A Stability Criterion for Two-Fluid Interfaces and Applications

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Abstract

We derive a new stability criterion for two-fluid interfaces that ensures the existence of "stable" local solutions that do not break down too fast due to Kelvin–Helmholtz instabilities. It can be seen both as a two-fluid generalization of the Rayleigh–Taylor criterion and as a nonlinear version of the Kelvin stability condition. We show that gravity can control the inertial effects of the shear up to frequencies that are high enough for the surface tension to play a relevant role. This explains why surface tension is a necessary condition for well-posedness while the (low frequency) main dynamics of interfacial waves are unaffected by it. In order to derive a practical version of this criterion, we work with a nondimensionalized version of the equations and allow for the possibility of various asymptotic regimes, such as the shallow water limit. This limit being singular, we have to derive a new symbolic analysis of the Dirichlet–Neumann operator that includes an infinitely smoothing "tail" accounting for the contribution of the bottom. We then validate our criterion by comparison with experimental data in two important settings: air–water interfaces and internal waves. The good agreement we observe allows us to discuss the scenario of wave breaking and the behavior of water-brine interfaces, and to propose a formula for the maximal amplitude of interfacial waves. We also show how to rigorously justify two-fluid asymptotic models used for applications and how to relate some of their properties to Kelvin–Helmholtz instabilities.

1. Introduction

1.1. General Setting and Overview of the Results

We are interested, here, in the motion of the interface between two incompressible fluids of different densities $\rho^+ > \rho^-$, with vorticity concentrated at the interface, and at rest at infinity. This problem is sometimes called the Rayleigh–Taylor problem; the limit cases $\rho^+ = \rho^-$ and $\rho^- = 0$ are also known as the Kelvin–Helmholtz and water-wave problems, respectively (see [9] for a recent review).

The mathematical analysis of the stability issue of such interfaces has raised considerable interest. The particular case of the water-wave problem is certainly the best understood. Following the pioneering works of [11,20,45,48,66], S. Wu established the well posedness of the water-wave equation for one-dimensional and two-dimensional surface waves [61,62] in infinite depth and without surface tension, provided that the *Rayleigh–Taylor criterion* is satisfied,

$$\text{(Rayleigh–Taylor)} \quad -\partial_z P \mid_{\text{surface}} > 0,$$

where z is the vertical coordinate and P the pressure (she also established that this condition is satisfied as long as the wave does not self-intersect). It is instructive to remark that the linearized version of this criterion (around the rest state) is simply $\rho^+ g > 0$, where g is the (vertical) acceleration of gravity—in other words, water must stand *below* the interface. Other approaches have also been developed, such as [39] (finite depth, Eulerian framework), [19,43,54] (fluid droplet in zero gravity, Lagrangian framework, irrotationality assumption removed), [18] (weak solutions in the whole space with density discontinuity at the surface), etc. More recently, various works have given some additional information on the solution. It was shown in [4,34] that the existence time is uniform with respect to the shallowness parameter (among others), allowing for the justification of various models used in coastal oceanography. In another direction, Wu [64,65] and Germain–Masmoudi–Shatah [26] showed almost global (two-dimensional) and global (three-dimensional) existence results under smallness conditions.

In presence of surface tension, the analysis is technically more difficult, but there is no striking difference for the behavior of the solution. Local well-posedness remains true and the solution of the water-wave problem with surface tension converges to the solution of the water-wave problem without surface tension when the surface tension coefficient goes to zero [6,7,44,67]. It is not known, however, whether this zero surface tension limit is compatible with the shallow water limit; the justification proposed in [32,33,53], for instance, of some shallow water models in the presence of surface tension is relevant for large values of the surface tension (more precisely, for Bond numbers of order $O(1)$) but provide an extremely short existence time when the Bond number is large (small surface tension), which is the case in coastal oceanography, for instance.

For the Rayleigh–Taylor problem ($\rho^- > 0$) the picture is completely different. It is known, at least for one-dimensional interfaces, that, outside the analytic framework of [59,60] (see also [31] for vortex sheets, where the authors are able to

obtain an existence time for analytic solutions that almost reaches the time where a curvature singularity appears), the evolution equations are ill-posed in absence of surface tension [24,35,36], even in very low regularity ($C^{1+\alpha}$ in [36,42] and chord-arc in [63])—note that [42,63] deal with the related Kelvin–Helmholtz problem where $\rho^+ = \rho^-$. The reason of this ill-posedness is that the nonlinearity creates locally a discontinuity of the tangential velocity at the interface that induces Kelvin–Helmholtz instabilities.

More recently, it has been proved that taking into account the surface tension restores the local well-posedness of the equations [5,8,14,55,56]. However, the existence time of the solution provided by these results is very small when the surface tension is small. The fact that the role of gravity (or gravity itself) is not considered in these references suggests that these general results can be improved in the "stable" configuration, where the heavier fluid is placed below the lighter one. In particular, we would like to explain two physical phenomena

♯1 Air–water interface. In coastal areas, for instance, waves propagate over several wavelengths (depending on various physical parameters introduced below). This observation is consistent with the results obtained in [4], where the air density is neglected ($\rho^- = 0$). If this density, which is small but nonzero, is taken into account, the existence results of [5,8,55,14] ensure that surface tension can prevent Kelvin–Helmholtz instabilities, but for times that allow the wave to propagate over only a few millimeters. We therefore want to understand the mechanism that controls Kelvin–Helmholtz instabilities for time scales consistent with the observations.

♯2 Internal waves. Waves at the interface of two immiscible fluids of possibly very similar densities have been widely investigated in the physics literature. Waves at the interface of two layers of water of different densities in the ocean are also commonly described with this formalism. As for the case of the air–water interface above, the known results only provide an existence time smaller by many orders of magnitude than what is observed.

These two phenomena raise an apparent paradox:

♯3 The asymptotic models describing waves in coastal areas or internal waves, and which provide very accurate results, all neglect surface tension. On the other hand, the analysis of the full equations tells us that surface tension is *crucial* for the existence of a solution. In other words, *surface tension allows interfacial waves to exist but does not play any role in their dynamics*.

Very recently, some progress has been made to understand ♯1. It has been proved in [15,49] that, the surface tension coefficient being kept fixed, the solution of the two-fluid system converges to the solution of the water-wave problem (with surface tension) as the density ρ^- goes to zero. In [50], the following important result was proved: the solution of the two-fluid system converges to the solution of the water-wave equations (without surface tension) when *both* the density ρ^- and the surface tension go to zero. It is required only that $(\rho^-)^2 \leqq \sigma^{7/3}$, where σ is the surface tension coefficient. A corollary of the result presented here is a slight improvement of this condition, showing that the result holds under the weaker condition $(\rho^-)^2 = o(\sigma)$ as suggested by the linear analysis. However, in physical

applications, the ratio $(\rho^-)^2/\sigma$ cannot be made arbitrarily small, and it would be interesting to have at our disposal a criterion to check whether a given interfacial wave will be "stable" (in the sense of "observable on a relevant time scale") or will almost instantaneously break down due to Kelvin–Helmholtz instabilities.

The need for such a criterion is also crucial to explain ♯2. Indeed, the smallness of the ratio $(\rho^-)^2/\sigma$ obviously cannot be invoked to explain the observation of such phenomena as internal waves since the relative density of the upper fluid is not small in that case (while σ remains very small), and a better understanding of the formation of Kelvin–Helmholtz instabilities is required.

A hint of what such a criterion should look like can be obtained by considering the related problem of the *linear* stability of two fluid layers moving with constant horizontal velocities $c^+ \neq c^-$ (with $c^\pm \in \mathbb{R}^d, d = 1, 2$). It is known since Kelvin that small perturbations of the interface are linearly stable if the following criterion is satisfied (see for instance [13])

$$\text{(Kelvin)} \quad g(\rho^+ - \rho^-) > \frac{1}{4\sigma} \frac{(\rho^+\rho^-)^2}{(\rho^+ + \rho^-)^2} \mathfrak{c}(0) |[\![c^\pm]\!]|^4,$$

where $[\![c^\pm]\!] = c^+ - c^-$ and $\mathfrak{c}(0)$ is a constant taking into account the geometry of the problem.[1] The equivalent of this linear criterion in the water-wave case is the linear Rayleigh–Taylor criterion already mentioned, namely, $\rho^+ g > 0$. In the problem we investigate here both fluids are at rest at infinity, but there is is a local discontinuity of the horizontal velocity created by the *nonlinear* motion of the interfacial waves. We can therefore expect that a local version of the Kelvin criterion may be derived to assess the local linear stability of interface perturbations. A natural question is then, *is there a nonlinear generalization of this expected local Kelvin criterion?* Or equivalently, *is there a generalization to two-fluid interfaces of the Rayleigh–Taylor criterion?*

The first main result of this paper is to show that such a nonlinear criterion exists. It can be stated as

$$[\![-\partial_z P^\pm_{|z=\zeta}]\!] > \frac{1}{4} \frac{(\rho^+\rho^-)^2}{\sigma(\rho^+ + \rho^-)^2} \mathfrak{c}(\zeta) |\omega|_\infty^4, \tag{1.1}$$

where ζ is the interface parametrization, $\omega = [\![V^\pm_{|z=\zeta}]\!]$ is the jump of the horizontal velocity at the interface, and $\mathfrak{c}(\zeta)$ is a constant that depends on the geometry of the problem (two layers of finite depth in this paper) and can be estimated quite precisely.[2]

The question is now to check whether the criterion (1.1) can explain (for instance) the observation of the phenomena ♯1 and ♯2 mentioned above.

For ♯1 (air–water interface), it is likely that the relative density $\rho^- = \rho^-/(\rho^+ + \rho^-)$ is small enough to make the right-hand side of (1.1) smaller than the left-hand

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[1] $\mathfrak{c}(0) = 1$ for the case of two layers of infinite depth considered in [13]; see Remark 23 for an explicit expression in the case of two layers of finite depth.
[2] For a flat interface $\zeta = 0$, this constant is the same as in the linear Kelvin criterion.

side, even though σ is small. This is at least the case if we consider the limit $(\underline{\rho}^-)^2/\sigma \to 0$ as in [50].

For $\sharp 2$ (internal waves), the relative density $\underline{\rho}^-$ is close to $1/2$ and cannot be used to explain why (1.1) should be satisfied in that case. Therefore, *the small-ness of the discontinuity on the horizontal velocity is the only reason that could explain* $\sharp 2$.

The main idea behind the stability criterion (1.1) is the following explanation of the apparent paradox $\sharp 3$, which we can state in rough mathematical terms as follows: *the Kelvin–Helmholtz instabilities appear above the frequency threshold for which surface tension is relevant, while the main (observable) part of the wave involves low frequencies located below this frequency threshold*. Consequently, the Kelvin–Helmholtz instabilities are regularized by surface tension, while the main part of the wave is unaffected by it. The role of criterion (1.1) is to ensure that the frequency threshold mentioned above is high enough. For low frequencies, we show that gravity can stabilize the inertial effects of the shear. The main task in this paper is to rigorously prove this scenario.

Remark 1 (Rayleigh–Taylor stability vs. Kelvin–Helmholtz instability). As already mentioned, surface tension has been shown to regularize the Kelvin–Helmholtz instabilities (wihch are due to the shear). As a matter of fact, it also stabilizes Rayleigh–Taylor instability in the sense that it allows for a local solution, even if the heavier fluid is placed above the lighter one. Such a solution is an example of what would be an unstable solution for us; even though the initial value problem is locally well-posed, it does not satisfy (1.1) and the existence time is very small. In the stable configuration (heavier fluid below), we said that we use gravity to control the destabilizing effects of the shear; equivalently, Rayleigh–Taylor *stability* controls the Kelvin–Helmholtz *instabilities* at low frequencies.

Having proved that (1.1) determines the stability of interfacial waves (and constructed such solutions to the two-fluid equations), one practical question arises: how can we have access to the quantities $[\![-\partial_z P^\pm]\!]$ and $|\omega|_\infty$, which we need to know to determine whether (1.1) is satisfied or not? Indeed, one would like to use this criterion to assess the stability of some given configuration. For instance, knowing the depth and density of both fluid layers, is it possible to have a stable perturbation of the interface of amplitude a and wavelength λ?

It is quite difficult to know $[\![-\partial_z P^\pm]\!]$ and $|\omega|_\infty$ experimentally (see however [28] for measurements of ω). A useful approach, therefore, is to perform an asymptotic analysis of the problem. Of particular importance for applications is the shallow water regime (the wavelength of the perturbation is large compared to the depths H^+ and H^- of the fluid layers). In this regime, the pressure satisfies the hydrostatic approximation with fairly good precision, and one therefore has $[\![-\partial_z P^\pm]\!] \sim (\rho^+ - \rho^-)g$. It is also possible to show that ω has a typical size of order $\frac{a}{H}\sqrt{g'H}$ (with $g' = (\underline{\rho}^+ - \underline{\rho}^-)g$ and $H = \frac{H^+H^-}{\rho^+H^-+\rho^-H^+}$), a prediction that is consistent with the measurements of [28], for instance. Plugging these approximations into (1.1) shows the relevance of the dimensionless parameter Υ defined as

$$\Upsilon = (\underline{\rho}^+ \underline{\rho}^-)^2 \frac{a^4}{H^2} \frac{(\rho^+ + \rho^-)g'}{4\sigma},$$

and suggests a very simple practical stability criterion in shallow water,

$$\Upsilon \ll 1 : \text{ Stable configuration}; \quad \Upsilon \gg 1 : \text{ Unstable configuration.} \quad (1.2)$$

When $\Upsilon \sim 1$, stability/instability is critical and it is necessary to look at the exact criterion (1.1).

In order to rigorously prove the relevance of the practical criterion (1.2), one must be able to handle the shallow water limit in the construction of the stable solutions allowed by (1.1). Unfortunately, this limit is singular, which complicates the proof. In particular, it restricts the number of tools at our disposal since standard symbolic analysis is not adapted to shallow water regimes. Roughly speaking, symbolic analysis neglects the information coming from the bottom since it is infinitely smoothing (this is the argument used in [1] to prove local well posedness of the water-wave equations over very exotic bottoms). However, in the shallow limit, the influence of the bottom is the main factor in the evolution of the wave; though infinitely smooth, its contribution becomes quantitatively very large. In the water-wave case, various techniques allow one to bypass symbolic analysis and justify some shallow water models [4,34]. In the two-fluid case under investigation here, it is not clear whether these techniques can be adapted. We therefore appealed to a technical result of independent interest, namely, a symbolic analysis "with tail" of the Dirichlet–Neumann operator, the tail corresponding to the infinitely smoothing component of this operator taking into account the influence of the bottom. This allows us to use some (rudimentary) symbolic analysis without suffering from the shallow water singularity.

