag{28}$$

(recall that d is already known from the previous step), one gets

$$F'(\varepsilon k) = \frac{(1-\varepsilon gk^2)(1-\varepsilon fk^2)}{(1+\varepsilon ek^2)(1+\varepsilon dk^2)} + O\big((\varepsilon k^2)^3\big). \tag{29}$$

Example 3. *The following choice of the parameters $\sigma, \theta, \lambda, \mu, \nu$ leads to coefficients a, b, c, d, e, f, g satisfying (26)–(28), with $a = -(1-\sqrt{23/35})/18$ and $d = (2+\sqrt{19/7})/9$:*

$$\sigma := \sqrt{1-2\frac{6-10d}{10-15d}}, \qquad \theta := \sqrt{1-2(a+b)},$$

$$\lambda := \frac{c}{\theta^2/2-1/6}, \qquad \mu := \frac{a}{a+b}, \qquad \nu := 0,$$

that is, numerically, $\sigma \sim 0.0820$, $\theta \sim 0.9709$, $\lambda \sim -0.3301$, $\mu \sim -0.3672$ and $\nu = 0$ (and $a \sim -0.0105$, $b \sim 0.0392$, $c \sim -0.1006$, $d \sim 0.4053$, $e \sim 0.4966$; $f = 0$, $g \sim -0.2353$).

Remark 4. (i) The 1D Boussinesq system corresponding to the choice of the coefficients a, b, c and d of example 3 coincides with the 1D Boussinesq system (41c) of [20].

(ii) Computations show that it is not possible to choose e, f, g in order to match the expansion of $F'(\varepsilon k)$ up to order $O\big((\varepsilon k^2)^4\big)$. However, the difference in the coefficient of $(\varepsilon k^2)^3$ in the expansion (29) is less than 10^{-2} if one chooses the coefficients as in example 3.

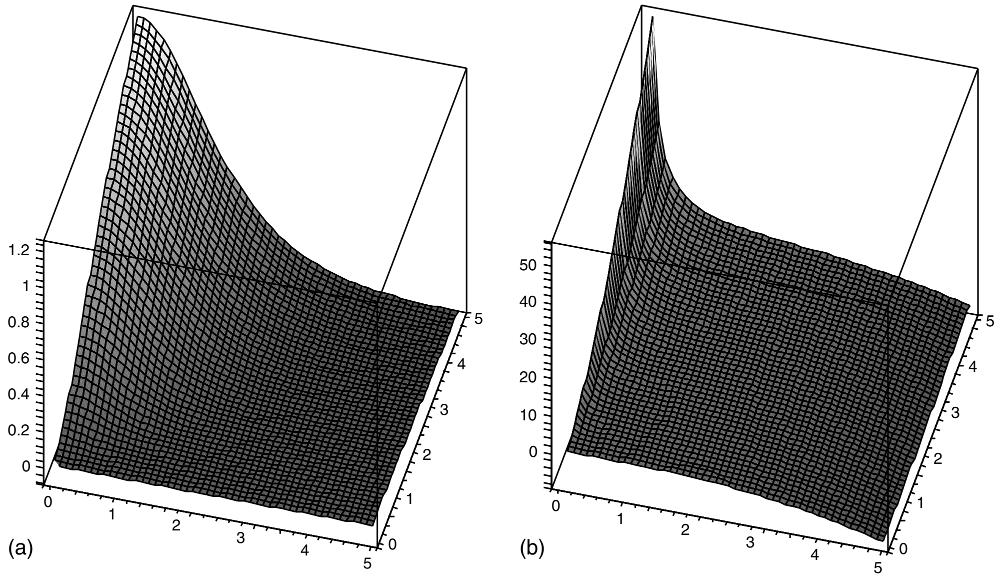

Figure 1. Left = δ_{bouss}, right = δ_{KP}.

(iii) The system associated to the choice of parameters of example 3 is linearly well-posed (the condition (i) of the previous section is satisfied).

Choosing the parameters as in example 3 gives a weakly transverse Boussinesq model with very good dispersive properties in the sense that the dispersion relation $\omega_{\text{bouss}}(k,l)$ given by (25) is very close to the dispersion relation $\omega_{\text{eul}}(k,l)$ of the linearized water-waves equations. This is not the case at all for the well-known KP approximation described in the next section and for which the dispersion relation is given by $\omega_{\text{KP}}(k,l) = k + \varepsilon((l^2/2k) - (k^3/6))$.

To illustrate this fact, we plot on figure 1 the relative error on the phase velocity for both models (see also [20] for such representations). More precisely, setting $K := \sqrt{\varepsilon}k$ and $L := \varepsilon l$, we define

$$c_{\text{eul}}(K,L) := \frac{\omega_{\text{eul}}}{\sqrt{K^2+L^2}} \qquad \left(= \frac{1}{\varepsilon}\left(\frac{\tanh(\sqrt{K^2+L^2})}{\sqrt{K^2+L^2}}\right)^{1/2}\right)$$

$$c_{\text{KP}}(K,L) := \frac{\omega_{\text{KP}}}{\sqrt{K^2+L^2}} \qquad \left(= \frac{1}{\varepsilon}\frac{K + \frac{L^2}{2K} - \frac{K^3}{6}}{\sqrt{K^2+L^2}}\right)$$

$$c_{\text{bouss}}(K,L) := \frac{\omega_{\text{bouss}}}{\sqrt{K^2+L^2}}.$$

Figure 1 plots the relative error $\delta_{\text{KP}} := ((c_{\text{KP}} - c_{\text{eul}})/c_{\text{eul}})$ and $\delta_{\text{bouss}} := (c_{\text{bouss}} - c_{\text{eul}})/c_{\text{eul}})$, for $(K,L) \in [0,5]$. As one sees, δ_{bouss} remains small, while δ_{KP} is not small at all, and even goes to infinity as $K \to 0$.

3. The KP approximation for Boussinesq systems

The aim of this section is to derive rigorously the uncoupled KP approximation from the weakly transverse Boussinesq systems obtained in the previous sections. We show in particular that all the coupled systems of the classes $\mathfrak{S}$ and $\mathfrak{S}'$ share the same asymptotics, that is, the same uncoupled KP approximation. We also prove a convergence theorem for the completely symmetric systems of the class Σ.

3.1. Statement of the results

Let us write $U = (v, w, \zeta)^{\mathrm{T}}$ and work in a slightly more general framework allowing us to deal with systems of $\mathfrak{S}$ and $\mathfrak{S}'$ at the same time. We consider initial value problems of the form

$$\begin{aligned} &P^{\varepsilon}(\partial, U)U = 0,\\ &U_{|_{t=0}} = U^{0,\varepsilon}, \end{aligned} \tag{30}$$

where the operator $P^{\varepsilon}(\partial_t, \partial_x, \partial_y, \cdot)$ can be written as

$$P^{\varepsilon}(\partial, \cdot) = (1 - \varepsilon G\partial_x^2)\partial_t + A\partial_x + \sqrt{\varepsilon}B\partial_y + \varepsilon\big(C\partial_x^3 + D(\cdot)\partial_x\big) + \varepsilon^{3/2}\big(E\partial_x^2\partial_y + F(\cdot)\partial_y\big), \tag{31}$$

with $A, B, C, E, G \in \mathcal{M}_3(\mathbb{R})$, while $D(\cdot)$ and $F(\cdot)$ are linear mappings with values in $\mathcal{M}_3(\mathbb{R})$.