Once this analysis is done, we know that interfacial waves are well behaved in the shallow water limit when the stability criterion (1.1)—or its practical version (1.2)—is satisfied. We then apply this result to the two phenomena ♯1 and ♯2 described above. The good agreement with experimental data allows us to confirm the relevance of one-fluid asymptotic models for air–water interfaces, and to give the first rigorous justification (on the relevant time scale) of two-fluid asymptotic models commonly used to describe internal waves. We also indicate how some singularities of these models can be related to Kelvin–Helmholtz instabilities. Finally, we discuss three phenomena:

♯1 Air–Water interfaces. Simple physical examples show that close to wave breaking, it is likely that (1.1) fails to be satisfied for certain configurations. This would mean that *the first singularity observed in wave breaking may not always be a singularity of the water-wave equations, but sometimes a two-fluid singularity*.

♯2 The description of water-brine interfaces (as for oceanic internal waves) does not fall into the range of equations (1.3)–(1.8) because water and brine are not immiscible. *We use the stability criterion (or criteria) derived in this paper to propose a two-fluid description of the water-brine interface* that includes an artificial surface tension σ. Using comparison with laboratory experiments, we are able to propose a numerical value for σ.

♯3 Formula for the maximal amplitude of interfacial waves. We have seen that the practical stability criterion (1.2) can be used to determine the surface tension; conversely, if the surface tension and other parameters (densities, depth) are known, it can be used to determine the order of the maximal amplitude that interfacial waves can reach before being destroyed by Kelvin–Helmholtz instabilities. The formula obtained this way is

$$a^4 = \frac{4\sigma H^2}{(\underline{\rho^+ \rho^-})^2 (\rho^+ + \rho^-) g'}.$$

In order to construct a more detailed description of the results described above, we first need to write the equations and to derive their nondimensionalized version. This is done in the following two subsections.

1.2. The Equations

We assume that the interface is parametrized by a function $\zeta(t, X)$ ($X \in \mathbb{R}^d$) and denote by Ω_t^+ and Ω_t^- the volume occupied by the lower and upper fluids, respectively, at time t. Choosing the origin of the vertical axis to correspond with the interface between the two fluids at rest, we assume that Ω_t^+ (respectively Ω_t^-) is bounded below (respectively above) by a horizontal wall located at $z = -H^+$ (respectively $z = H^-$). We also denote by Γ_t the interface $\Gamma_t = \{(X, z), z = \zeta(t, X)\}$ and by $\Gamma^\pm$ the upper and lower boundaries $\Gamma^\pm = \{z = \mp H^\pm\}$.
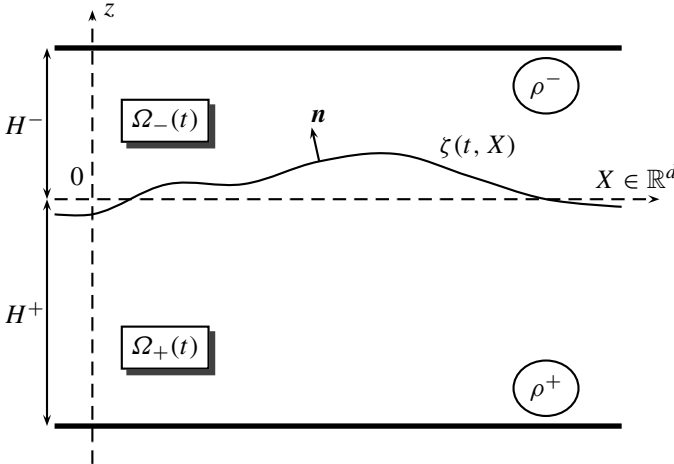
Finally, we denote by $\mathbf{U}^\pm$ the velocity field in $\Omega_t^\pm$; the horizontal component of $\mathbf{U}^\pm$ is written $V^\pm$ and its vertical one $w^\pm$. The pressure is denoted by $P^\pm$.

For the sake of clarity, it is also convenient to introduce some notation to express the difference and average of these quantities across the interface.

Notation 1. *If A^+ and A^- are two quantities (real numbers, functions, etc.), the notations $[\![A^\pm]\!]$ and $\langle A^\pm \rangle$ stand for*

$$[\![A^\pm]\!] = A^+ - A^- \quad \text{and} \quad \langle A^\pm \rangle = \frac{A^+ + A^-}{2}.$$

We can now state the equations of motion.

– Equations in the fluid layers. In both fluid layers, the velocity field $\mathbf{U}^\pm$ and the pressure $P^\pm$ satisfy the equations

$$\text{div }\mathbf{U}^\pm(t, \cdot) = 0, \qquad \text{curl }\mathbf{U}^\pm(t, \cdot) = 0, \qquad \text{in } \Omega_t^\pm \quad (t \geqq 0), \qquad (1.3)$$

which express the incompressibility and irrotationality assumptions, and

$$\rho^\pm\big(\partial_t \mathbf{U}^\pm + (\mathbf{U}^\pm \cdot \nabla_{X,z})\mathbf{U}^\pm\big) = -\nabla_{X,z} P^\pm - \rho^\pm g \mathbf{e}_z \qquad \text{in } \Omega_t^\pm \quad (t \geqq 0), \tag{1.4}$$

which expresses the conservation of momentum (Euler equation).

– Boundary conditions at the rigid bottom and lid. Impermeability of these two boundaries is classically rendered by

$$w^\pm(t, \cdot)_{|\Gamma^\pm} = 0, \qquad (t \geqq 0). \tag{1.5}$$

– Boundary conditions at the moving interface. The fact that the interface is a bounding surface (the fluid particles do not cross it) yields the equations

$$\partial_t \zeta - \sqrt{1 + |\nabla \zeta|^2}\, \mathbf{U}_n^\pm = 0, \qquad (t \geqq 0), \tag{1.6}$$

where $\mathbf{U}_n^\pm := \mathbf{U}^\pm{}_{|\Gamma_t} \cdot \mathbf{n}$, $\mathbf{n}$ being the upward unit normal vector to the interface Γ_t. A direct consequence of (1.6) is that *there is no jump of the normal component of the velocity at the interface*. Finally, the continuity of the stress tensor at the interface gives in our particular case

$$[\![P^\pm(t, \cdot)_{|\Gamma_t}]\!] = \sigma k(\zeta), \qquad (t \geqq 0), \tag{1.7}$$

where σ is the surface tension coefficient and $k(\zeta)$ denotes the mean curvature of the interface,

$$k(\zeta) = -\nabla \cdot \Big(\frac{\nabla \zeta}{\sqrt{1 + |\nabla \zeta|^2}} \Big).$$

It is quite convenient for the mathematical analysis of the equations (1.3)–(1.7) to transform them into a set of scalar equations on the interface. From the irrotationality assumption, we can write the velocities $\mathbf{U}^\pm$ in terms of velocity potentials $\Phi^\pm$, namely,

$$\mathbf{U}^\pm(t, \cdot) = \nabla_{X,z} \Phi^\pm(t, \cdot) \qquad \text{in } \Omega_t^\pm \quad (t \geqq 0).$$

Defining the trace of $\Phi^\pm$ at the interface by

$$\psi^\pm(t, \cdot) = \Phi^\pm(t, \cdot)_{|\Gamma_t} \qquad (t \geqq 0),$$

we can now reduce the equations (1.3)–(1.7) to a set of equations on ζ and $\psi^\pm$, namely,

$$\partial_t \zeta - G^+[\zeta]\psi^+ = 0, \tag{1.8}$$

$$G^+[\zeta]\psi^+ = G^-[\zeta]\psi^-, \tag{1.9}$$

$$\rho^\pm\left(\partial_t \psi^\pm + g\zeta + \frac{1}{2}|\nabla\psi^\pm|^2 - \frac{(G^\pm[\zeta]\psi^\pm + \nabla\zeta \cdot \nabla\psi^\pm)^2}{2(1+|\nabla\zeta|^2)}\right) = -P^\pm_{|\Gamma_t}, \tag{1.10}$$

$$[\![P^\pm(t,\cdot)_{|\Gamma_t}]\!] = \sigma \, \kappa(\zeta), \tag{1.11}$$

where $G^\pm[\zeta]$ are the Dirichlet–Neumann operators corresponding to the two fluid layers and defined (under reasonable assumptions on ζ and $\psi^\pm$) by

$$G^\pm[\zeta]\psi^\pm = \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi^\pm_{|\Gamma_t},$$

where $\Phi^\pm$ solve

$$\begin{cases} \Delta_{X,z}\Phi^\pm = 0 & \text{in } \Omega^\pm_t, \\ \Phi^\pm_{|\Gamma_t} = \psi^\pm, \quad \partial_z\Phi^\pm_{|\Gamma^\pm} = 0 \end{cases} \tag{1.12}$$

and $\partial_n\Phi^\pm_{|\Gamma_t}$ stands for the *upward* normal partial derivative of $\Phi^\pm$ at the interface.

We now show[3] that it is possible to reduce the two-fluid equations to a set of two equations on the surface elevation ζ and of the quantity ψ defined as

$$\psi := \rho^+\psi^+ - \rho^-\psi^-,$$

where $\rho^\pm$ stands for the relative density, $\rho^\pm = \frac{\rho^\pm}{\rho^+ + \rho^-}$ (in particular, $\rho^+ + \rho^- = 1$ and $\rho^+ - \rho^-$ is the Atwood number). Defining the operator $G[\zeta]$ by

$$G[\zeta] = G^-[\zeta]\left(\rho^+ G^-[\zeta] - \rho^- G^+[\zeta]\right)^{-1}G^+[\zeta], \tag{1.13}$$

one can write $G^+[\zeta]\psi^+ = G^-[\zeta]\psi^-$ in terms of ζ and ψ only,

$$G[\zeta]\psi = G^+[\zeta]\psi^+ = G^-[\zeta]\psi^- \tag{1.14}$$

and one can also get $\psi^\pm$ in terms of ζ and ψ,

$$\psi^\pm = G^\pm[\zeta]^{-1} \circ G[\zeta]\psi. \tag{1.15}$$

A formulation of the two-fluid equations as a system of two scalar evolution equations on ζ and ψ can then be given,

$$\begin{cases} \partial_t \zeta - G[\zeta]\psi = 0, \\ \partial_t \psi + g'\zeta \dfrac{1}{2}[\![\rho^\pm|\nabla\psi^\pm|^2]\!] \\ \quad -\dfrac{1}{2}\dfrac{[\![\rho^\pm(G^\pm[\zeta]\psi^\pm + \nabla\zeta \cdot \nabla\psi^\pm)^2]\!]}{1+|\nabla\zeta|^2} = -\dfrac{\sigma}{\rho^+ + \rho^-}\kappa(\zeta), \end{cases} \tag{1.16}$$

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[3] We do not justify the different steps of the derivation here; this will be done in Section 2.

where $\psi^\pm = \mathcal{G}^\pm[\zeta]^{-1} \circ \mathcal{G}[\zeta]\psi$ and g' stands for the reduced gravity,

$$g' = (\underline{\rho}^+ - \underline{\rho}^-)g.$$

Remark 2. In the case $\rho^- = 0$, one has $\psi = \psi^+$ and $\mathcal{G}[\zeta]\psi = \mathcal{G}^+[\zeta]\psi^+$ and (1.16) reduces to the formulation of the water-wave equations in terms of the surface elevation ζ and the velocity potential at the surface ψ^+ due to ZAKHAROV [68] and CRAIG–SULEM [22]. In the case $\rho^- \neq 0$, ζ and ψ are the two canonical variables of the Hamiltonian formulation exhibited by BENJAMIN and BRIDGES [10] (see also [21,38])

1.3. Nondimensionalized Equations

The order of magnitude of the existence time of the solution is our main concern in this paper. It is therefore crucial to get some information on the size of the different quantities appearing in the equation, and in particular of those that play a role in our stability criterion (1.1). This information can be obtained by experimental measurements, but it is of course preferable to know them *a priori* in terms of the physical characteristics of the flow under consideration, namely,

– The typical size a of the interfacial waves
– The typical wavelength λ of these waves
– The depth $H^\pm$ of the upper and lower fluid layers at rest
– The value of the surface tension coefficient σ
– The densities $\rho^\pm$ of the two fluids

We show in Appendix 6.3 how linear theory can be used to obtain such an *a priori* estimate of the order of magnitude of the different unknowns. The best way to exploit this information is to nondimensionalize the equations, that is, to perform a linear change of unknowns and variables such that all the unknowns are now dimensionless and of size $O(1)$ for typical configurations.

Writing (1.16) in such a dimensionless form requires the introduction of various parameters; the first two are given by

$$\varepsilon = \frac{a}{H}, \qquad \mu = \frac{H^2}{\lambda^2}, \quad \text{with} \quad H = \frac{H^+ H^-}{\underline{\rho}^+ H^- + \underline{\rho}^- H^+},$$

in the water-wave configuration, that is, when $\underline{\rho}^- = 0$, ε is called the amplitude or nonlinearity parameter and μ the shallowness parameter. Throughout this article, μ is assumed to remain bounded, as is ε. For notational convenience, and without loss of generality, we take

$$0 \leqq \mu \leqq 1, \quad 0 \leqq \varepsilon \leqq 1.$$

The above definition of ε and μ takes into account both fluid layers at the same time. For a more specific description of each fluid layer, it is convenient to introduce

$$\varepsilon^\pm = \frac{a}{H^\pm}, \qquad \mu^\pm = \frac{(H^\pm)^2}{\lambda^2}.$$

We also need to define the relative depth $\underline{H}^{\pm}$ and the Bond number[4] Bo as

$$\underline{H}^{\pm} = \frac{H^+}{H}, \qquad \mathrm{Bo} = \frac{(\rho^+ + \rho^-)g'\lambda^2}{\sigma}.$$

Remark 3. In full generality, one should allow H^+ to be much larger or much smaller than H^- (as in [12] for instance). For the sake of clarity, we assume throughout this paper that H^+ and H^- are of the same order (so that $\varepsilon^+ \sim \varepsilon^- \sim \varepsilon$ and $\mu^+ \sim \mu^- \sim \mu$). There does not seem to be any obstruction other than technical to generalizing our method to more general configurations.

The next step is to introduce dimensionless Dirichlet–Neumann operators,

$$\mathcal{G}_{\mu}^{\pm}[\varepsilon\zeta, \underline{H}^{\pm}]\psi^{\pm} = \sqrt{1 + |\varepsilon\nabla\zeta|^2}\partial_n \Phi_{|z=\varepsilon\zeta}^{\pm},$$

where $\Phi^{\pm}$ solve

$$\begin{cases} (\mu\Delta + \partial_z^2)\Phi^{\pm} = 0 & \text{in } -\underline{H}^{\pm} < \pm z < \pm\varepsilon\zeta, \\ \Phi^{\pm}{}_{|z=\varepsilon\zeta} = \psi^{\pm}, \qquad \partial_z\Phi^{\pm}{}_{|z=\mp\underline{H}^{\pm}} = 0, \end{cases} \qquad (1.17)$$

and $\partial_n\Phi^{\pm}{}_{|z=\varepsilon\zeta}$ stands for the *upward* conormal derivative of $\Phi^{\pm}$ at the interface (that is, $\mathcal{G}_{\mu}^+[\varepsilon\zeta]\psi^{\pm} = \partial_z\Phi^{\pm}{}_{|z=\varepsilon\zeta} - \varepsilon\mu\nabla\zeta \cdot \nabla\phi^{\pm}{}_{|z=\varepsilon\zeta}$). The operator $\mathcal{G}[\zeta]$ defined in (1.13) has similarly a dimensionless version given by

$$\mathcal{G}_{\mu}[\varepsilon\zeta] = \mathcal{G}_{\mu}^-[\varepsilon\zeta, \underline{H}^-]\left(\underline{\rho}^+ \mathcal{G}_{\mu}^-[\varepsilon\zeta, \underline{H}^-] - \underline{\rho}^- \mathcal{G}_{\mu}^+[\varepsilon\zeta, \underline{H}^+]\right)^{-1} \mathcal{G}_{\mu}^+[\varepsilon\zeta, \underline{H}^+]. \tag{1.18}$$

We show in Appendix 6.3 that the equations (1.16) can then be written in dimensionless form as

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}\mathcal{G}_{\mu}[\varepsilon\zeta]\psi = 0, \\ \partial_t\psi + \zeta + \varepsilon\dfrac{1}{2}[\![\underline{\rho}^{\pm}|\nabla\psi^{\pm}|^2]\!] \\ \quad -\dfrac{1}{2}\dfrac{\varepsilon}{\mu}\dfrac{[\![\underline{\rho}^{\pm}(\mathcal{G}_{\mu}^{\pm}[\varepsilon\zeta, \underline{H}^{\pm}]\psi^{\pm} + \varepsilon\mu\nabla\zeta \cdot \nabla\psi^{\pm})^2]\!]}{1 + \varepsilon^2\mu|\nabla\zeta|^2} = -\dfrac{1}{\mathrm{Bo}}\dfrac{1}{\varepsilon\sqrt{\mu}}\mathcal{K}(\varepsilon\sqrt{\mu}\zeta), \end{cases} \qquad (1.19)$$

with $\psi^{\pm} = \mathcal{G}_{\mu}^+[\varepsilon\zeta, \underline{H}^+]^{-1} \circ \mathcal{G}_{\mu}[\varepsilon\zeta]\psi$.

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[4] The Bond number measures the ratio of gravity forces over capillary forces. For some reason, it often appears as the inverse of this quantity in the mathematics literature.

1.4. Description of the Results and Organization of the Paper

In Section 2 we give some preliminary results that provide a rigorous basis to the manipulations performed in Section 1.2 and 1.3, above, to derive the equations (1.16) and (1.19). General notations and definitions are given in Section 2.1 and the Dirichlet–Neumann operators $G_\mu^\pm[\varepsilon\zeta, \underline{H}^\pm]$ are studied in Section 2.2. Their invertibility properties are investigated in (Section 2.2.3); a technical difficulty is that the inverse of the DN operator takes values in Beppo–Levi spaces (that is, functions with gradient in Sobolev spaces, but not necessarily in L^2 since we consider unbounded domains). The structure of the shape derivatives of the DN operators are discussed in Section 2.2.4. We are then able to address in Section 2.3 a transmission problem that is crucial to rigorously deriving the two-fluid equations (1.19), since it states that the trace of the potential at the upper and lower parts of the interface, ψ^- and ψ^+, are fully determined by ζ and $\psi = \rho^+\psi^+ - \rho^-\psi^-$. This allows us to give a rigorous definition to the operator $G = \widetilde{G_\mu}[\varepsilon\zeta]$ introduced in (1.18). Since we want to be able to handle the shallow water limit, we need to give two versions for (almost) all the estimates given in this section: one that is sharp with respect to regularity but not optimal with respect to the dependence on μ, and one that is sharp with respect to μ but not optimal with respect to regularity. For instance, $G_\mu^\pm[\varepsilon\zeta, \underline{H}^\pm]$ has an operator norm of size $O(\sqrt{\mu})$ if it is seen, as usual, as a first order operator; but if we consider it as a second order operator, we get a better control of size $O(\mu)$ on its operator norm (see Remark 5 below).

Section 3 is devoted to the symbolic analysis of the Dirichlet–Neumann operator $G^+ = \underline{H}^+ G_\mu^+[\varepsilon\zeta, \underline{H}^+]$. It is standard [3,39,54,57] that the principal symbol of G^+ can be written in terms of the Laplace–Beltrami associated to the surface. More precisely, if we define the symbol $g(X, \xi)$ as

$$g(x, \xi) = \sqrt{|\xi|^2 + \varepsilon^2\mu(|\nabla\zeta|^2|\xi|^2 - (\nabla\zeta \cdot \xi)^2)},$$

then the following holds,

$$\forall 0 \leqq s \leqq t_0, \quad |G^+\psi - \sqrt{\mu^+}g(x, D)\psi|_{H^{s+1/2}} \leqq M(t_0 + 3)|\nabla\psi|_{H^{s-1/2}}, \quad (1.20)$$

where $t_0 > d/2$ and $M(t_0 + 3)$ is a constant depending on $|\zeta|_{H^{t_0+3}}$ and the minimal depth of the lower fluid. Since G^+ and $g(x, D)$ are first order operators, this identity shows that G^+ can be replaced by $g(x, D)$ up to a more regular (zero order) operator. Unfortunately, such a substitution is not possible in the shallow water limit, since this zero order operator is of size $O(1)$ with respect to μ, while both G^+ and $g(x, D)$ have operator norms of size $O(\sqrt{\mu})$. Symbolic analysis induces, therefore, a singularity of order $O(\mu^{-1/2})$ when the shallow water limit is considered. The explanation of this behavior is that the shallow water regime corresponds to configurations where the bottom plays a very important role; however, the contribution of the bottom to the Dirichlet–Neumann operator is analytic and therefore neglected (at any order) by symbolic analysis. In order to handle this difficulty, we propose a symbolic analysis "with tail" of the DN operator, which takes into account the infinitely smoothing (but very large) contribution of the bottom. This leads us to

replace the symbol $g(X, \xi)$ by $g(X, \xi)t^+(X, \xi)$, where

$$t^+(X, \xi) = (1 + \varepsilon^+ \zeta) \int_{-1}^0 \frac{\sqrt{|\xi|^2 + \varepsilon^2 \mu (z + 1)^2 (|\nabla \zeta|^2 |\xi|^2 - (\nabla \zeta \cdot \xi)^2)}}{1 + \varepsilon^2 \mu (z + 1)^2 |\nabla \zeta|^2} dz,$$

(when $d = 1$, this simplifies into $t^+(x, \xi) = (1 + \varepsilon^+ \zeta) \dfrac{\arctan(\varepsilon \sqrt{\mu} \partial_x \zeta)}{\varepsilon \sqrt{\mu} \partial_x \zeta} |\xi|$). We then show in Theorem 4 that (1.20) can be improved to

$$\left| \mathcal{G}^+ \psi - \sqrt{\mu^+} \mathrm{Op} \big[g \tanh \big(\sqrt{\mu^+} t^+ \big) \big] \psi \right|_{H^{s+1/2}} \leqq \varepsilon \sqrt{\mu} M (t_0 + 3) |\nabla \psi|_{H^{s-1/2}}.$$

The cost in terms of derivatives of ψ is the same as in (1.20), but the behavior with respect to the parameters ε and μ is much better, since there is a gain of size $O(\varepsilon \sqrt{\mu})$. This allows us to handle the shallow water limit. We also give in Section 3 the symbolic analysis (with tail) of $(\mathcal{G}^-)^{-1} \mathcal{G}^+$, $(\mathcal{G}^-)^{-1} \mathcal{G}$ and $\mathcal{G}(\mathcal{G}^-)^{-1} \partial_j$ that are needed for the analysis of the two-fluid equations.

In Section 4, we show that it is possible to quasilinearize the two-fluid equations (1.19) by differentiating them and writing them in terms of the relevant unknowns. We first introduce and study in Section 4.1 some new operators that are needed to write the quasilinearized equations. An important step is a linearization formula for the operator $\mathcal{G}$ that is given in Section 4.2; the main step in establishing this linearization formula is an explicit formula for the derivatives of the mapping $\zeta \mapsto \mathcal{G}_\mu[\varepsilon \zeta] \psi$ (shape derivative). This linearization formula also suggests what the good unknown should be. More precisely, the equations (1.19) differentiated α-times should not be written in terms of $\partial^\alpha \zeta$ and $\partial^\alpha \psi$, but in terms of $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$ defined as

$$\zeta_{(\alpha)} = \partial^\alpha \zeta, \qquad \psi_{(\alpha)} = [\![\underline{\rho}^\pm \partial^\alpha \psi^\pm]\!] - [\![\underline{\rho}^\pm \underline{w}^\pm]\!] \partial^\alpha \zeta,$$

where $\underline{w}^\pm$ is the vertical component of the velocity at the interface in the $\pm$ fluid, while $\psi^\pm$ is as in (1.19). When $\rho^- = 0$, these unknowns coincide, of course, with the ones used in [34,51] to write the quasilinearized water-wave equations. The quasilinearized equations are then derived in Section 4.3. Differentiating α times the second equation of (1.19) and rewriting it in terms of $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$, one gets (without surface tension)

$$\partial_t \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta + \varepsilon \langle \underline{V}^\pm \rangle \cdot \nabla \psi_{(\alpha)} + \varepsilon [\![V^\pm]\!] \cdot \nabla \langle \underline{\rho}^\pm \psi_{(\alpha)}^\pm \rangle \sim 0$$

(the symbol ~ 0 means that harmless terms are omitted); in this expression, $\mathfrak{a}$ can be related to the (dimensionless) jump of the vertical derivative of the pressure at the interface, and $\underline{V}^\pm$ stands for the horizontal velocity at the interface in the fluid $\pm$. In order to write this equation in terms of the good unknowns $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$, it is necessary to write $\langle \underline{\rho}^\pm \psi_{(\alpha)}^\pm \rangle$ in terms of $[\![\underline{\rho}^\pm \psi_{(\alpha)}^\pm]\!] = \psi_{(\alpha)}$ and $\zeta_{(\alpha)}$. This is completely trivial in the water-wave case $\rho^- = 0$. The dependence of this term on $\zeta_{(\alpha)}$ is therefore specific to the two-fluid system, *and it is responsible for the Kelvin–Helmholtz instabilities*—through the operator $\mathcal{E}[\zeta]$ in the quasilinear system given in Proposition 8.

The main existence and stability results are then given in Section 5. We first state in Section 5.1 several versions of the stability criterion. The first one (see Section 5.1.1) differs slightly from (1.1) because it also involves the L^∞-norm of first order space-time derivatives of the jump of the horizontal velocity. In Section 5.1.2, we show how to recover (1.1) when the jump of velocity is non-zero. The practical criterion (1.2) is then derived in Section 5.1.3. After some considerations on the conditions we impose on the initial data (see Section 5.2), we state in Section 5.3 the main results of this paper, which show that *"stable" solutions exist under the criterion given in Section 5.1.1 and 5.1.2* (we give two versions of the theorem). By stable, we mean that the existence time depends on the surface tension σ through the stability criterion (5.1) only. In particular, it is not necessarily small if σ is small. We want to emphasize the fact that the theorems are given under a nondimensionalized form and that the existence time is uniform with respect to the parameters ε and μ. In particular, this allows one to handle the shallow water limit (or other regimes) straightforwardly. Other kinds of limits (zero density and zero surface tension) are also direct corollaries. We then prove in Section 5.4 that these stable solutions persist over large times (that is of size $O(1/\varepsilon)$) if a stronger version of the stability criterion is satisfied.

The proofs of the theorems are then given in Section 5.5. The stability criterion appears in the key Lemma 11, where it is used to control the destabilizing effects of the shear by the surface tension and gravity terms. As in the water-wave case, some technical difficulties arise because of the second order surface tension term, which requires special care in the treatment of the subprincipal terms in the symmetrization process. We chose to adapt the technique introduced in [51] (see also [52] for a synthetic presentation), which is probably not the sharpest one in terms of regularity (see for instance [1,6,7,17,44,54] for alternative methods) but seems to be the most robust with respect to the shallow water limit, in particular because it does not require symbolic analysis. This technique requires that time derivatives be treated as space derivatives, and this is the reason why this is done throughout this paper.

Finally, in Section 6 we sketch some applications for the stability criterion (1.1) and its practical version (1.2). We consider in Section 6.1 the case of air–water interfaces characterized by a very small density of the upper fluid. We first confirm in Section 6.1.1 that the density of the air can be neglected in the asymptotic models used in coastal oceanography to describe the propagation of waves (we consider here the example of a so called long wave). For waves close to the breaking point, we show in Section 6.1.2 that this might not be true anymore and that Kelvin–Helmholtz instabilities may appear. We therefore suggest that wave breaking may sometimes be a two-fluid singularity rather than a singularity of the water-wave (one-fluid) equations. Links with physical phenomena such as spilling breakers and white caps are also mentioned.

We then consider in Section 6.2 the case of internal waves at the interface of two fluids of comparable density (stability of the fluid is therefore ensured by a small shear velocity). We first check that some "stable" configurations reported in

experimental works do satisfy our stability criterion. We then consider the case of water-brine interfaces, of interest for the study of oceanographic internal waves. There is no natural value for the surface tension for such an interface (water and brine are not immiscible), but investigating the occurrence of Kelvin–Helmholtz instabilities, thanks to the experiments of [28], we show how to use our stability criterion to propose a value for the artificial surface tension which must be introduced to use the two-fluid formalism. Finally, we propose a formula for the maximal amplitude that interfacial waves can reach before being destroyed by Kelvin–Helmholtz instabilities.

We end this section by showing how to rigorously justify the two-fluid asymptotic models used in applications, and on a time scale consistent with the observations. We give the details for a shallow-water/shallow-water model and explain how one of its singularities can be related to Kelvin–Helmholtz instabilities.