The uncoupled KP asymptotics is derived from (30) under the following assumption.

Assumption 1. *The matrices A and B are symmetric. Moreover, the eigenvalues of A are* ± 1 *and* 0, *and if* $\boldsymbol{e}_\pm$ *and* $\boldsymbol{e}_0$ *denote associated unit eigenvectors, one has* $\boldsymbol{e}_\pm^{\mathrm{T}}B = (\boldsymbol{e}_\pm \cdot B\boldsymbol{e}_0)\boldsymbol{e}_0^{\mathrm{T}}$, $\boldsymbol{e}_0 \cdot B\boldsymbol{e}_0 = 0$ *and* $\boldsymbol{e}_0 \cdot G\boldsymbol{e}_\pm = 0$.

Moreover, the initial condition of (30) is such that

$$U^{0,\varepsilon} = U_0^0 + \sqrt{\varepsilon}U_1^0, \qquad \textit{with} \qquad U_0^0 = u_+^0\boldsymbol{e}_+ + u_-^0\boldsymbol{e}_- \qquad \textit{and} \qquad U_1^0 = u_0^0\boldsymbol{e}_0.$$

Remark 5. For all the systems of $\mathfrak{S}$ and $\mathfrak{S}'$, one has

$$A = \begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0 \end{pmatrix}, \qquad \boldsymbol{e}_\pm = \begin{pmatrix} \frac{1}{\sqrt{2}}\\ 0\\ \pm\frac{1}{\sqrt{2}} \end{pmatrix}, \quad \boldsymbol{e}_0 = \begin{pmatrix} 0\\ 1\\ 0 \end{pmatrix},$$

and one can check easily that the above assumption is satisfied, provided that the initial condition is taken under the form $(v^0, \sqrt{\varepsilon}w^0, \zeta^0)$, which is a natural scaling since by definition, $w = \sqrt{\varepsilon}\partial_y\psi$.

In order to control the secular growth of the corrector terms in the formal KP approximation which can be constructed under assumption 1, we need to impose that the nonlinearities of (31) have a particular structure.

Assumption 2. *Under assumption 1, one has* $\boldsymbol{e}_0 \cdot D(\cdot)\boldsymbol{e}_\pm = 0$, *and for all* $U \in C^1(\mathbb{R}^d)^n$ *such that* $\boldsymbol{e}_0 \cdot U = 0$, *one has*

$$D(U)\partial_x U = \tfrac{1}{2}\partial_x\big(D(U)U\big).$$

Remark 6. For the systems of the classes $\mathfrak{S}$ and $\mathfrak{S}'$, the mapping $D(\cdot)$ is given, respectively, by $D_{\mathfrak{S}}(\cdot)$ and $D_{\mathfrak{S}'}(\cdot)$, where for all $U = (\zeta, v, w)^{\mathrm{T}}$,

$$D_{\mathfrak{S}}(U) = \begin{pmatrix} v & w/2 & 0\\ 0 & v/2 & 0\\ \zeta & 0 & v \end{pmatrix} \qquad \textit{and} \qquad D_{\mathfrak{S}'}(U) = \begin{pmatrix} 3v/2 & w/2 & \zeta/2\\ w/2 & v/2 & 0\\ \zeta/2 & 0 & v/2 \end{pmatrix},$$

and it is easy to check that assumption 2 is satisfied in both cases.

Finally, a third assumption is needed for the proof of the convergence of the KP approximation.

Assumption 3. *The matrices C, E and G are symmetric, and G is positive. Moreover, for all $U \in \mathbb{R}^3$, $D(U)$ and $F(U)$ are symmetric matrices.*

Remark 7. This last assumption is satisfied by the systems of the class Σ.

Let us now describe the so-called KP approximation. First set some notation.

Notation 1. *(i) Let $s \in \mathbb{R}$. We use the notation $\partial_x H^s(\mathbb{R}^2)$ to refer to the space of all the distributions f such that there exists $\widetilde{f} \in H^s(\mathbb{R}^2)$ with $\partial_x \widetilde{f} = f$. We naturally write $|f|_{\partial_x H^s} := |\widetilde{f}|_{H^s}$.*

(ii) We define similarly $\partial_x^2 H^s(\mathbb{R}^2)$.

(iii) The operator ∂_x^{-1} denotes the Fourier multiplier of symbol $-i/\xi$ (with ξ denoting the dual variable of x). This operator is well defined on the spaces $\partial_x H^s(\mathbb{R}^2)$, $s \geqslant 0$, defined above.

We can now define the KP approximation $U^{\varepsilon}_{\mathrm{KP}}(t, x, y)$ as follows

$$U^{\varepsilon}_{\mathrm{KP}}(t, x, y) = u_+(\varepsilon t, x - t, y)\boldsymbol{e}_+ + u_-(\varepsilon t, x + t, y)\boldsymbol{e}_-,$$

where $u_\pm(\tau, \widetilde{x}, y)$ solve the *uncoupled* KP equations

$$\partial_\tau u_\pm \pm \alpha_\pm \partial_{\widetilde{x}}^{-1}\partial_y^2 u_\pm + \beta_\pm \partial_{\widetilde{x}}^3 u_\pm + \gamma_\pm u_\pm \partial_{\widetilde{x}} u_\pm = 0, \qquad u_{\pm|_{\tau=0}} = u^0_\pm, \tag{32}$$

with $\alpha_\pm = (\boldsymbol{e}_\pm \cdot B\boldsymbol{e}_0)^2$, $\beta_\pm = (\boldsymbol{e}_\pm \cdot C\boldsymbol{e}_\pm)$ and $\gamma_\pm = \boldsymbol{e}_\pm \cdot D(\boldsymbol{e}_\pm)\boldsymbol{e}_\pm$.

The remainder of this section is devoted to the proof of the following theorem.

Theorem 1. *Suppose that assumptions 1–2 are satisfied and let $s > 1$. Assume moreover that $u^0_\pm \in \partial_x H^{s+7}(\mathbb{R}^2)$ and $\partial_y^2 u^0_\pm \in \partial_x^2 H^{s+3}(\mathbb{R}^2)$; assume also $u^0_0 \in H^{s+4}(\mathbb{R}^2)$. Then, for all $T > 0$,*

(i) One can construct an approximate solution U_{app} to (30) such that

$$\left|U_{\mathrm{app}} - U_{\mathrm{KP}}\right|_{L^\infty([0,\frac{T}{\varepsilon}]\times\mathbb{R}^2)} = O(\sqrt{\varepsilon}) \quad \text{and} \quad \left|P^\varepsilon(\partial, U^\varepsilon_{\mathrm{app}})U_{\mathrm{app}}\right|_{L^\infty([0,\frac{T}{\varepsilon}];H^s(\mathbb{R}^2))} = o(\varepsilon),$$

(ii) If moreover assumption 3 is satisfied, then there exists a unique exact solution $U_{\mathrm{exact}} \in C([0, \frac{T}{\varepsilon}], H^s(\mathbb{R}^2))$ to (30) and one has

$$\left|U_{\mathrm{exact}} - U_{\mathrm{KP}}\right|_{L^\infty([0,\frac{T}{\varepsilon}]\times\mathbb{R}^2)} = o(1).$$

Remark 8. Assumption 1 is necessary to construct a formal KP asymptotics for (30). Assumption 2 is needed to prove that this formal asymptotics is consistent with (30) (first point of the theorem). Finally, we must make assumption 3 to prove the second point of the theorem, that is, that the KP approximation converges to an exact solution of (30).