1.5. Basic Notations

– We denote by $C(\lambda_1, \lambda_2, \dots)$ a constant depending on the parameters $\lambda_1, \lambda_2, \dots$ and *whose dependence on the λ_j is always assumed to be nondecreasing*.
– We denote by $X = (X_1, \dots, X_d) \in \mathbb{R}^d$ the horizontal variables, by z the vertical one, and by t the time variable.
– We denote by ∂_j ($1 \leqq j \leqq d$) partial differentiation with respect to X_j; partial differentiation with respect to t and z is denoted by ∂_t and ∂_z. - We denote by ∇ and Δ the standard gradient and Laplace operators in the horizontal coordinates. These operators are denoted by $\nabla_{X,z}$ and $\Delta_{X,z}$ when they also take into account the vertical variable. - We denote by $\nabla^\mu = (\sqrt{\mu}\nabla^T, \partial_z)^T$, that is, ∇^μ corresponds to $\nabla_{X,z}$ with a factor $\sqrt{\mu}$ in front of each *horizontal* derivative.
– We denote by $\Lambda := (1 - \Delta)^{1/2}$ and $H^s(\mathbb{R}^d)$ ($s \in \mathbb{R}$) the usual Sobolev space $H^s(\mathbb{R}^d) = \{u \in \mathcal{S}'(\mathbb{R}^d), |u|_{H^s} < \infty\}$, where $|u|_{H^s} = |\Lambda^s u|_{L^2}$. We keep this notation if u is a vector or a matrix with coefficients in $H^s(\mathbb{R}^d)$.
– We denote by $(\cdot, \cdot)$ the standard L^2-scalar product.
– The notation $a \vee b$ stands for $\max\{a, b\}$.
– We denote by $\mathrm{Op}(\sigma)$ or $\sigma(x, D)$ the pseudodifferential operator of symbol $\sigma(x, \xi)$,

$$\mathrm{Op}(\sigma)u(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \sigma(x, \xi)\widehat{u}(\xi)\mathrm{d}\xi.$$

2. Preliminary Results

2.1. Notations and Definitions

2.1.1. General Notations Throughout this section, we always assume that the interface deformation $\zeta \in H^{t_0+1}(\mathbb{R}^d)$ ($t_0 > d/2$) touches neither the bottom nor the lid. In dimensionless variables, this condition reads

$$\exists h^\pm_{\min} > 0, \qquad \inf_{X\in\mathbb{R}^d} (1 \pm \varepsilon^\pm \zeta(X)) \geqq h^\pm_{\min}. \tag{2.1}$$

We also denote by $\Omega^\pm$ the domains

$$\Omega^+ = \{(X, z) \in \mathbb{R}^{d+1}, -1 < z < \varepsilon^+ \zeta(X)\}$$
$$\Omega^- = \{(X, z) \in \mathbb{R}^{d+1}, \varepsilon^- \zeta(X) < z < 1\}.$$

We finally introduce, for notational convenience, a constant M defined as[5]

$$M = C\left(\frac{1}{h^\pm_{\min}}, \frac{H^+}{H^-}, \frac{H^-}{H^+}, |\zeta|_{H^{t_0+2}}\right), \tag{2.2}$$

as well as

$$M(s) = C(M, |\zeta|_{H^s}), \tag{2.3}$$

where we recall that $C(\cdot)$ denotes generically a nondecreasing, positive, function of its arguments.

2.1.2. Diffeomorphisms It is often convenient to transform boundary value problems on the fluid domains $\Omega^\pm$ into boundary value problems on flat strips $\mathcal{S}^\pm$, defined as

$$\mathcal{S}^+ = \{(X, z), -1 < z < 0\} \quad \text{and} \quad \mathcal{S}^- = \{(X, z), 0 < z < 1\},$$

so that $\mathcal{S}^\pm$ correspond to the two fluid domains at rest. Various diffeomorphisms can be used for such domain transformations. We use here only *admissible* diffeomorphisms in the following sense:

Definition 1. (*Admissible diffeomorphisms*) Let $t_0 > d/2$ and let $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ be such that (2.1) is satisfied.
   We say that $\Sigma^\pm : \mathcal{S}^\pm \to \Omega^\pm$ is an *admissible diffeomorphism* if

1. $\Sigma^\pm$ can be extended to the boundaries in such a way that

$$\Sigma^\pm(\{z = 0\}) = \{z = \varepsilon^\pm \zeta\} \quad \text{and} \quad \Sigma^\pm(\{z = \mp 1\}) = \{\zeta = \mp 1\}.$$

2. The coefficients of the Jacobian matrix $J_{\Sigma^\pm} = d_{X,z}\Sigma^\pm$ are bounded on $\mathcal{S}^\pm$, and $|J_{\Sigma^\pm}|_{L^\infty(\mathcal{S}^\pm)} \leqq M$.
3. The determinant $|\det J_{\Sigma^\pm}|$ is uniformly bounded from below on $\mathcal{S}^\pm$ by a non-negative constant $c^\pm$ such that $\frac{1}{c^\pm} \leqq M$.

   If $\Phi^\pm$ is defined on $\Omega^\pm$, then one can define $\phi^\pm = \Phi^\pm \circ \Sigma^\pm$ on $\mathcal{S}^\pm$; a direct application of the chain rule shows that

$$\nabla^{\mu^\pm}\Phi^\pm \circ \Sigma^\pm = I^{\mu^\pm}(J^{-1}_{\Sigma^\pm})^T(I^{\mu^\pm})^{-1}\nabla^{\mu^\pm}\phi^\pm,$$

where $I^{\mu^\pm}$ is the $(d+1) \times (d+1)$ diagonal matrix with entries $\sqrt{\mu^\pm}$ on the d-first diagonal coefficients, and 1 on the last one.

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[5] The dependence on H^+/H^- and H^-/H^+ is harmless because we assumed that H^+ and H^- are of same order (see Remark 3). For more general configurations, a finer analysis on the dependence on $H^\pm$ of the solution is needed.

In particular, the (variational formulation of the) equation $(\partial_z^2 + \mu^{\pm}\Delta)\Phi^{\pm} = 0$ in $\Omega^{\pm}$ is transformed by $\Sigma^{\pm}$ into the (variational formulation of the) following variable coefficient elliptic equation on $\mathcal{S}^{\pm}$,

$$\nabla^{\mu^{\pm}} \cdot P(\Sigma^{\pm})\nabla^{\mu^{\pm}}\phi^{\pm} = 0 \quad \text{in } \mathcal{S}^{\pm}, \tag{2.4}$$

where

$$P(\Sigma^{\pm}) = |\det J_{\Sigma^{\pm}}|(I^{\mu^{\pm}})^{-1}J_{\Sigma^{\pm}}^{-1}(I^{\mu^{\pm}})^2(J_{\Sigma^{\pm}}^{-1})^T(I^{\mu^{\pm}})^{-1}. \tag{2.5}$$

It is easy to check (Prop. 2.3 of [4]) that the matrix $P(\Sigma^{\pm})$ is uniformly coercive with coercivity constant $k(\Sigma^{\pm})$. We assume further that

$$\|P(\Sigma^{\pm})\|_{\infty} \leqq M \quad \text{and} \quad \frac{1}{k(\Sigma^{\pm})} \leqq M. \tag{2.6}$$

It is also convenient to assume that $P(\Sigma^{\pm})$ is a Sobolev perturbation of the identity matrix (this is, in general, a direct consequence of the fact that ζ has Sobolev regularity) in the sense that

$$\|\Lambda^{t_0+1}(\text{Id} - P(\Sigma^{\pm}))\|_{L_z^{\infty}L_X^2} \leqq M. \tag{2.7}$$

Let us also mention that we use the notation ∂_n for the transformed equations to denote the upward *conormal* derivative naturally associated to any elliptic equation. For instance, the upward conormal derivative associated to (2.4) is

$$\partial_n\phi^{\pm} = \mathbf{e_z} \cdot P(\Sigma^{\pm})\nabla^{\mu^{\pm}}\phi^{\pm}. \tag{2.8}$$

We now introduce as in [39] (see also [11]) a very useful class of admissible diffeomorphisms called *regularizing* diffeomorphisms. They allow the gain of half a derivative on ζ in the elliptic estimates necessary to the study the Dirichlet–Neumann operator.[6]

Definition 2. (*Regularizing diffeomorphisms*) Let $t_0 > d/2, s \geqq 0$ and $\zeta \in H^{t_0+2} \cap H^s(\mathbb{R}^d)$ be such that (2.1) is satisfied.

We say that an admissible diffeomorphism $\Sigma^{\pm} : \mathcal{S}^{\pm} \to \Omega^{\pm}$ is *regularizing* if it is of the form $\Sigma^{\pm}(X, z) = (X, z + \sigma^{\pm}(X, z))$, with

$$\|\Lambda^s\nabla^{\mu}\sigma^{\pm}\|_{L^2(\mathcal{S}^{\pm})} + \|\Lambda^{s-1}\partial_z\nabla^{\mu}\sigma^{\pm}\|_{L^2(\mathcal{S}^{\pm})} \leqq M(s + 1/2). \tag{2.9}$$

Note that if $\Sigma^{\pm}$ is regularizing, then one automatically has from (2.5) that

$$\|\Lambda^s(P(\Sigma^{\pm}) - \text{Id})\|_{L^2(\mathcal{S}^{\pm})} \leqq M(s + 1/2). \tag{2.10}$$

Example 1. The simplest example of admissible diffeomorphism is given by

$$\Sigma^{\pm}(X, z) = (X, \varepsilon^{\pm}(1 \pm z)\zeta(X) + z).$$

It satisfies (2.6) and (2.7), but not (2.9) (the right-hand side should be $M(s + 1)$ in this inequality).

---

[6] With a standard diffeomorphism, one would, for instance, have $|h_l|_{H^{s+1}}$ instead of $|h_l|_{H^{s+1/2}}$ in (2.33).

Example 2. For small enough $\delta > 0$ (and with $\chi : \mathbb{R} \to \mathbb{R}$ a smooth, positive, compactly supported function such that $\chi(0) = 1$), the functions $\Sigma^{\pm}$ defined as

$$\Sigma^{\pm}(X, z) = (X, z + \sigma^{\pm}(X, z)) \quad \text{with} \quad \sigma^{\pm}(\cdot, z) = \varepsilon^{\pm}(1 \pm z)\chi(\delta z|D|)\zeta$$

are regularizing diffeomorphisms (see [39]) also satisfying (2.6) and (2.7).

2.1.3. Functional Spaces We introduce here two functional spaces that are closely related to the energy norm. For $N \in \mathbb{N}$, the space $H_\sigma^{N+1}(\mathbb{R}^d)$ is the standard $H^{N+1}(\mathbb{R}^d)$ Sobolev space endowed with a norm adapted to the measure of the extra control (with respect to the H^N-norm) provided by the surface tension term,

$$H_\sigma^{N+1}(\mathbb{R}^d) = \left\{ u \in H^N(\mathbb{R}^d), \frac{1}{\sqrt{\text{Bo}}} \nabla u \in H^N(\mathbb{R}^d)^d \right\}, \qquad (2.11)$$

with the norm $|u|^2_{H_\sigma^{N+1}} = |u|^2_{H^N} + \frac{1}{\text{Bo}} |\nabla u|^2_{H^N}$.

Since the velocity potential does not necessary decay at infinity (while the velocity does), we have to work in Beppo–Levi spaces [23] on open subsets $\Omega \subset \mathbb{R}^{d+1}$, defined for $N \geqq 0$ as

$$\dot{H}^{N+1}(\Omega) = \{u \in L^2_{loc}(\Omega), \nabla_{X,z} u \in H^N(\Omega)^{d+1}\},$$

endowed with the (semi)-norm $\|u\|_{\dot{H}^{N+1}} = \|\nabla u\|_{H^N}$. We also define similar spaces over $\mathbb{R}^d$,

$$\dot{H}^{s+1/2}(\mathbb{R}^d) = \{u \in L^2_{loc}(\mathbb{R}^d), \nabla u \in H^{s-1/2}(\mathbb{R}^d)^d\} \qquad (s \in \mathbb{R}),$$

endowed with the (semi)-norm $|u|_{\dot{H}^{s+1/2}} = |\nabla u|_{H^{s-1/2}}$. In order to capture the shallow-water dynamics, it is convenient to introduce a variant of this space that depends on the shallowness parameter μ,

$$\dot{H}_\mu^{s+1/2}(\mathbb{R}^d) = \dot{H}^{s+1/2}(\mathbb{R}^d) \quad \text{endowed with} \quad |u|_{\dot{H}_\mu^{s+1/2}} = |\mathfrak{P} u|_{H^s}, \quad (2.12)$$

where $\mathfrak{P}$ is the nonhomogeneous Fourier multiplier of order $1/2$ defined as

$$\mathfrak{P} = \frac{|D|}{(1 + \sqrt{\mu}|D|)^{1/2}}. \qquad (2.13)$$

2.2. A Few Results on Dirichlet–Neumann Operators

2.2.1. Two Equivalent Definitions If $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ ($t_0 > d/2$) satisfies (2.1), then it is well known that for all $\psi^{\pm} \in \dot{H}^{3/2}(\mathbb{R}^d)$, there exists a unique solution $\Phi^{\pm} \in \dot{H}^2(\Omega^{\pm})$ to the elliptic boundary value problem

$$\begin{cases} (\partial_z^2 + \mu^{\pm}\Delta)\Phi^{\pm} = 0 & \text{in } \Omega^{\pm}, \\ \Phi^{\pm}\big|_{z=\varepsilon\pm\zeta} = \psi^{\pm}, \qquad \partial_z\Phi\big|_{z=\mp 1} = 0 \end{cases} \qquad (2.14)$$

(since the interface is a graph, it is possible to view the trace of $\Phi^\pm$ on the interface as a function defined on $\mathbb{R}^d$) so that the following definition of the Dirichlet–Neumann operators $\mathcal{G}^\pm_{\mu\pm}[\varepsilon^\pm\zeta, 1]$ makes sense,

$$\mathcal{G}^\pm_{\mu\pm}[\varepsilon^\pm\zeta, 1] : \begin{array}{ccc} \dot{H}^{3/2}(\mathbb{R}^d) & \to & H^{1/2}(\mathbb{R}^d) \\ \psi^\pm & \mapsto & \sqrt{1 + |\varepsilon^\pm\nabla\zeta|^2}\partial_n\Phi^\pm_{|_{z=\varepsilon^\pm\zeta}}, \end{array}$$

where $\Phi^\pm$ solves (2.14) and ∂_n denotes the *upwards* conormal derivative.

Remark 4. The operators $\mathcal{G}^\pm_\mu[\varepsilon\zeta, \underline{H}^\pm]$ that appear in the introduction are related to $\mathcal{G}^\pm_{\mu\pm}[\varepsilon^\pm\zeta, 1]$ through the scaling law

$$\mathcal{G}^\pm_\mu[\varepsilon\zeta, \underline{H}^\pm] = \frac{1}{\underline{H}^\pm}\mathcal{G}^\pm_{\mu\pm}[\varepsilon^\pm\zeta, 1]. \tag{2.15}$$

Notation 2. *For the sake of simplicity, we simply write*

$$\mathcal{G}^\pm = \mathcal{G}^\pm_{\mu\pm}[\varepsilon^\pm\zeta, 1],$$

when no confusion is possible.

From the discussion in Section 2.1.2, $\mathcal{G}^\pm\psi^\pm$ can be equivalently defined as

$$\mathcal{G}^\pm\psi^\pm = \partial_n\phi^\pm_{|_{z=0}}, \tag{2.16}$$

where ∂_n now stands for the upwards conormal derivative associated to the elliptic operator in the flat strip (see (2.8)) solved by $\phi^\pm$,

$$\begin{cases} \nabla^{\mu^\pm} \cdot P(\Sigma^\pm)\nabla^{\mu^\pm}\phi^\pm = 0 & \text{in } \mathcal{S}^\pm, \\ \phi^\pm_{|_{z=0}} = \psi^\pm, \qquad \partial_n\phi^\pm_{|_{z=\mp 1}} = 0 \end{cases} \tag{2.17}$$

($\Sigma^\pm$ being an admissible diffeomorphism in the sense of Definition 1).

2.2.2. Basic Estimates Let $t_0 > d/2$ and $\zeta \in H^{t_0+2}(\mathbb{R}^d)$. If $\Sigma^\pm$ is an admissible diffeomorphism in the sense of Definition 1, then for all $\psi^\pm \in \dot{H}^{3/2}(\mathbb{R}^d)$, there exists a unique solution $\phi^\pm \in \dot{H}^2(\mathcal{S}^\pm)$ to (2.17). Moreover, one has (see [39,34]),

$$\forall 0 \leqq s \leqq t_0 + 1, \qquad \|\Lambda^s\nabla^{\mu^\pm}\phi^\pm\|_{L^2(\mathcal{S}^\pm)} \leqq M\sqrt{\mu}|\psi^\pm|_{\dot{H}^{s+1/2}_\mu}, \tag{2.18}$$

where M is as in (2.2) and $\dot{H}^{s+1/2}_\mu(\mathbb{R}^d)$ as in (2.12). The reverse inequality also holds (see Appendix 6.3),

$$\sqrt{\mu}|\psi^\pm|_{\dot{H}^{s+1/2}_\mu} \leqq M\|\Lambda^s\nabla^{\mu^\pm}\phi^\pm\|_{L^2(\mathcal{S}^\pm)}. \tag{2.19}$$

For all $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ ($s \geqq 0$), (2.16) and (2.18) allow one to extend $\mathcal{G}^\pm$ as a mapping

$$\forall 0 \leqq s \leqq t_0 + 1, \qquad \mathcal{G}^\pm : \dot{H}^{s+1/2}(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^d).$$

Moreover, the following estimates hold (they correspond to the particular case $j = 0$ of (2.31)–(2.32) below)

$$\forall 0 \leqq s \leqq t_0 + 1, \ |\mathcal{G}^\pm\psi|_{H^{s-1/2}} \leqq \mu^{3/4}M|\mathfrak{P}\psi|_{H^s}, \tag{2.20}$$

$$\forall 0 \leqq s \leqq t_0 + 1/2, \ |\mathcal{G}^\pm\psi|_{H^{s-1/2}} \leqq \mu M|\mathfrak{P}\psi|_{H^{s+1/2}}. \tag{2.21}$$

Remark 5. From the definition (2.13) of $\mathfrak{P}$, it follows easily that $\frac{1}{\sqrt{\mu}}\mathcal{G}^{\pm}$ remains uniformly bounded as a *first* order operator as $\mu \to 0$. However, the quantity $\frac{1}{\mu}\mathcal{G}^{\pm}$ involved in the first equation of (1.19) remains bounded only if we consider $\mathcal{G}^{\pm}$ as a *second* order operator. This fact makes the shallow limit singular and requires some specific attention.

Remark 6. The estimates provided in this section are not the sharpest possible in terms of regularity (see [2] and [41] for improvements). Working with sharp estimates would be technically more involved but would not lead to an improvement of the main theorems; for the sake of simplicity, we therefore work with the nonoptimal estimates given in this section.

It also follows upon integrating by parts in (2.17) that for all $\psi_1, \psi_2 \in \dot{H}^{1/2}(\mathbb{R}^d)$,

$$\int_{\mathcal{S}^{\pm}} \nabla^{\mu\pm} \phi_1^{\pm} \cdot P(\Sigma^{\pm}) \nabla^{\mu\pm} \phi_2^{\pm} = \pm \int_{\mathbb{R}^d} \psi_1(\mathcal{G}^{\pm}\psi_2), \qquad (2.22)$$

where $\phi_1^{\pm}$ and $\phi_2^{\pm}$ the (variational) solutions to (2.17) with Dirichlet condition ψ_1 and ψ_2. This identity is the key ingredient in proving the following estimate (see Appendix 6.3),

$$\forall 0 \leqq s \leqq t_0 + 1, \qquad \left| (\Lambda^s \mathcal{G}^{\pm}\psi_1, \Lambda^s \psi_2) \right| \leqq M\mu |\psi_1|_{\dot{H}_{\mu}^{s+1/2}} |\psi_2|_{\dot{H}_{\mu}^{s+1/2}}, \quad (2.23)$$

for all $\psi_1, \psi_2 \in \dot{H}^{s+1/2}(\mathbb{R}^d)$. Finally, we will also need the following commutator estimate (see Appendix 6.3 for a proof),

$$\forall 0 \leqq s \leqq t_0 + 1, \qquad \left| ([\Lambda^s, \mathcal{G}^{\pm}]\psi_1, \Lambda^s \psi_2) \right| \leqq \mu M |\psi_1|_{\dot{H}_{\mu}^{s-1/2}} |\psi_2|_{\dot{H}_{\mu}^{s+1/2}}, \quad (2.24)$$

and the inequality (see Proposition 3.7 of [4]),

$$\left(\mathbf{v} \cdot \nabla u, \mathcal{G}^{\pm} u \right) \leqq \mu M |\mathbf{v}|_{W^{1,\infty}} |\mathfrak{P}u|_2^2, \qquad (2.25)$$

that holds for all $\mathbf{v} \in W^{1,\infty}(\mathbb{R}^d)^d$ and $u \in \dot{H}_{\mu}^{1/2}(\mathbb{R}^d)$.

2.2.3. Invertibility It follows quite easily from (2.22) that $\mathcal{G}^+$ (resp. $\mathcal{G}^-$) is a symmetric, positive (resp. negative) operator. We can also deduce that they are injective (up to constants of course) so that the inverse $(\mathcal{G}^{\pm})^{-1}$ is well defined on the range of $\mathcal{G}^{\pm}$ and with values in $\dot{H}^{1/2}(\mathbb{R}^d)$. The next proposition states that $(\mathcal{G}^-)^{-1} \circ \mathcal{G}^+$ is well defined and uniformly bounded (with respect to ε and μ) as a family of operators mapping $\dot{H}_{\mu}^{s+1/2}(\mathbb{R}^d)$ into itself.

Proposition 1. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ be such that (2.1) is satisfied. For all $0 \leqq s \leqq t_0 + 1$, the mapping*

$$(\mathcal{G}^-)^{-1} \circ \mathcal{G}^+ : \begin{array}{ccc} \dot{H}_{\mu}^{s+1/2}(\mathbb{R}^d) & \to & \dot{H}_{\mu}^{s+1/2}(\mathbb{R}^d) \\ \psi & \mapsto & (\mathcal{G}^-)^{-1}(\mathcal{G}^+\psi) \end{array}$$

is well defined and one has, for all $\psi \in \dot{H}_{\mu}^{s+1/2}(\mathbb{R}^d)$,

$$|(\mathcal{G}^-)^{-1} \circ \mathcal{G}^+\psi|_{\dot{H}_{\mu}^{s+1/2}} \leqq M|\psi|_{\dot{H}_{\mu}^{s+1/2}},$$

where M is as defined in (2.2). *Moreover, one also has the estimate*

$$\forall 0 \leqq s \leqq t_0 + 1/2, \quad |\nabla (G^-)^{-1} \circ G^+ \psi|_{H^s} \leqq M|\nabla \psi|_{H^s}.$$

Remark 7. It is straightforward to check that the proposition still holds if one switches the $+$ and $-$ signs everywhere.

Remark 8. Let $A : H^{s+1/2}(\mathbb{R}^d) \mapsto H^{s-1/2}(\mathbb{R}^d)$ be a linear operator such that

$$\forall 0 \leqq s \leqq t_0 + 1, \quad (\Lambda^s A \psi_1, \Lambda^s \psi_2) \leqq \mu M_A(\psi_1) |\psi_2|_{\dot{H}_\mu^{s+1/2}},$$

for some positive constant $M_A(\psi_1)$. Replacing (2.23) by this inequality everywhere in the proof of the Proposition shows that $(G^-)^{-1} \circ A : H^{s+1/2} \to \dot{H}_\mu^{s+1/2}$ is well defined and

$$\forall 0 \leqq s \leqq t_0 + 1, \quad |(G^-)^{-1} \circ A \psi|_{\dot{H}_\mu^{s+1/2}} \leqq M \, M_A(\psi).$$

For instance, one has, for all $1 \leqq j \leqq d$,

$$\forall 0 \leqq s \leqq t_0 + 1, \quad |(G^-)^{-1} \circ \partial_j \psi|_{\dot{H}_\mu^{s+1/2}} \leqq \frac{1}{\mu} M |(1 + \sqrt{\mu} |D|)^{1/2} \psi|_{H^s}.$$

Remark 9. With a little more work, it is possible to show (using regularizing diffe-omorphisms in the proof) that for all $s \geqq 0$, the mapping $(G^-)^{-1} \circ G^+ : \dot{H}_\mu^{s+1/2} \to \dot{H}_\mu^{s+1/2}$ is well defined and that

$$|(G^-)^{-1} \circ G^+ \psi|_{\dot{H}_\mu^{s+1/2}} \leqq M(s + 1/2) |\psi|_{\dot{H}_\mu^{s+1/2}},$$

with $M(s + 1/2)$ as in (2.3), but since we do not need such a high order estimate here, we do not prove it.

Proof. Let us prove that $(G^-)^{-1} G^+ \psi$ is well defined in $\dot{H}^{1/2}(\mathbb{R}^d)$. We first prove that there exists a unique variational solution $\Phi^- \in \dot{H}^1(\Omega^-)$ to the boundary value problem

$$\begin{cases} (\partial_z^2 + \mu^- \Delta) \Phi^- = 0 & \text{in } \Omega^-, \\ \sqrt{1 + |\varepsilon^- \nabla \zeta|^2} \partial_n \Phi^-_{|_{z = \varepsilon^- \zeta}} = G^+ \psi, \quad \partial_z \Phi^-_{|_{z=1}} = 0, \end{cases} \tag{2.26}$$

or equivalently (by Section 2.1.2), we seek a unique variational solution $\phi^- \in \dot{H}^1(\mathcal{S}^-)$ to

$$\begin{cases} \nabla^{\mu^-} \cdot P(\Sigma^-) \nabla^{\mu^-} \phi^- = 0 & \text{in } \mathcal{S}^-, \\ \partial_n \phi^-_{|_{z=0}} = G^+ \psi, \quad \partial_n \phi^-_{|_{z=1}} = 0, \end{cases} \tag{2.27}$$

where Σ^- is an admissible diffeomorphism in the sense of Definition 1 and satis-fying (2.7).

For all $\varphi \in C^\infty(\overline{\mathcal{S}^-}) \cap \dot{H}^1(\mathcal{S}^-)$, one deduces from (2.23) that

$$(G^+ \psi, \varphi_{|_{z=0}}) \leqq M \mu |\psi|_{\dot{H}_\mu^{1/2}} |\varphi_{|_{z=0}}|_{\dot{H}_\mu^{1/2}}.$$

Since, moreover, $|\varphi_{|z=0}|_{\dot{H}_\mu^{1/2}} \lesssim \mu^{-1/2}\|\nabla^{\mu^-}\varphi\|_{L^2(\mathcal{S}^-)}$ (this is a variant of the trace lemma, see the proof of Proposition 3.4 of [4]), one has

$$(\mathcal{G}^+\psi, \varphi_{|z=0}) \leqq M\sqrt{\mu}|\psi|_{\dot{H}_\mu^{1/2}}\|\nabla^{\mu^-}\varphi\|_{L^2(\mathcal{S}^-)}.$$

The linear form $\varphi \mapsto (\mathcal{G}^+\psi, \varphi_{|z=0})$ is thus continuous on $\dot{H}^1(\mathcal{S}^-)$; moreover, $\nabla^{\mu^-} \cdot P(\Sigma^-)\nabla^{\mu^-}$ is obviously coercive on this space, so that existence/unique-ness of a variational solution to (2.27) follows classically from the Lax-Milgram theorem.

We can thus define $\psi^- = \phi^-_{|z=0}$ so that one obviously has $\mathcal{G}^-\psi^- = \mathcal{G}^+\psi$ and $\psi^- = (\mathcal{G}^-)^{-1} \circ \mathcal{G}^+\psi$. Since such a solution is obviously unique, the operator $(\mathcal{G}^-)^{-1} \circ \mathcal{G}^+$ is well defined.