The following corollary deals with the case when the i.v.p. (30) is associated to a system of the class $\mathfrak{S}$ or $\mathfrak{S}'$. In particular, it shows that the uncoupled KP equations are the same for all these systems, namely

$$\partial_\tau \zeta_\pm \pm \frac{1}{2}\partial_{\widetilde{x}}^{-1}\partial_y^2 \zeta_\pm \pm \frac{1}{6}\partial_{\widetilde{x}}^3 \zeta_\pm + \frac{3}{2\sqrt{2}}\zeta_\pm \partial_{\widetilde{x}} \zeta_\pm = 0 \qquad (\mathrm{KP})_\pm.$$

Corollary 1. *Let $s > 1$ and $v^0, \zeta^0 \in \partial_x H^{s+7}(\mathbb{R}^2)$, $w^0 \in H^{s+4}(\mathbb{R}^2)$ and assume moreover that $\partial_y^2 v^0, \partial_y^2 \zeta^0 \in \partial_x^2 H^{s+3}(\mathbb{R}^2)$.*

(i) For all the systems of the class $\mathfrak{S}$ and $\mathfrak{S}'$, the KP equations given by theorem 1; (i) are always the same, namely, (KP)$_\pm$.

(ii) For any $T > 0$, all the systems of the class Σ have a unique solution $(v, w, \zeta)^T \in C([0, T/\varepsilon]; H^s(\mathbb{R}^2))$ with initial condition $(v^0, \sqrt{\varepsilon}w^0, \zeta^0)^T$ and one has

$$\zeta(t,x,y) - \frac{1}{\sqrt{2}}\Big(\zeta_+(\varepsilon t, x-t, y) - \zeta_-(\varepsilon t, x+t, y)\Big) = o(1)$$

in $L^\infty([0, \frac{T}{\varepsilon}]\times\mathbb{R}^2_{x,y})$, where $\zeta_\pm(\tau, \xi, y)$ denotes the solution of (KP)$_\pm$ *with initial condition $\zeta_\pm(t=0) = (\zeta^0 \pm v^0/\sqrt{2})$.*

Proof. One just has to identify the quantities $\boldsymbol{e}_\pm$, $\boldsymbol{e}_0$, A, B, C, $D(\cdot)$ and G when (30) is a system of $\mathfrak{S}$ or $\mathfrak{S}'$. One can check easily that

$$C = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad G = \begin{pmatrix} b & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & d \end{pmatrix},$$

and we refer to remarks 5 and 6 for the other terms.

Simple computations then show that $\alpha_\pm = 1/2$, $\beta_\pm = \pm(a+b+c+d)/2$ and $\gamma_\pm = 3\sqrt{2}/4$. Since the systems of $\mathfrak{S}$ and $\mathfrak{S}'$ satisfy assumptions 1 and 2 and since the systems of Σ also satisfy assumption 3, the corollary follows from theorem 1 and the observation that $a+b+c+d = 1/3$ (recall that a, b, c and d are given by (16)–(17)). □

3.2. Construction of the approximate solution

3.2.1. The ansatz. We look for an approximate solution to (30) under the form

$$U^\varepsilon_{\text{app}}(t,x,y) = U_0(\varepsilon t, t, x, y) + \sqrt{\varepsilon}U_1(\varepsilon t, t, x, y) + \varepsilon U_2(\varepsilon t, t, x, y), \tag{33}$$

with $U_j = (v_j, w_j, \zeta_j)^T$, $j = 0, 1, 2$. In order to match the initial value of (30), and with the notations of assumption 1, we impose

$$U_{0|_{\tau=t=0}} = U_0^0, \qquad U_{1|_{\tau=t=0}} = U_1^0 \qquad \text{and} \qquad U_{2|_{\tau=t=0}} = 0. \tag{34}$$

The profiles $U_j(\tau, t, x, y)$ are then found by a classical BKW method; more precisely, writing

$$P^\varepsilon(\partial, U^\varepsilon_{\text{app}})U^\varepsilon_{\text{app}}(t,x,y) = \sum_{k=0}^{7}(\sqrt{\varepsilon})^k R_k(\varepsilon t, t, x, y), \tag{35}$$

with $R_j(\tau, t, x, y)$ $(0 \leqslant j \leqslant 7)$ given by

$$R_0 = \big(\partial_t + A\partial_x\big)U_0, \tag{36}$$

$$R_1 = \big(\partial_t + A\partial_x\big)U_1 + B\partial_y U_0, \tag{37}$$

$$R_2 = \big(\partial_t + A\partial_x\big)U_2 + B\partial_y U_1 + \partial_\tau U_0 + C\partial_x^3 U_0 - G\partial_x^2\partial_t U_0 + D(U_0)\partial_x U_0, \tag{38}$$

$$\begin{aligned} R_3 = {} & B\partial_y U_2 + \partial_\tau U_1 + C\partial_x^3 U_1 - G\partial_x^2\partial_t U_1 + D(U_1)\partial_x U_0 + D(U_0)\partial_x U_1 \\ & + E\partial_x^2\partial_y U_0 + F(U_0)\partial_y U_0, \end{aligned} \tag{39}$$

$$\begin{aligned} R_4 = {} & \partial_\tau U_2 - G\partial_x^2\partial_\tau U_0 + C\partial_x^3 U_2 - G\partial_x^2\partial_t U_2 + D(U_2)\partial_x U_0 + D(U_0)\partial_x U_2 \\ & + D(U_1)\partial_x U_1 + E\partial_x^2\partial_y U_1 + F(U_1)\partial_y U_0 + F(U_0)\partial_y U_1, \end{aligned} \tag{40}$$

$$\begin{aligned} R_5 = {} & -G\partial_x^2\partial_\tau U_1 + D(U_2)\partial_x U_1 + D(U_1)\partial_x U_2 + E\partial_x^2\partial_y U_2 \\ & + F(U_2)\partial_y U_0 + F(U_0)\partial_y U_2 + F(U_1)\partial_y U_1 \end{aligned} \tag{41}$$

$$R_6 = -G\partial_x^2\partial_\tau U_2 + D(U_2)\partial_x U_2 + F(U_1)\partial_y U_2, \tag{42}$$

$$R_7 = F(U_2)\partial_y U_2, \tag{43}$$

one seeks the profiles U_j in order to cancel R_0, R_1 and R_2.