We now turn to proving the estimates given in the statement of the proposi-tion. Remarking that $|(\mathcal{G}^-)^{-1} \circ \mathcal{G}^+\psi|_{\dot{H}_\mu^{s+1/2}} = |\psi^-|_{\dot{H}_\mu^{s+1/2}}$, with $\psi^- = \phi^-_{|z=0}$ as constructed above, we deduce from (2.19) that

$$\sqrt{\mu}|(\mathcal{G}^-)^{-1} \circ \mathcal{G}^+\psi|_{\dot{H}_\mu^{s+1/2}} \leqq M\|\Lambda^s\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)}. \tag{2.28}$$

In order to get an estimate on the right-hand side of this inequality, let us apply Λ^s to the left-hand side of (2.27), multiply it by $\Lambda^s\phi^-$ and integrate by parts, to get

$$\int_{\mathcal{S}^-} P(\Sigma^-)\nabla^{\mu^-}\Lambda^s\phi^- \cdot \nabla^{\mu^-}\Lambda^s\phi^- = -\int_{\mathbb{R}^d} \Lambda^s(\mathcal{G}^+\psi)\Lambda^s\psi^-$$
$$+ \int_{\mathcal{S}^-} [\Lambda^s, P(\Sigma^-)]\nabla^{\mu^-}\phi^- \cdot \nabla^{\mu^-}\Lambda^s\phi^-.$$

We can therefore deduce from the coercivity of $P(\Sigma^-)$, (2.23) and the Cauchy–Schwarz inequality that

$$k(\Sigma^-)\|\Lambda^s\nabla^{\mu^-}\phi^-\|^2_{L^2(\mathcal{S}^-)} \leqq M\mu|\psi|_{\dot{H}_\mu^{s+1/2}}|\psi^-|_{\dot{H}_\mu^{s+1/2}}$$
$$+ \|[\Lambda^s, P(\Sigma^-)]\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)}\|\Lambda^s\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)}.$$

Since $\sqrt{\mu}|\psi^-|_{\dot{H}_\mu^{s+1/2}} \leqq M\|\Lambda^s\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)}$ by (2.19), one can deduce, recalling that $k(\Sigma^-)$ satisfies (2.6), that

$$\|\Lambda^s\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)} \leqq M\sqrt{\mu}|\psi|_{\dot{H}_\mu^{s+1/2}} + M\|[\Lambda^s, P(\Sigma^-)]\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)}.$$

Using the classical commutator estimate

$$\forall 0 \leqq s \leqq t_0 + 1, \qquad |[\Lambda^s, f]g|_2 \lesssim |f|_{H^{t_0+1}}|g|_{H^{s-1}},$$

it is easy to deduce from (2.7) that

$$\|[\Lambda^s, P(\Sigma^-)]\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)} \leqq M\|\Lambda^{s-1}\nabla^{\mu^-}\phi^-\|_2, \tag{2.29}$$

and therefore

$$\|\Lambda^s\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)} \leqq M\left(\sqrt{\mu}|\psi|_{\dot{H}_\mu^{s+1/2}} + \|\Lambda^{s-1}\nabla^{\mu^-}\phi^-\|_2\right).$$

The first estimate of the proposition follows, therefore, from a continuous induction on s and (2.28). The second estimate is then a direct consequence of the following lemma.

Lemma 1. *Let A be a linear operator mapping $\dot{H}_\mu^{s+1/2}$ into itself for all $0 \leqq s \leqq t_0 + 1$ and such that $\|A\|_{\dot{H}_\mu^{s+1/2} \to \dot{H}_\mu^{s+1/2}} \leqq M$. Then one has*

$$\forall 0 \leqq s \leqq t_0 + 1/2, \qquad |\nabla(A\psi)|_{H^s} \leqq M|\nabla\psi|_{H^s}.$$

Proof of the lemma. From the definition of $\mathfrak{P}$, one deduces that

$$\begin{aligned}
|\nabla(A\psi)|_{H^s} &= |(1 + \sqrt{\mu}|D|)^{1/2}\mathfrak{P}A\psi|_{H^s} \\
&\leqq |\mathfrak{P}(A\psi)|_{H^s} + \mu^{1/4}|\mathfrak{P}(A\psi)|_{H^{s+1/2}}.
\end{aligned}$$

Using the assumption made on A, we then get

$$\begin{aligned}
|\nabla(A\psi)|_{H^s} &\leqq M\big(|\mathfrak{P}\psi|_{H^s} + \mu^{1/4}|\mathfrak{P}\psi|_{H^{s+1/2}}\big) \\
&\leqq M\Big|\frac{1 + \mu^{1/4}|D|^{1/2}}{(1 + \sqrt{\mu}|D|)^{1/2}}|D|\psi\Big|_{H^s},
\end{aligned}$$

and the result follows from the observation that $\frac{1+\mu^{1/4}|D|^{1/2}}{(1+\sqrt{\mu}|D|)^{1/2}}$ is uniformly bounded (as a zero order operator) for $\mu \in (0, 1)$. $\square$

$\square$

2.2.4. Shape Derivatives   It is known (for example, [39,34]) that for all $0 \leqq s \leqq t_0 + 1$ and $\psi^\pm \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, the mapping

$$\begin{aligned}
H^{t_0+2}(\mathbb{R}^d) &\to H^{s-1/2}(\mathbb{R}^d) \\
\zeta &\mapsto \mathcal{G}^\pm\psi^\pm = \mathcal{G}_{\mu\pm}^\pm[\varepsilon^\pm\zeta, 1]\psi^\pm
\end{aligned}$$

is smooth in a neighborhood of any $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ satisfying (2.1). Let us denote by $d^j\mathcal{G}^\pm(\mathbf{h})\psi^\pm$ ($j \in \mathbb{N}^*$, $\mathbf{h} = (h_1, \ldots, h_j) \in H^{t_0+2}(\mathbb{R}^d)^j$) its j-th derivative at ζ and in the direction $\mathbf{h} = (h_1, \ldots, h_j)$. Such derivatives are called *shape derivatives*.

A first important result is that there exists an exact formula for the first order shape derivative (Theorem 3.20 of [39]),

$$d\mathcal{G}^\pm(h)\psi^\pm = -\varepsilon\mathcal{G}^\pm\big(h\underline{w}^\pm\big) - \varepsilon^\pm\mu^\pm\nabla \cdot (h\underline{V}^\pm), \tag{2.30}$$

where $\underline{w}^\pm$ and $\underline{V}^\pm$ are the vertical and horizontal velocities at the surface,

$$\underline{w}^\pm = \frac{1}{1 + \varepsilon^2\mu|\nabla\zeta|^2}\Big(\frac{1}{\underline{H}^\pm}\mathcal{G}^\pm\psi^\pm + \varepsilon\mu\nabla\zeta \cdot \nabla\psi^\pm\Big), \qquad \underline{V}^\pm = \nabla\psi^\pm - \varepsilon\underline{w}^\pm\nabla\zeta.$$

Remark 10. Consistently with the notations used in the introduction, $\underline{V}^\pm$ and $\underline{w}^\pm$ stand for the horizontal and vertical components of the (nondimensionalized) velocity fields in the two layers, evaluated at the interface.

Formula (2.30) is not convenient for giving control of the operator norm of the shape derivatives of $\mathcal{G}^{\pm}$ because it looks more singular than expected (in fact, the identity $\varepsilon \mathcal{G}^{\pm}[\varepsilon \zeta]\underline{w}^{\pm} = -\varepsilon^{\pm}\mu^{\pm}\nabla \cdot \underline{V}^{\pm}$ [1,12] shows that the most singular terms cancel one another). Using direct methods, however, it is possible to obtain some estimates. Let us first set some notation.

Notation 3. *For all* $\mathbf{h} = (h_1, \ldots, h_j) \in X^j$ *(X Banach space), and* $1 \leqq l \leqq j$, *we write*

$$\langle \mathbf{h} \rangle_X = \prod_{m=1}^{j} |h_j|_X \quad \text{and} \quad \langle \check{\mathbf{h}}_l \rangle_X = \prod_{m \neq l}^{j} |h_j|_X.$$

The first estimates are given by (see Proposition 3.3 of [4] or adapt the proof of (2.33) given in the Appendix)

$$\forall 0 \leqq s \leqq t_0 + 1, \quad |d^j \mathcal{G}^{\pm}(\mathbf{h})\psi|_{H^{s-1/2}} \leqq M \varepsilon^j \mu^{3/4} \langle \mathbf{h} \rangle_{H^{s \vee t_0 + 1}} |\psi|_{\dot{H}_\mu^{s+1/2}}, \quad (2.31)$$

and (see Remark 3.3 of [4] or adapt the proof of (2.34) given in the Appendix)

$$\forall 0 \leqq s \leqq t_0 + 1/2, \quad |d^j \mathcal{G}^{\pm}(\mathbf{h})\psi|_{H^{s-1/2}} \leqq M \varepsilon^j \mu \langle \mathbf{h} \rangle_{H^{(s+1/2) \vee t_0 + 1}} |\psi|_{\dot{H}_\mu^{s+1}}. \quad (2.32)$$

For small values of s, (2.31) and (2.32) require much more regularity on the h_j terms than on ψ. This is the reason why we also need the following two estimates, where the s-depending norm in the right-hand side is on one of the h_j terms rather than on $\psi^{\pm}$ (see Appendix 6.3 for a proof): for all $1 \leqq l \leqq j$,

$$|d^j \mathcal{G}^{\pm}(\mathbf{h})\psi|_{H^{s-1/2}} \leqq M \varepsilon^j \mu^{3/4} |h_l|_{H^{s+1/2}} \langle \check{\mathbf{h}}_l \rangle_{H^{s \vee t_0 + 3/2}} |\psi|_{\dot{H}_\mu^{s \vee t_0 + 1}}, \quad (2.33)$$

for all $0 \leqq s \leqq t_0 + 1/2$, and

$$|d^j \mathcal{G}^{\pm}(\mathbf{h})\psi|_{H^{s-1/2}} \leqq M \varepsilon^j \mu |h_l|_{H^{s+1}} \langle \check{\mathbf{h}}_l \rangle_{H^{(s+1/2) \vee t_0 + 3/2}} |\psi|_{\dot{H}_\mu^{(s+1/2) \vee t_0 + 1}}, \quad (2.34)$$

for all $0 \leqq s \leqq t_0$.

Remark 11. Formulas (2.31) and (2.33) require less regularity on ψ or $\mathbf{h}$ than (2.32) and (2.34), respectively, but they give a control of size $O(\mu^{3/4})$ while (2.32) and (2.34) give an $O(\mu)$ control. This is reminiscent of the comments made in Remark 5.

In the same spirit, is also possible to give two generalizations of (2.23) to shape derivatives (see Appendix 6.3 for a proof). One has

$$\left|(\Lambda^s d^j \mathcal{G}^{\pm}(\mathbf{h})\psi_1, \Lambda^s \psi_2)\right| \leqq M \varepsilon^j \mu |\psi_1|_{\dot{H}_\mu^{s+1/2}} |\psi_2|_{\dot{H}_\mu^{s+1/2}} \langle \mathbf{h} \rangle_{H^{s \vee t_0 + 1}}, \quad (2.35)$$

for all $0 \leqq s \leqq t_0 + 1$, and, for $1 \leqq l \leqq j$,

$$\left|(\Lambda^s d^j \mathcal{G}^{\pm}(\mathbf{h})\psi_1, \Lambda^s \psi_2)\right| \leqq M \varepsilon^j \mu |h_l|_{H^{s+1/2}} \langle \check{\mathbf{h}}_l \rangle_{H^{s \vee t_0 + 3/2}} |\psi_1|_{\dot{H}_\mu^{s \vee t_0 + 1}} |\psi_2|_{\dot{H}_\mu^{s+1/2}}, \quad (2.36)$$

for all $0 \leqq s \leqq t_0 + 1/2$.

Remark 12. A consequence of these estimates is that one can use Remark 8 to prove that $(\mathcal{G}^-)^{-1} \circ d^j \mathcal{G}^+(\mathbf{h})$ is well defined and give some estimate on its operator norm. For instance, one gets with (2.36)

$$|(\mathcal{G}^-)^{-1} d^j \mathcal{G}^+(\mathbf{h})\psi|_{\dot{H}_\mu^{s+1/2}} \lesssim \varepsilon^j M |h_l|_{H^{s+1/2}} |\psi|_{\dot{H}_\mu^{s \vee t_0 + 1}} \langle \check{\mathbf{h}}_l \rangle_{H^{s \vee t_0 + 3/2}}.$$

2.3. A Transmission Problem

Attention is given here to the following transmission problem, whose resolution ensures that the velocity potentials in both fluid layers can be recovered from the knowledge of ζ (that is the shape of the interface) and ψ.

$$\begin{cases} (\partial_z^2 + \mu\Delta)\Phi^+ = 0 & \text{for } -\underline{H}^+ < z < \varepsilon\zeta, \\ (\partial_z^2 + \mu\Delta)\Phi^- = 0 & \text{for } \varepsilon\zeta < z < \underline{H}^-, \\ (\underline{\rho}^+\Phi^+ - \underline{\rho}^-\Phi^-)_{|z=\varepsilon\zeta} = \psi, \\ \partial_n\Phi^+{}_{|z=\varepsilon\zeta} - \partial_n\Phi^-{}_{|z=\varepsilon\zeta} = 0, & \partial_z\Phi^\pm{}_{|z=\mp\underline{H}^\pm} = 0. \end{cases}$$

From the discussion in Section 2.1.2, we can equivalently investigate the following straightened version,

$$\begin{cases} \nabla^{\mu\pm} \cdot P(\Sigma^\pm)\nabla^{\mu\pm}\phi^\pm = 0 & \text{in } \mathcal{S}^\pm, \\ \underline{\rho}^+\phi^+{}_{|z=0} - \underline{\rho}^-\phi^-{}_{|z=0} = \psi, \\ \dfrac{1}{\underline{H}^+}\partial_n\phi^+{}_{|z=0} - \dfrac{1}{\underline{H}^-}\partial_n\phi^-{}_{|z=0} = 0, & \partial_n\phi^\pm{}_{|z=\mp1} = 0, \end{cases} \tag{2.37}$$

where ∂_n denotes, as usual, the upwards conormal derivative (see (2.8)).

Transmission problems of this kind are usually studied with harmonic analysis tools and the solution is then given in terms of the single and double layer potentials (see for instance [25]). Such an approach gives very sharp results in terms of regularity but is not adapted to our problem (influence of the boundaries, asymptotics, . . .). We thus propose another approach, more elementary, but more robust with respect to the applications we have in mind.

Proposition 2. *Let* $t_0 > d/2, 0 \leq s \leq t_0 + 1$ *and* $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ *be such that* (2.1) *is satisfied. Then for all* $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, *there exists a unique solution* $\phi^\pm \in \dot{H}^1(\mathcal{S}^\pm)$ *to* (2.37) *such that* $\Lambda^s\nabla^{\mu\pm}\phi^\pm \in L^2(\mathcal{S}^\pm)$ *and*

$$\|\Lambda^s\nabla^{\mu\pm}\phi^\pm\|_{L^2(\mathcal{S}^\pm)} \leq M\sqrt{\mu}|\psi|_{\dot{H}_\mu^{s+1/2}}.$$

Remark 13. As with Proposition 1 it is possible to extend the result to all $s \geq 0$ in Proposition 2 above, and Lemma 2 and Corollaries 1 and 2 below. The estimates given in the statement of these results still hold for all $s \geq 0$, provided that M is replaced by $M(s + 1/2)$ in the right-hand side.

Proof. If the result of the proposition holds true, then, denoting $\psi^\pm = \phi^\pm{}_{|z=0}$, one has

$$\underline{\rho}^+\psi^+ - \underline{\rho}^-\psi^- = \psi \quad \text{and} \quad \frac{1}{\underline{H}^+}\mathcal{G}^+\psi+ = \frac{1}{\underline{H}^-}\mathcal{G}^-\psi^-,$$

and thus $\psi^- = \frac{H^-}{H^+}(\mathcal{G}^-)^{-1}\mathcal{G}^+\psi^+$ (well defined in $\dot{H}^{s+1/2}$ by Proposition 1) and

$$\underline{\rho}^+\psi^+ - \underline{\rho}^- \frac{H^-}{H^+}(\mathcal{G}^-)^{-1}\mathcal{G}^+\psi^+ = \psi.$$

We now need the following lemma showing that the operator $J[\zeta]$ mapping ψ^+ to ψ is one-to-one and onto.

Lemma 2. *For all $0 \leqq s \leqq t_0 + 1$, the mapping*

$$J[\zeta] : \begin{array}{l} \dot{H}_\mu^{s+1/2}(\mathbb{R}^d) \to \dot{H}_\mu^{s+1/2}(\mathbb{R}^d) \\ \psi \qquad\qquad \mapsto \big(\underline{\rho}^+ - \underline{\rho}^- \frac{H^-}{H^+}(\mathcal{G}^-)^{-1}\mathcal{G}^+\big)\psi \end{array}$$

is one-to-one and onto; moreover, one has

$$|J[\zeta]^{-1}\psi|_{\dot{H}_\mu^{s+1/2}} \leqq M|\psi|_{\dot{H}_\mu^{s+1/2}}.$$

Proof. Let us first consider the case $s = 0$. For all $\psi^+ \in \dot{H}_\mu^{1/2}(\mathbb{R}^d)$, let $\psi^- = \frac{H^-}{H^+}(\mathcal{G}^-)^{-1} \circ \mathcal{G}^+\psi^+ \in \dot{H}_\mu^{1/2}(\mathbb{R}^d)$ (as provided by Proposition 1), and let $\phi^\pm \in \dot{H}^1(\mathcal{S}^\pm)$ be the unique solution of (2.17). Let us now multiply (2.22)$_+$ by $\frac{1}{H^+}\underline{\rho}^+$, (2.22)$_-$ by $\frac{1}{H^-}\underline{\rho}^-$ and sum up the two identities to get (with $P^\pm = P(\Sigma^\pm)$)

$$\frac{\underline{\rho}^+}{H^+}\int_{\mathcal{S}^+} P^+\nabla^{\mu^+}\phi^+ \cdot \nabla^{\mu^+}\phi^+ + \frac{\underline{\rho}^-}{H^-}\int_{\mathcal{S}^-} P^-\nabla^{\mu^-}\phi^- \cdot \nabla^{\mu^-}\phi^-$$
$$= \int_{\mathbb{R}^d} \frac{1}{H^+}(\mathcal{G}^+\psi^+)J[\zeta]\psi^+. \tag{2.38}$$

Using the coercivity of $P^\pm$ (with coercivity constants $k^\pm = k(\Sigma^\pm)$ as in (2.6)) to get a lower bound on the left-hand side, and (2.23) to derive an upper bound for the right-hand side, one obtains

$$\frac{\underline{\rho}^+}{H^+}k^+\|\nabla^{\mu^+}\phi^+\|_{L^2(\mathcal{S}^+)}^2 + \frac{\underline{\rho}^-}{H^-}k^-\|\nabla^{\mu^-}\phi^-\|_{L^2(\mathcal{S}^-)}^2 \leqq \frac{M}{H^+}\mu|\psi^+|_{\dot{H}_\mu^{1/2}}|J[\zeta]\psi^+|_{\dot{H}_\mu^{1/2}}.$$

Now, since $\psi^+ = (\mathcal{G}^+)^{-1} \circ \mathcal{G}^-\psi^-$, we can deduce from Remark 7 that $|\psi^+|_{\dot{H}_\mu^{1/2}} \leqq M|\psi^-|_{\dot{H}_\mu^{1/2}}$; using (2.6) and (2.19) we then deduce

$$|\psi^+|_{\dot{H}_\mu^{1/2}} \leqq M|J[\zeta]\psi^+|_{\dot{H}_\mu^{1/2}}.$$

This tells us that for all $0 \leqq \underline{\rho}^+ \leqq 1$ and $\underline{\rho}^- = 1 - \underline{\rho}^+$, $J[\zeta]$ is a closed, one-to-one operator; it is thus semi-Fredholm. Moreover, since $J[\zeta]$ is clearly invertible (by a Neumann series expansion) for small enough values of $\underline{\rho}^-$, we deduce from the homotopic invariance of the index that it is Fredholm of index zero, and thus invertible (since it is one-to-one). We now turn to proving the Lemma for the general

case $0 \leq s \leq t_0 + 1$. It is easy to deduce from (2.17) the following generalization of (2.22),

$$\int_{\mathcal{S}^\pm} P^\pm \nabla^{\mu^\pm} \Lambda^s \phi^\pm \cdot \nabla^{\mu^\pm} \Lambda^s \phi^\pm = \pm \int_{\mathbb{R}^d} (\Lambda^s \mathcal{G}^\pm \psi^\pm) \Lambda^s \psi^\pm$$
$$+ \int_{\mathcal{S}^\pm} [\Lambda^s, P^\pm] \nabla^{\mu^\pm} \phi^\pm \cdot \Lambda^s \nabla^{\mu^\pm} \phi^\pm.$$

Proceeding as for the case $s = 0$, one gets

$$\underline{\rho}^+ \|\Lambda^s \nabla^{\mu^+} \phi^+\|_{L^2(\mathcal{S}^+)} + \underline{\rho}^- \|\Lambda^s \nabla^{\mu^-} \phi^-\|_{L^2(\mathcal{S}^-)} \leq M \sqrt{\mu} |J[\zeta] \psi^+|_{\dot{H}_\mu^{s+1/2}}$$
$$+ M \left(\underline{\rho}^+ \|[\Lambda^s, P^+] \nabla^{\mu^+} \phi^+\|_{L^2(\mathcal{S}^+)} + \underline{\rho}^- \|[\Lambda^s, P^-] \nabla^{\mu^-} \phi^-\|_{L^2(\mathcal{S}^-)} \right).$$

Using (2.29) and a continuous induction as in the proof of Proposition 1, we then get

$$\underline{\rho}^+ \|\Lambda^s \nabla^{\mu^+} \phi^+\|_{L^2(\mathcal{S}^+)} + \underline{\rho}^- \|\Lambda^s \nabla^{\mu^-} \phi^-\|_{L^2(\mathcal{S}^-)} \leq M \sqrt{\mu} |J[\zeta] \psi^+|_{\dot{H}_\mu^{s+1/2}},$$

and the result follows from (2.19) as in the case $s = 0$. □

Thanks to the lemma (and Proposition 1), one can define $\psi^+, \psi^- \in \dot{H}_\mu^{s+1/2}$ by

$$\psi^+ = J[\zeta]^{-1} \psi, \qquad \psi^- = \frac{H^-}{H^+} (\mathcal{G}^-)^{-1} \circ \mathcal{G}^+ \psi^+. \tag{2.39}$$

Taking $\phi^\pm$ as the solution to (2.17) with $\psi^\pm$ as above concludes the proof. □

As a first corollary to Proposition 2, we can prove that the quantities that appear in the equations (1.19) are well defined in the sense that $\nabla \psi^\pm$ can be viewed as a function of ζ and ψ. Since $\nabla \psi^\pm$ corresponds to the horizontal components of the tangential velocity at the interface, we write this function under the form $\nabla \psi^\pm = \mathcal{V}_{/\!/}^\pm[\varepsilon\zeta] \psi$.

Corollary 1. *Under the assumptions of Proposition 2, and denoting by $\psi^\pm$ the trace of $\phi^\pm$ at the interface $\{z = 0\}$, the mappings*

$$\mathcal{V}_{/\!/}^\pm[\varepsilon\zeta] : \begin{array}{c} \dot{H}^{s+1/2}(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^d) \\ \psi \mapsto \nabla \psi^\pm \end{array}$$

and

$$w^\pm[\varepsilon\zeta] : \begin{array}{c} \dot{H}^{s+1/2}(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^d) \\ \psi \mapsto \dfrac{1}{1 + \varepsilon^2 \mu |\nabla\zeta|^2} \left(\dfrac{1}{\underline{H}^\pm} \mathcal{G}^\pm \psi^\pm + \varepsilon\mu\nabla\zeta \cdot \nabla\psi^\pm \right) \end{array}$$

are well defined and one has,

$$\forall 0 \leq s \leq t_0 + 1/2, \ |\mathcal{V}_{/\!/}^\pm[\varepsilon\zeta] \psi|_{H^s} \leq M |\nabla\psi|_{H^s},$$

$$\forall 0 \leq s \leq t_0 + 1, \ |w^\pm[\varepsilon\zeta] \psi|_{H^{s-1/2}} \leq \mu^{3/4} M |\psi|_{\dot{H}_\mu^{s+1/2}},$$

$$\forall 0 \leq s \leq t_0 + 1/2, \ |w^\pm[\varepsilon\zeta] \psi|_{H^{s-1/2}} \leq \mu M |\psi|_{\dot{H}_\mu^{s+1}}.$$

Proof. This is an immediate consequence of (2.39), Lemma 2, Proposition 1 and Lemma 1 (and of (2.20)–(2.21) for the estimate on $w^{\pm}[\varepsilon\zeta]\psi$). $\quad\square$

A second important corollary to Proposition 2 concerns the operator $\mathcal{G}_{\mu}[\varepsilon\zeta]$ defined in (1.18).

Corollary 2. *Under the assumptions of Proposition 2, the mapping*

$$\mathcal{G}_{\mu}[\varepsilon\zeta]: \begin{array}{l} \dot{H}^{s+1/2}(\mathbb{R}^d) \to H^{s-1/2}(\mathbb{R}^d) \\ \psi \qquad\qquad \mapsto \mathcal{G}^-\big(\underline{\rho}^+\underline{H}^+\mathcal{G}^- - \underline{\rho}^-\underline{H}^-\mathcal{G}^+\big)^{-1}\mathcal{G}^+\psi \end{array}$$

is well defined for all $0 \leqq s \leqq t_0 + 1/2$ and one has

$$\forall\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d), \qquad |\mathcal{G}_{\mu}[\varepsilon\zeta]\psi|_{H^s} \leqq \sqrt{\mu}M|\nabla\psi|_{H^s}.$$

Proof. Let us check first that $(\rho^+\underline{H}^+\mathcal{G}^- - \underline{H}^-\rho^-\mathcal{G}^+)^{-1}\circ\mathcal{G}^+$ is well defined. We can deduce from the positivity of $(\rho^+\underline{H}^+\mathcal{G}^- - \rho^-\underline{H}^-\mathcal{G}^+)$ and the identity

$$(\rho^+\underline{H}^+\mathcal{G}^- - \rho^-\underline{H}^-\mathcal{G}^+)\circ(\mathcal{G}^-)^{-1}\circ\mathcal{G}^+\circ J[\zeta]^{-1} = \underline{H}^+\mathcal{G}^+$$

that $(\rho^+\underline{H}^+\mathcal{G}^- - \rho^-\underline{H}^-\mathcal{G}^+)^{-1}\circ\mathcal{G}^+$ is well defined and given by the formula

$$(\rho^+\underline{H}^+\mathcal{G}^- - \rho^-\underline{H}^-\mathcal{G}^+)^{-1}\circ\mathcal{G}^+ = \frac{1}{\underline{H}^+}(\mathcal{G}^-)^{-1}\circ\mathcal{G}^+\circ J[\zeta]^{-1}.$$

The estimate given in the statement of the corollary is thus a direct consequence of (2.20) and Lemmas 2 and 1. $\quad\square$

Remark 14. Proceeding as in the proof of Corollary 2, one can derive several equivalent expressions for $\mathcal{G}_{\mu}[\varepsilon\zeta]$, for instance,

$$\mathcal{G}_{\mu}[\varepsilon\zeta] = \frac{1}{\underline{H}^+}\mathcal{G}^+\circ J[\zeta]^{-1}$$

$$\underline{\rho}^+\mathcal{G}_{\mu}[\varepsilon\zeta] = \frac{1}{\underline{H}^+}\big(1 + \underline{\rho}^-\underline{H}^-\mathcal{G}_{\mu}[\varepsilon\zeta]\circ(\mathcal{G}^-)^{-1}\big)\mathcal{G}^+.$$

Remark 15. In the undisturbed case ($\zeta = 0$), it is easy to check by Fourier analysis that $\mathcal{G}^{\pm} = \pm\mathcal{G}_{\mu^{\pm}}^{\pm}[0, 1] = \pm\sqrt{\mu^{\pm}}|D|\tanh(\sqrt{\mu^{\pm}}|D|)$. It follows therefore that

$$\mathcal{G}_{\mu}[0] = \sqrt{\mu}|D|\frac{\tanh(\sqrt{\mu^+}|D|)\tanh(\sqrt{\mu^-}|D|)}{\underline{\rho}^+\tanh(\sqrt{\mu^-}|D|) + \underline{\rho}^-\tanh(\sqrt{\mu^+}|D|)}.$$

Some important features of the operator $\mathcal{G}_{\mu}[\varepsilon\zeta]$ are gathered in the following proposition.