3.2.2. The profile equations. The profile equations are the equations found on the profiles U_j in the cancellation of the residual terms R_0, R_1 and R_2.

Cancelling R_0. The equation $R_0 = 0$ reads, according to (36), $\big(\partial_t + A\partial_x\big)U_0 = 0$. Under assumption 1, this is equivalent to the three transport equations

$$(\partial_t \pm \partial_x)(\boldsymbol{e}_\pm \cdot U_0) = 0, \qquad \partial_t(\boldsymbol{e}_0 \cdot U_0) = 0.$$

Under assumption 1, we can therefore take

$$\boldsymbol{e}_\pm \cdot U_0(\tau, t, x, y) = u_\pm(\tau, x \mp t, y) \qquad \text{and} \qquad \boldsymbol{e}_0 \cdot U_0(\tau, t, x, y) = 0, \tag{44}$$

where $u_\pm(\tau, \widetilde{x}, y)$ are two scalar functions such that $u_{\pm|_{\tau=0}} = u^0_\pm$.

Cancelling R_1. According to (37), (44) and under assumption 1, the equation $R_1 = 0$ reduces to

$$(\partial_t \pm \partial_x)(\boldsymbol{e}_\pm \cdot U_1) = 0$$

and

$$\partial_t(\boldsymbol{e}_0 \cdot U_1) + (\boldsymbol{e}_0 \cdot B\boldsymbol{e}_+)\partial_y(\boldsymbol{e}_+ \cdot U_0) + (\boldsymbol{e}_0 \cdot B\boldsymbol{e}_-)\partial_y(\boldsymbol{e}_- \cdot U_0) = 0;$$

according to (34) and (44), and under assumption 1, we can take therefore

$$U_1(\tau, t, x, y) = u_0(\tau, t, x, y)\boldsymbol{e}_0, \tag{45}$$

u_0 being a scalar valued function of its arguments, and

$$u_0(\tau, t, x, y) = u_0^0(x, y) - (\boldsymbol{e}_0 \cdot B\boldsymbol{e}_+)\int_0^t \partial_y u_+(\tau, x - s, y)\,\mathrm{d}s$$
$$-(\boldsymbol{e}_0 \cdot B\boldsymbol{e}_-)\int_0^t \partial_y u_-(\tau, x + s, y)\,\mathrm{d}s,$$

or, with $\partial_x{}^{-1}$ defined as in notation 1,

$$u_0(\tau, t, x, y) = u_0^0(x, y) - (\boldsymbol{e}_0 \cdot B\boldsymbol{e}_+)\big(\partial_{\widetilde{x}}^{-1}\partial_y u_+(\tau, x, y) - \partial_{\widetilde{x}}^{-1}\partial_y u_+(\tau, x - t, y)\big)$$
$$-(\boldsymbol{e}_0 \cdot B\boldsymbol{e}_-)\big(\partial_{\widetilde{x}}^{-1}\partial_y u_-(\tau, x + t, y) - \partial_{\widetilde{x}}^{-1}\partial_y u_-(\tau, x, y)\big). \tag{46}$$

Cancelling R_2. According to (38), (44) and (45), and under assumptions 1–2, the equation $R_2 = 0$ reduces to

$$(\partial_t \pm \partial_x)(\boldsymbol{e}_\pm \cdot U_2) + (\boldsymbol{e}_\pm \cdot B\boldsymbol{e}_0)\partial_y(\boldsymbol{e}_0 \cdot U_1) + \partial_\tau(\boldsymbol{e}_\pm \cdot U_0) + \boldsymbol{e}_\pm \cdot C\partial_x^3 U_0 - \boldsymbol{e}_\pm \cdot G\partial_x^2\partial_t U_0$$
$$+\frac{1}{2}\boldsymbol{e}_\pm \cdot \partial_x\big(D(U_0)U_0\big) = 0, \tag{47}$$

and

$$\partial_t(\boldsymbol{e}_0 \cdot U_2) + \boldsymbol{e}_0 \cdot C\partial_x^3(u_{+|_{\widetilde{x}=x-t}}\boldsymbol{e}_+ + u_{-|_{\widetilde{x}=x+t}}\boldsymbol{e}_-) = 0, \tag{48}$$

where we used the identity $D(U_0)\partial_x U_0 = \frac{1}{2}\partial_x\big(D(U_0)U_0\big)$ which comes from assumption 2 and (44).

We then replace $\boldsymbol{e}_0 \cdot U_1$ by its expression (46) and recall that $U_0 = u_+\boldsymbol{e}_+ + u_-\boldsymbol{e}_-$; as in [3, 13], we can therefore split (47) into

$$\partial_\tau u_\pm \pm \alpha_\pm\partial_{\widetilde{x}}^{-1}\partial_y^2 u_\pm + \beta_\pm\partial_{\widetilde{x}}^3 u_\pm + \gamma_\pm u_\pm\partial_{\widetilde{x}} u_\pm = 0 \tag{49}$$

and

$$(\partial_t \pm \partial_x)(\boldsymbol{e}_\pm \cdot U_2) = \pm\alpha_\pm\partial_{\widetilde{x}}^{-1}\partial_y^2 u_{\pm|_{\xi=x}} \mp \delta\big(\partial_{\widetilde{x}}^{-1}\partial_y^2 u_{\mp|_{\xi=x}} - \partial_{\widetilde{x}}^{-1}\partial_y^2 u_{\mp|_{\xi=x\pm t}}\big)$$
$$-\lambda_\pm\partial_{\widetilde{x}}^3 u_{\mp|_{\xi=x\pm t}} - \mu_\pm\partial_x\big(u_{+|_{\xi=x-t}}u_{-|_{\xi=x+t}}\big)$$
$$-\tfrac{1}{2}\gamma_\mp\big(\partial_{\widetilde{x}}u_\mp^2\big)_{|_{\xi=x\pm t}}, \tag{50}$$

with $\alpha_\pm = (\boldsymbol{e}_\pm \cdot B\boldsymbol{e}_0)^2$, $\beta_\pm = (\boldsymbol{e}_\pm \cdot C\boldsymbol{e}_\pm) \pm (\boldsymbol{e}_\pm \cdot G\boldsymbol{e}_\pm)$, $\gamma_\pm = \boldsymbol{e}_\pm \cdot D(\boldsymbol{e}_\pm)\boldsymbol{e}_\pm$, $\delta = (\boldsymbol{e}_+ \cdot B\boldsymbol{e}_0)(\boldsymbol{e}_- \cdot B\boldsymbol{e}_0)$, $\lambda_\pm = (\boldsymbol{e}_\pm \cdot C\boldsymbol{e}_\mp) \mp (\boldsymbol{e}_\pm \cdot G\boldsymbol{e}_\mp)$ and $\mu_\pm = \frac{1}{2}\boldsymbol{e}_\pm \cdot \big(D(\boldsymbol{e}_+)\boldsymbol{e}_- + D(\boldsymbol{e}_-)\boldsymbol{e}_+\big)$.

Note that this splitting is in fact necessary; this can be seen as in [3] using the average projectors introduced in lemmas 3–6 of [11].

Finally, equation (48) can be solved explicitly:

$$\boldsymbol{e}_0 \cdot U_2 = -\boldsymbol{e}_0 \cdot C\Big(-\partial_{\bar{x}}^2 u_{+|_{x-t}} \boldsymbol{e}_+ + \partial_{\bar{x}}^2 u_{+|_x} \boldsymbol{e}_+ + \partial_{\bar{x}}^2 u_{-|_{x+t}} \boldsymbol{e}_- - \partial_{\bar{x}}^2 u_{-|_x} \boldsymbol{e}_- \Big). \quad (51)$$

3.2.3. Properties of the approximate solution. We prove here that the profile equations derived above can be solved, which allows us to define the approximate solution (33); we then prove that the corrector terms $\sqrt{\varepsilon}U_1$ and εU_2 remain of size $o(1)$ for times $t = O(1/\varepsilon)$ (i.e. they remain corrector terms).

Proposition 5. *Suppose that assumptions 1 and 2 are satisfied and assume moreover that* $u_\pm^0 \in \partial_x H^{s+1}(\mathbb{R}^2)$ *and* $u_0^0 \in H^{s-1}(\mathbb{R}^2)$, *with* $s > 3$. *Then for all* $T > 0$,

- *(i) There exists a unique solution* $u_\pm \in C([0,T]; H^s(\mathbb{R}^2)) \cap Lip([0,T]; H^{s-3}(\mathbb{R}^2))$ *to the KP equation (49) with initial condition* $u_\pm^0$. *Moreover, one has* $u_\pm \in L^\infty([0,T], \partial_x H^s(\mathbb{R}^2))$.
- *(ii) The function* u_0 *given by (46) satisfies* $u_0 \in L^\infty([0,T]\times \mathbb{R}_t^+; H^{s-1}(\mathbb{R}^2_{x,y}))$.
- *(iii) One can construct the approximate solution* $U^\varepsilon_{\rm app}$ *defined in (33), with*

$$U_0(\tau, t, x, y) = u_+(\tau, x-t, y)\boldsymbol{e}_+ + u_-(\tau, x+t, y)\boldsymbol{e}_-,$$
$$U_1(\tau, t, x, y) = u_0(\tau, t, x, y)\boldsymbol{e}_0$$

and U_2 *solving the profile equations (51) and (50). Moreover,*

$$\boldsymbol{e}_\pm \cdot U_{\rm app}(t,x,y) - u_\pm(\varepsilon t, x \mp t, y) = o(1),$$
$$\boldsymbol{e}_0 \cdot U_{\rm app}(t,x,y) - \sqrt{\varepsilon} u_0(\varepsilon t, t, x, y) = O(\varepsilon)$$

in $L^\infty([0, T/\varepsilon]\times \mathbb{R}^2_{x,y})$.
- *(iv) If moreover* $\partial_y^2 u_\pm^0 \in \partial_x^2 H^{s-3}(\mathbb{R}^2)$ *and* $s > 4$, *then the* $o(1)$ *error estimate given in (iii) can be improved to* $o(\sqrt{\varepsilon})$.

Proof

- (i) Existence and uniqueness of the functions $u_\pm$ is provided by classical results on the KP equations (e.g. [21]). The fact that $u_\pm \in \partial_x H^s(\mathbb{R}^2)$ is proved in [19].
- (ii) From (46) one deduces easily

$$|u_0(\tau, t, \cdot)|_{H^{s-1}(\mathbb{R}^d)} \leqslant C(|u_0^0|_{H^{s-1}} + |u_+(\tau,\cdot)|_{\partial_x H^s} + |u_-(\tau,\cdot)|_{\partial_x H^s}),$$

for all $\tau \in [0,\tau]$ and $t \geqslant 0$, so that the result follows from the first point.
- (iii) First remark that $\boldsymbol{e}_0 \cdot U_{\rm app} - \sqrt{\varepsilon} u_0 = \varepsilon \boldsymbol{e}_0 \cdot U_2(\varepsilon t, t, x, y)$. From (51), one obtains easily that $|\boldsymbol{e}_0 \cdot U_2(\tau, t, \cdot)|_{H^{s-2}(\mathbb{R}^2)} \leqslant C\big(|u_+(\tau,\cdot)|_{H^s} + |u_-(\tau,\cdot)|_{H^s}\big)$, so that the estimate on $\boldsymbol{e}_0 \cdot U_{\rm app}$ follows from the first point and the Sobolev embedding $H^{s-2}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$.

Since $\boldsymbol{e}_\pm \cdot U_{\rm app} - u_\pm = \varepsilon \boldsymbol{e}_\pm \cdot U_2(\varepsilon t, t, x, y)$, we now seek an estimate on $\boldsymbol{e}_\pm \cdot U_2$, which is given by (50). We need the following lemma.

Lemma 2. *Let* $c, c_1 \in \mathbb{R}$, $c \neq c_1$ *and* $f \in H^r(\mathbb{R}^2)$ *for some* $r > 1$, *and define* $F(t,x,y) = f(x - c_1 t, y)$. *Then*

- *(i) If* $u(t,x,y)$ *solves* $(\partial_t + c\partial_x)u = F$, *with* $u(t=0) = 0$, *then*

$$\lim_{t\to\infty} \frac{1}{t}|u(t,\cdot)|_{H^r(\mathbb{R}^2)} = 0.$$

If moreover $f \in \partial_x H^r(\mathbb{R}^2)$, *then for all* $t \geqslant 0$, $|u(t,\cdot)|_{H^r} \leqslant C|f|_{\partial_x H^r}$.

(ii) Let also $g \in H^r(\mathbb{R}^2)$ *and* $G(t,x,y) = g(x-ct,y)$. *Then if* $v(t,x,y)$ *solves* $(\partial_t + c\partial_x)v = FG$, *with* $v(t=0) = 0$, *then*

$$\lim_{t\to\infty} \frac{1}{\sqrt{t}} |v(t,\cdot)|_{H^r(\mathbb{R}^2)} = 0.$$

Proof. The first estimate of (i) is classical (see, e.g. proposition 2.2. of [14]). Remark now that under the assumption $f \in \partial_x H^r$, one can write $f = \partial_x \widetilde{f}$, with $\widetilde{f} \in H^r(\mathbb{R}^2)$; the solution u of the i.v.p. under consideration is then given by

$$u(t,x,y) = \frac{1}{c-c_1}\big(\widetilde{f}(x-c_1t,y) - \widetilde{f}(x-ct,y)\big),$$

so that the second estimate follows.

For the proof of (ii), we refer to proposition 3.6 of [14]. □

From the lemma and (50), it is obvious that for all $0 \leqslant \tau \leqslant T$,

$$|\boldsymbol{e}_\pm \cdot U_2(\tau,t,\cdot)|_{H^{s-2}} \leqslant C(|u_+|_{H^s_T}, |u_-|_{H^s_T}) + C(|u_+|_{\partial_x H^s_T}, |u_-|_{\partial_x H^s_T})\epsilon(t)t$$
$$+C(|u_+|_{H^s_T}, |u_-|_{H^s_T})\epsilon(t)\sqrt{t}, \tag{52}$$

where $\epsilon(t)$ denotes a function such that $\lim_{t\to\infty}\epsilon(t) = 0$. It is then easy to conclude that $\varepsilon\boldsymbol{e}_\pm \cdot U_2(\varepsilon t,t,\cdot) = o(1)$ in $L^\infty([0,T/\varepsilon]\times\mathbb{R}^2)$, which concludes the proof of (iii).

Before proving (iv), let us prove the following lemma.

Lemma 3. *If* $u^0_\pm \in \partial_x H^{s+1}(\mathbb{R}^2)$, *with* $s > 3$, *and if moreover one has* $\partial_y^2 u^0_\pm \in \partial_x^2 H^{s-3}(\mathbb{R}^2)$, *then* $\partial_y^2 u_\pm \in L^\infty([0,T], \partial_x^2 H^{s-3}(\mathbb{R}^2))$ *and*

$$|\partial_y^2 u_\pm|_{\partial_x^2 H^{s-3}_T} \leqslant C(|u_\pm|_{H^s_T}).$$

Proof. We must prove that $\partial_{\widetilde{x}}^{-2}\partial_y^2 u_\pm \in L^\infty([0,T], H^{s-3}(\mathbb{R}^2))$. Remarking that

$$\partial_{\widetilde{x}}^{-2}\partial_y^2 u_\pm = \mp\frac{1}{\alpha_\pm}\left(\partial_{\widetilde{x}}^{-1}\partial_\tau u_\pm + \beta_\pm \partial_{\widetilde{x}}^2 u_\pm + \frac{\gamma_\pm}{2}u_\pm^2\right),$$

it suffices to prove that $\partial_\tau u_\pm \in L^\infty([0,T], \partial_x H^{s-3})$. This can be deduced from the identities (3.9) and (3.10) of [18] under the assumption that $\partial_y^2 u^0_\pm \in \partial_x^2 H^{s-3}(\mathbb{R}^2)$. □

Owing to this lemma, we can use the second estimate of point (i) of lemma 2 to control the first terms of the rhs of (50). It follows that one can replace (52) by

$$|\boldsymbol{e}_\pm \cdot U_2(\tau,t,\cdot)|_{H^{s-3}} \leqslant C(|u_+|_{H^s_T}, |u_-|_{H^s_T})$$
$$+C(|u_+|_{H^s_T}, |u_-|_{H^s_T})\epsilon(t)\sqrt{t},$$

so that the proof of (iv) follows easily.

3.2.4. Proof of theorem 1. We first prove that the approximate solution $U^\varepsilon_{\rm app}(\varepsilon t,t,x,y)$ solves the i.v.p. (30) up to a small residual term.

Proposition 6. *Let* $s > 1$. *Suppose that assumptions 1 and 2 are satisfied and assume moreover that* $u^0_\pm \in \partial_x H^{s+7}(\mathbb{R}^2)$ *and* $\partial_y^2 u^0_\pm \in \partial_x^2 H^{s+3}(\mathbb{R}^2)$; *assume also that* $u^0_0 \in H^{s+4}(\mathbb{R}^2)$.

Then, for all $T > 0$, *the approximate solution* $U^\varepsilon_{\rm app}$ *given in (33) is defined on* $[0,T/\varepsilon]\times\mathbb{R}^2$ *and one has*

$$P^\varepsilon(\partial, U^\varepsilon_{\rm app})U^\varepsilon_{\rm app} = o(\varepsilon) \qquad \text{in} \quad L^\infty([0,T/\varepsilon]; H^s(\mathbb{R}^2)),$$

and $U^\varepsilon_{\rm app}(t=0) = U^{0,\varepsilon}$.

Proof. Recall first that $U_0 = u_+\boldsymbol{e}_+ + u_-\boldsymbol{e}_-$, $U_1 = u_0\boldsymbol{e}_0$ and that we showed in the proof of proposition 5 that $U_2(\varepsilon t, t, \cdot) = o(1/\sqrt{\varepsilon})$ in $L^\infty([0, T/\varepsilon]; H^r(\mathbb{R}^2))$ provided that $u^0_\pm \in \partial_x H^{r+4}$ and $\partial_y^2 u^0_\pm \in \partial_x^2 H^r$, for any $r > 1$. Except for the terms $\partial_\tau U_1$ and $\partial_\tau U_2$, it is thus easy to check that all the terms appearing in the expansion (39)–(43) and evaluated at $\tau = \varepsilon t$ are $o(1/\sqrt{\varepsilon})$ (or better) in $L^\infty([0, T/\varepsilon]; H^s(\mathbb{R}^2))$ under the assumptions on the initial data made in the statement of the proposition.

To give an estimate on $\partial_\tau U_1(\varepsilon t, t, \cdot)$ and $\partial_\tau U_2(\varepsilon t, t, \cdot)$, we need the following lemma.

Lemma 4. *One has, for all* $(\tau, t) \in [0, T] \times \mathbb{R}_+$,

$$|\partial_\tau U_1(\tau, t, \cdot)|_{H^s} \leqslant C(|u_+|_{H^{s+4}_T}, |u_-|_{H^{s+4}_T})$$

and

$$|\partial_\tau U_2(\tau, t, \cdot)|_{H^s} \leqslant C(|u_+|_{H^{s+6}_T}, |u_-|_{H^{s+6}_T})(1 + \epsilon(t)t),$$

with $\epsilon(t) \to 0$ *when* $t \to \infty$.

Proof. One has $\partial_\tau U_1 = \sqrt{\varepsilon}\partial_\tau u_0 \boldsymbol{e}_0$, and we recall that u_0 is given by (46). Differentiating (46) with respect to τ and using (49) to express $\partial_\tau u_\pm$ in terms of spatial derivatives of $u_\pm$, one can express $\partial_\tau u_0$ in terms of $\partial_{\tilde{x}}^{-2}\partial_y^3 u_\pm$, $\partial_{\tilde{x}}^2 \partial_y u_\pm$ and $\partial_y(u_\pm^2)$, and the first estimate of the lemma follows easily from lemma 3.

Similarly, differentiating (51) and (50) with respect to τ and using lemma 2 yields the second estimate. □

Using the lemma, one gets that $\varepsilon^{3/2}\partial_\tau U_1(\varepsilon t, t, \cdot) = O(\varepsilon^{3/2})$ and $\varepsilon^2 \partial_\tau U_2(\varepsilon t, t, \cdot) = o(\varepsilon)$ in $L^\infty([0, T/\varepsilon]; H^s(\mathbb{R}^2))$ and it is thus easy to conclude the proof. □

Remarking that

$$\begin{aligned} U^\varepsilon_{\text{app}} - U_{\text{KP}} = &(\boldsymbol{e}_+ \cdot U^\varepsilon_{\text{app}}(t, x, y) - u_+(\varepsilon t, x - t, y))\boldsymbol{e}_+ \\ &+(\boldsymbol{e}_- \cdot U^\varepsilon_{\text{app}}(t, x, y) - u_-(\varepsilon t, x + t, y))\boldsymbol{e}_- \\ &+(\boldsymbol{e}_0 \cdot U^\varepsilon_{\text{app}}(t, x, y) - \sqrt{\varepsilon} u_0(\varepsilon t, x, y))\boldsymbol{e}_0 \\ &+\sqrt{\varepsilon} u_0 \boldsymbol{e}_0(t, x, y), \end{aligned}$$

the first assertion of theorem 1(i) follows from proposition 5, while the second assertion is a consequence of proposition 6.

Moreover, under assumption 3, the system (30) is completely symmetric and thus well-posed in Sobolev spaces. Denoting U_{exact} the exact solution of the i.v.p. (30) and using classical arguments as in [3, 10], one can prove that its existence time is at least the same as for the approximate solution $U^\varepsilon_{\text{app}}$ constructed above (that is, $[0, T/\varepsilon]$)) and that $U_{\text{exact}} - U^\varepsilon_{\text{app}} = o(1)$ in $L^\infty([0, T/\varepsilon] \times \mathbb{R}^2)$. From the first part of the theorem, one also has $U^\varepsilon_{\text{app}} - U_{\text{KP}} = o(1)$ in $L^\infty([0, T/\varepsilon] \times \mathbb{R}^2)$, so that the second part of the theorem follows easily.

4. Convergence results for the water-waves equations

We describe in this section some asymptotic properties of the solutions of the water-waves equations. The first theorem shows that one can describe the solution of the water-waves equations using any of the systems of the class $\mathfrak{S}$ introduced in this paper. More precisely, if $\boldsymbol{p} = (\theta, \sigma, \lambda, \mu, \nu) \in [0,1]^2 \times \mathbb{R}^3$ and $U_p^\varepsilon := (v_p^\varepsilon, w_p^\varepsilon, \zeta_p^\varepsilon)$ solves the i.v.p

$$\begin{aligned} &S_{\boldsymbol{p}}(\partial)U_p^\varepsilon = 0,\\ &v^\varepsilon_{p\,|_{t=0}} := (1 - \tfrac{\varepsilon}{2}(1-\theta^2)\partial_x^2)^{-1} v^0,\\ &w^\varepsilon_{p\,|_{t=0}} := (1 - \tfrac{\varepsilon}{2}(1-\sigma^2)\partial_x^2)^{-1} w^0,\\ &\zeta^\varepsilon_{p\,|_{t=0}} := \zeta^0, \end{aligned} \tag{53}$$

one can approximate the solution of the water-waves equations in terms of U_p^ε.

Theorem 2. *Let $0 < \varepsilon_0 < 1$, $T > 0$, s large enough, and (ψ^0, ζ^0) be such that $(\nabla\psi^0, \zeta^0) \in H^s(\mathbb{R}^2)^{2+1}$.*

Let also $(\psi^\varepsilon, \zeta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be a family of solutions of (5) and define $v^\varepsilon := \partial_x\psi^\varepsilon$ and $w^\varepsilon := \sqrt{\varepsilon}\partial_y\psi^\varepsilon$. Assume moreover that

- *for all $0 < \varepsilon < \varepsilon_0$, $(\psi^\varepsilon, \zeta^\varepsilon)_{|_{t=0}} = (\psi^0, \zeta^0)$;*
- *the family $(v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is bounded in $L^\infty([0, T/\varepsilon], H^s(\mathbb{R}^2)^{2+1})$.*

For all $\boldsymbol{p} = (\theta, \sigma, \lambda, \mu, \nu) \in [0,1]^2 \times \mathbb{R}^3$, if $U_p^\varepsilon := (v_p^\varepsilon, w_p^\varepsilon, \zeta_p^\varepsilon)_\varepsilon$ solves (53) and is bounded in $L^\infty([0, T/\varepsilon], H^s(\mathbb{R}^2)^3)$, one has, for $\varepsilon > 0$ small enough,

$$\left|(v^\varepsilon, w^\varepsilon, \zeta^\varepsilon) - (v^\varepsilon_{\rm app}, w^\varepsilon_{\rm app}, \zeta^\varepsilon_p)\right|_{L^\infty([0,t],H^s)} \leqslant C\varepsilon^2 t,$$

where $v^\varepsilon_{\rm app} := (1 - \frac{\varepsilon}{2}(1-\theta^2)\partial_x^2)v_p^\varepsilon$ and $w^\varepsilon_{\rm app} := (1 - \frac{\varepsilon}{2}(1-\sigma^2)\partial_x^2)w_p^\varepsilon$.

Proof. Let $\underline{\boldsymbol{p}} \in [0,1]^2 \times \mathbb{R}^3$ be such that $S'_{\underline{\boldsymbol{p}}}(\partial) \in \Sigma$ (i.e. is a completely symmetric system).

Writing $\boldsymbol{p}_0 = (1,1,1,0,0)$, we know from the results of section 2.1 (see also example 1) that $U^\varepsilon := (v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is consistent with $S_{\boldsymbol{p}_0}(\partial)$. It follows therefore from propositions 2 and 3 that $(U_1^\varepsilon)_\varepsilon := (\widetilde{v}_1^\varepsilon, \widetilde{w}_1^\varepsilon, \zeta^\varepsilon)_\varepsilon$ is consistent with $S'_{\underline{\boldsymbol{p}}}(\partial) \in \Sigma$, where $\widetilde{v}_1^\varepsilon := (1 + \frac{\varepsilon}{2}\zeta^\varepsilon)v_1^\varepsilon$, $\widetilde{w}_1^\varepsilon := (1 + \frac{\varepsilon}{2}\zeta^\varepsilon)w_1^\varepsilon$ and

$$v_1^\varepsilon = (1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\partial_x^2)^{-1}v^\varepsilon, \qquad w_1^\varepsilon = (1 - \frac{\varepsilon}{2}(1-\underline{\sigma}^2)\partial_x^2)^{-1}w^\varepsilon.$$

Similarly, one obtains that $(U_2^\varepsilon)_\varepsilon := (\widetilde{v}_2^\varepsilon, \widetilde{w}_2^\varepsilon, \zeta_p^\varepsilon)_\varepsilon$ is consistent with $S'_{\underline{\boldsymbol{p}}}(\partial)$, where $\widetilde{v}_2^\varepsilon := (1 + \frac{\varepsilon}{2}\zeta_p^\varepsilon)v_2^\varepsilon$, $\widetilde{w}_2^\varepsilon := (1 + \frac{\varepsilon}{2}\zeta_p^\varepsilon)w_2^\varepsilon$ and

$$\begin{aligned} v_2^\varepsilon &= (1 - \frac{\varepsilon}{2}(1-\underline{\theta}^2)\partial_x^2)^{-1}(1 - \frac{\varepsilon}{2}(1-\theta^2)\partial_x^2)v_p^\varepsilon,\\ w_2^\varepsilon &= (1 - \frac{\varepsilon}{2}(1-\underline{\sigma}^2)\partial_x^2)^{-1}(1 - \frac{\varepsilon}{2}(1-\sigma^2)\partial_x^2)w_p^\varepsilon. \end{aligned}$$

It follows therefore from proposition 4 that

$$\forall 0 \leqslant t \leqslant \frac{T}{\varepsilon}, \qquad |U_1^\varepsilon - U_2^\varepsilon|_{L^\infty([0,t],H^s)} \leqslant C\varepsilon^2 t.$$

Remarking that $|U^\varepsilon - (v^\varepsilon_{\rm app}, w^\varepsilon_{\rm app}, \zeta^\varepsilon_{\rm app})|_{L^\infty([0,t],H^{s-2})} \leqslant C|U_1 - U_2|_{L^\infty([0,t],H^s)}$ for all $0 \leqslant t \leqslant T/\varepsilon$, one concludes easily the proof. □

Finally, we state a convergence theorem concerning the KP approximation: under suitable assumptions, any solution of the water waves approximation existing over times $O(1/\varepsilon)$ is well approximated by the uncoupled KP approximation.

Theorem 3. *Let $0 < \varepsilon_0 < 1$, $T > 0$, s large enough, and (ψ^0, ζ^0) be such that $(\nabla\psi^0, \zeta^0) \in H^s(\mathbb{R}^2)^{2+1}$.*

Let also $(\psi^\varepsilon, \zeta^\varepsilon)_{0<\varepsilon<\varepsilon_0}$ be a family of solutions of (5) and define $v^\varepsilon := \partial_x\psi^\varepsilon$ and $w^\varepsilon := \sqrt{\varepsilon}\partial_y\psi^\varepsilon$. Assume moreover that

- *for all* $0 < \varepsilon < \varepsilon_0$, $(\psi^\varepsilon, \zeta^\varepsilon)_{|_{t=0}} = (\zeta^0, \psi^0)$*;*
- *the family* $(v^\varepsilon, w^\varepsilon, \zeta^\varepsilon)_\varepsilon$ *is bounded in* $L^\infty([0, T/\varepsilon], H^{s+8}(\mathbb{R}^2)^{2+1})$.

If moreover $\partial_y^2 v^0, \partial_y^2 \zeta^0 \in \partial_x^2 H^s(\mathbb{R}^2)$*, then*

$$\zeta^\varepsilon(t, x, y) - \frac{1}{\sqrt{2}}\Big(\zeta_+(\varepsilon t, x - t, y) - \zeta_-(\varepsilon t, x + t, y)\Big) = o(1)$$

in $L^\infty([0, \frac{T}{\varepsilon}] \times \mathbb{R}^2_{x,y})$*, where* $\zeta_\pm(\tau, \xi, y)$ *denotes the solution of (KP)*$_\pm$ *with initial condition* $\zeta_\pm(t = 0) = ((\zeta^0 \pm v^0)/\sqrt{2})$.

Proof. Using the same notations as in the proof of theorem 2, and denoting by $U_3^\varepsilon := (v_3^\varepsilon, w_3^\varepsilon, \zeta_3^\varepsilon)$ the exact solution of $S'_{(\delta)}$ with initial condition $(\widetilde{v}_2^\varepsilon, \widetilde{w}_2^\varepsilon, \zeta^\varepsilon)_{|_{t=0}}$, one gets as in the proof of theorem 2 that $|\zeta^\varepsilon - \zeta_3^\varepsilon|_{L^\infty([0,t]\times\mathbb{R}^2)} \leqslant C\varepsilon^2 t$.

Remark now that $(\widetilde{v}_2^\varepsilon, \widetilde{w}_2^\varepsilon, \zeta^\varepsilon)_{|_{t=0}} = (\partial_x\psi^0, \sqrt{\varepsilon}\partial_y\psi^0, \zeta^0) + O(\varepsilon)$, so that if U_4^ε denotes the solution of $S'_{\boldsymbol{p}}(\delta)$ with initial condition $(\partial_x\psi^0, \sqrt{\varepsilon}\partial_y\psi^0, \zeta^0)$, one has $|U_3^\varepsilon - U_4^\varepsilon|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^2)} \leqslant C\varepsilon$. It follows that $\zeta^\varepsilon - \zeta_4^\varepsilon = O(\varepsilon)$ in $L^\infty([0, T/\varepsilon] \times \mathbb{R}^2)$. Since corollary 1 asserts that $\zeta_4^\varepsilon - \frac{1}{\sqrt{2}}(\zeta_+(\varepsilon t, x - t, y) - \zeta_-(\varepsilon t, x + t, y)) = o(1)$, the end of the proof is straightforward. □

Remark 9. Even though the above furnishes a convergence theorem for the KP approximation, this latter has two important drawbacks.

- It requires a non-realistic zero mass and zero momentum condition on the initial condition (precisely, $\partial_y^2\zeta^0 \in \partial_x^2 H^s$). This is due to the $\partial_x^{-1}\partial_y^2$ term in the KP equation which makes it very singular for low frequencies.
- The convergence rate is weak, namely $o(1)$. Since the physical values of ε are not very low (up to 0.1), the precision of the approximation can be quite weak.

These two drawbacks of the KP approximation underline the interest of the weakly transverse Boussinesq systems introduced in this paper: no zero mass assumption is needed, and the convergence rate is $O(\varepsilon^2 t)$. Moreover, as we saw in section 2.5.2, their dispersion relation is very close to the one of the linearized water-waves equations.

Remark 10. The $\partial_x^{-1}\partial_y^2$ term of the KP equation comes from the strong uni-directionalization made in its derivation. In the weakly transverse Boussinesq systems, the uni-directionalization is less strong (it is modelled by the introduction of the transverse wavelength μ in the nondimensionalization); this yields less disastrous consequences: the zero mass constraints are replaced by the possible growth in the second component of the velocity, taken into account by the assumption that $\sqrt{\varepsilon}\partial_y\psi$ remains bounded. In classical (isotropic) 2DH Boussinesq systems, no uni-directionalization is made, and these models are therefore free of constraints ($\partial_y\psi$ remains bounded). Unfortunately, these isotropic systems do not provide a good convergence rate in the present case: indeed, it is shown in [6] that the error made by these models in approximating the full water-waves equations is of size $\varepsilon^2 C(|\zeta^0|_{H^s}, |\nabla\psi^0|_{H^s})t$, for s large enough, and where (ζ^0, ψ^0) is the initial condition. For weakly transverse initial conditions of the form

$$\zeta^0(x, y) = \underline{\zeta}^0(x, \sqrt{\varepsilon}y), \qquad \psi^0(x, y) = \underline{\psi}^0(x, \sqrt{\varepsilon}y),$$

with $\underline{\zeta}^0$ and $\underline{\psi}^0$ bounded in $H^s(\mathbb{R}^2)$, the error estimates of [6] are therefore $\varepsilon^2 C(\varepsilon^{-1/4})t$ and may thus grow to infinity when $\varepsilon \to 0$.

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