Proposition 3. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ satisfying (2.1). Then*

1. *The operator $\frac{1}{\mu}\mathcal{G}_{\mu}[\varepsilon\zeta]$ is positive and uniformly coercive on $\dot{H}_{\mu}^{1/2}(\mathbb{R}^d)$,*

$$\forall\psi \in \dot{H}^{1/2}(\mathbb{R}^d), \qquad |\mathfrak{P}\psi|_2^2 \leqq M\big(\psi, \frac{1}{\mu}\mathcal{G}_{\mu}[\varepsilon\zeta]\psi\big).$$

2. *The bilinear form $(\frac{1}{\mu} \mathcal{G}_\mu[\varepsilon\zeta]\cdot, \cdot)$ is symmetric on $\dot{H}_\mu^{1/2} \times \dot{H}_\mu^{1/2}$,*

$$\forall \psi_1, \psi_2 \in \dot{H}^{1/2}(\mathbb{R}^d), \qquad (\mathcal{G}_\mu[\varepsilon\zeta]\psi_1, \psi_2) = (\mathcal{G}_\mu[\varepsilon\zeta]\psi_2, \psi_1).$$

3. *For all $0 \leqq s \leqq t_0 + 1$ and $\psi_1, \psi_2 \in \dot{H}_\mu^{s+1/2}$, one has*

$$\left| (\Lambda^s \mathcal{G}_\mu[\varepsilon\zeta]\psi_1, \Lambda^s \psi_2) \right| \leqq \mu M |\psi_1|_{\dot{H}_\mu^{s+1/2}} |\psi_2|_{\dot{H}_\mu^{s+1/2}}.$$

Proof. Thanks to Remark 14, one has $(\mathcal{G}_\mu[\varepsilon\zeta]\psi, \psi) = \frac{1}{\underline{H}}(\mathcal{G}^+\psi^+, J[\zeta]\psi^+)$, with $\psi^+ = J[\zeta]^{-1}\psi$. It follows therefore from (2.38) and (2.19) that one has $\mu|\mathfrak{P}\psi^+| \leqq M(\mathcal{G}_\mu[\varepsilon\zeta]\psi, \psi)$. Since, moreover, $|\mathfrak{P}\psi| \leqq M|\mathfrak{P}\psi^+|$ by Proposition 1, the first point of the proposition is proved.

To prove the second point, we proceed as for (2.38) to get

$$(\mathcal{G}_\mu[\varepsilon\zeta]\psi_1, \psi_2) = \frac{\rho^+}{\underline{H}^+} \int_{\mathcal{S}^+} P^+ \nabla^{\mu^+} \phi_1^+ \cdot \nabla^{\mu^+} \phi_2^+ + \frac{\rho^-}{\underline{H}^-} \int_{\mathcal{S}^-} P^- \nabla^{\mu^-} \phi_1^- \cdot \nabla^{\mu^-} \phi_2^-,$$

with obvious notations for $\phi_j^\pm$ ($j = 1, 2$). The symmetry of $\mathcal{G}$ follows from the symmetry of $P^\pm$.

The last point of the proposition follows from the first equivalent expression of $\mathcal{G}_\mu[\varepsilon\zeta]$ given in Remark 14, together with (2.23) and Lemma 2. $\square$

3. Symbolic Analysis "with tail" of the Dirichlet–Neumann Operator and Consequences

3.1. Symbolic Analysis of $\mathcal{G}^\pm$

It is known that the principal symbol of the Dirichlet–Neumann operator can be expressed in terms of the Laplace–Beltrami operator associated to the surface. More precisely, if $\mathcal{G}^+[\zeta]$ denotes the Dirichlet–Neumann on the original (without nondimensionalization) problem (1.12), one has typically

$$\left| \mathcal{G}^+[\zeta]\psi - \sqrt{|D|^2 + (|D|^2|\nabla\zeta|^2 - (D \cdot \nabla\zeta)^2)}\psi \right|_{H^{s+1/2}} \leqq C(s, \zeta)|\nabla\psi|_{H^{s-1/2}}, \tag{3.1}$$

where $C(s, \zeta)$ depends on the norm of ζ in some Sobolev space depending on s. We refer for instance to [39,58] for a proof with PDE tools, to [54] for a more geometric approach, and to [3] for an interesting paradifferential extension.

The question that interests us here is the way this identity behaves asymptotically in the shallow water regime. Since this regime physically corresponds to configurations where the effect of the bottom is "felt" at the surface, it is possible to guess without computation that (3.1) becomes singular in shallow water; indeed, the influence of the bottom on $\mathcal{G}^+[\zeta]$ is analytic (by standard elliptic theory) and thus not taken into account in the symbolic expansion of $\mathcal{G}^+[\zeta]\psi$ (even at the next orders).

In order to make this singular behavior more precise, let us look at the form taken by (3.1) in the nondimensionalized setting, and in the case $\zeta = 0$. We recall that in this case, one has $\mathcal{G}_\mu^\pm[0] = \pm\sqrt{\mu^\pm}|D|\tanh(\sqrt{\mu^\pm}|D|)$, whose principal symbol is $\pm\sqrt{\mu^\pm}|D|$. It is thus straightforward to check that

$$|\mathcal{G}_\mu^\pm[0]\psi \mp \sqrt{\mu^\pm}|D|\psi|_{H^{s+1/2}} \lesssim |\nabla\psi|_{H^{s-1/2}}. \tag{3.2}$$

Since $\mathcal{G}_\mu^\pm[0]$ and $\sqrt{\mu^\pm}|D|$ are both of size $O(\sqrt{\mu^\pm})$ (as first order operators) while the residual is of size $O(1)$, the singularity of the symbolic approximation of the DN operator is of order $O(1/\sqrt{\mu^\pm})$ (a similar behavior also arises in semiclassical analysis). This is the reason why one cannot use standard symbolic analysis in a shallow water regime (though this is of course legitimate in other regimes, as in [1,44,55]).

Our purpose here is to show that it is, however, possible to adapt the symbolic analysis approach to the present case. As said previously, it is necessary to take into account the information coming from the bottom, that is, to include the smooth "tail" of the symbol that is neglected in any expansion into homogeneous symbols. In the flat case, this "tail" corresponds to $\left(1 - \tanh(\sqrt{\mu^\pm}|D|)\right)$. In order to describe the general case, it is necessary to define the usual principal symbol $g(X, \xi)$ and another symbol $t^\pm(X, \xi)$ associated to its tail; more precisely, we define

$$S^\pm(X, \xi) = \sqrt{\mu^\pm}g(X, \xi)\tanh(\sqrt{\mu^\pm}t^\pm(X, \xi)), \tag{3.3}$$

with

$$g(x, \xi) = \sqrt{|\xi|^2 + \varepsilon^2\mu(|\nabla\zeta|^2|\xi|^2 - (\nabla\zeta \cdot \xi)^2)},$$
$$t^\pm(x, \xi) = (1 \pm \varepsilon^\pm\zeta)\int_{-1}^0 \frac{\sqrt{|\xi|^2 + \varepsilon^2\mu(z+1)^2(|\nabla\zeta|^2|\xi|^2 - (\nabla\zeta \cdot \xi)^2)}}{1 + \varepsilon^2\mu(z+1)^2|\nabla\zeta|^2}\,dz.$$

We can then state the main result of this section.

Theorem 4. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+3}(\mathbb{R}^d)$ be such that (2.1) is satisfied. Then for all $0 \leqq s \leqq t_0$ and $\psi \in \dot{H}^{s+1/2}(\mathbb{R}^d)$, one has*

$$\left|\mathcal{G}^\pm\psi \mp S^\pm(x, D)\psi\right|_{H^{s+1/2}} \leqq \varepsilon\mu^{3/4}M(t_0 + 3)|\psi|_{\dot{H}_\mu^{s+1/2}}.$$

The following inequality also holds

$$\left|\mathcal{G}^\pm\psi \mp S^\pm(x, D)\psi\right|_{H^s} \leqq \varepsilon\mu M(t_0 + 3)|\psi|_{\dot{H}_\mu^{s+1/2}}.$$

Remark 16. In the case $d = 1$, one has $g(x, \xi) = |\xi|$ and one can compute explicitly $t^\pm(x, \xi) = (1 \pm \varepsilon^\pm\zeta)\dfrac{\arctan(\varepsilon\sqrt{\mu}\partial_x\zeta)}{\varepsilon\sqrt{\mu}\partial_x\zeta}|\xi|$.

Remark 17. The "tail" $1 - \tanh(\sqrt{\mu^\pm}t^\pm(X, \xi))$ in (3.3) accounts for the effects of the bottom. In particular, it vanishes in infinite depth (take formally $\mu^\pm = \infty$). Note also that (3.3) provides an exact expression for $\mathcal{G}^\pm$ in the undisturbed case $\zeta = 0$; the explicit computations that can be performed in that case also show that the hyperbolic tangent is a bottom effect.

Remark 18. One readily deduces from the theorem the following estimate for the standard symbolic approximation,

$$|\mathcal{G}^{\pm}\psi \mp \sqrt{\mu^{\pm}}g(x, D)\psi|_{H^{s+1/2}} \lesseqgtr M(t_0 + 3)|\nabla\psi|_{H^{s-1/2}},$$

which is the nondimensionalized version of (3.1) generalizing (3.2) to the nonflat case. The interest of the new symbolic expansion of Theorem 4 is that it gives a control of order $O(\varepsilon\sqrt{\mu})$ of $|\nabla\psi|_{H^{s-1/2}}$ (deduced from the $O(\varepsilon\mu^{3/4})$ control of $|\psi|_{\dot{H}^{s+1/2}_{\mu}}$); it is therefore more precise in its dependence on μ *and* ε. Even outside the shallow water regime (that is, if $\mu = O(1)$), the estimate of the theorem is more precise than the standard expansion if ε is small. In particular, the formula is exact in the flat case ($\varepsilon = 0$). Note also that the improvement factor, namely $\varepsilon\sqrt{\mu} = \varepsilon^{\pm}\sqrt{\mu^{\pm}}$, is the typical *slope* of the wave, a quantity that plays an important role in many asymptotic expansions. In the second estimate of the theorem, we have a better control of order $O(\varepsilon\mu)$ but in terms of regularity, it is half-a-derivative worse ($|\psi|_{\dot{H}^{s+1/2}_{\mu}}$ instead of $|\psi|_{\dot{H}^{s}_{\mu}}$).

3.2. Proof of Theorem 4

Since the proof of the result for $\mathcal{G}^-$ follows exactly the same lines as for $\mathcal{G}^+$, we only do it in the latter case.

We also assume throughout this proof that $\Sigma(X, z) = (X, z + \sigma(X, z))$ is the trivial diffeomorphism given in Example 1; in particular, $1 + \partial_z\sigma = 1 + \varepsilon^+\zeta$ does not depend on z.

Let us denote by $\mathbf{P} = \mathbf{P}(X, z, \nabla, \partial_z)$ the elliptic operator

$$\mathbf{P} = \nabla^{\mu^+} \cdot P(\Sigma)\nabla^{\mu^+}.$$

From the explicit expression of $P(\Sigma)$ given in (B.7), we can decompose $\mathbf{P}$ into

$$\mathbf{P} = \mathbf{P}_I + \varepsilon^+\mu^+\mathbf{P}_{II},$$

with

$$\mathbf{P}_I = \frac{1 + \mu^+|\nabla\sigma|^2}{1 + \partial_z\sigma}\partial_z^2 - 2\mu^+\nabla\sigma \cdot \nabla\partial_z + \mu^+(1 + \partial_z\sigma)\Delta,$$

$$\mathbf{P}_{II} = \frac{1}{\varepsilon^+}\Big(\frac{1}{1 + \partial_z\sigma}\partial_z(|\nabla\sigma|^2) - \Delta\sigma\Big)\partial_z.$$

The strategy for proving Theorem 4 is the following. Since by definition

$$\mathcal{G}^+\psi = \partial_n\phi_{|z=0}, \quad \text{with} \quad \begin{cases} \mathbf{P}\phi = 0, \\ \phi_{|z=0} = \psi, \qquad \partial_n\phi_{|z=-1} = 0 \end{cases}$$

(recall that ∂_n stands for the upward conormal derivative, see (2.8)), we expect that at leading order, $\mathcal{G}^+\psi \sim \partial_n\phi_{\text{app}\,|z=0}$, if ϕ_{app} solves the same boundary value problem as ϕ up to lower order terms. This is the approach used in [3,39] which leads to the standard symbolic analysis of $\mathcal{G}^+[\zeta]\psi$. The difference here is that we

want the approximation ϕ_{app} to be nonsingular with respect to μ^+. This is this extra constraint which imposes the presence of the "tail" in the symbolic analysis of the DN operator.

Since $\mathbf{P} = \mathbf{P}_I$ up to first order terms that are also of size $O(\mu^+)$, the idea is to look for an explicit function ϕ_{app} satisfying $\mathbf{P}_I \phi_{\mathrm{app}} = 0$ (up to lower order terms). In order to construct such a ϕ_{app}, we first remark that if we freeze the coefficients of $\mathbf{P}_I$ and take the Fourier transform with respect to the horizontal variables, $\mathbf{P}_I$ becomes a second order differential equation with respect to z, with characteristic polynomial

$$\frac{1 + \mu^+ |\nabla \sigma|^2}{1 + \partial_z \sigma} \eta^2 - 2i\mu^+ \nabla \sigma \cdot \xi \eta - \mu^+ (1 + \partial_z \sigma)|\xi|^2.$$

The roots of this polynomial are given by

$$\eta^{\pm}(X, z, \xi) = \pm \sqrt{\mu^+} a(X, z, \xi) + i\mu^+ b(X, z, \xi),$$

with

$$a(X, z, \xi) = \frac{1 + \partial_z \sigma}{1 + \mu^+ |\nabla \sigma|^2} \sqrt{|\xi|^2 + \mu^+(|\nabla \sigma|^2 |\xi|^2 - (\nabla \sigma \cdot \xi)^2)},$$

$$b(X, z, \xi) = \frac{1 + \partial_z \sigma}{1 + \mu^+ |\nabla \sigma|^2} \nabla \sigma \cdot \xi.$$

We then define ϕ_{app} as the inverse Fourier transform of the solution of this frozen coefficient differential operator satisfying the boundary conditions at $z = 0$ and $z = -1$, namely,

$$\phi_{\mathrm{app}}(X, z) = \Xi(X, z, D)\psi,$$

where the symbol of the pseudodifferential operator $\Xi(X, z, D)$ is given by

$$\Xi(X, z, \xi) = \frac{\cosh(\sqrt{\mu^+} \int_{-1}^{z} a(X, z', \xi) dz')}{\cosh(\sqrt{\mu^+} \int_{-1}^{0} a(X, z', \xi) dz')} \exp\left(-i\mu^+ \int_{z}^{0} b(X, z', \xi) dz'\right).$$

The following lemma quantifies the accuracy of the approximation of ϕ by ϕ_{app}.

Lemma 3. *The approximation ϕ_{app} of ϕ solves*

$$\begin{cases} \mathbf{P}\phi_{\mathrm{app}} = \varepsilon^+ \mu^+ h, \\ \phi_{\mathrm{app}}|_{z=0} = \psi, \qquad \partial_n \phi_{\mathrm{app}}|_{z=-1} = 0, \end{cases}$$

where, for all $0 \leq s \leq t_0$, h satisfies

$$\|\Lambda^s h\|_2 \leq M(t_0 + 3)|\psi|_{\dot{H}_\mu^{s+1/2}}.$$

Proof. One can decompose $\mathbf{P}\phi_{\mathrm{app}}$ under the form

$$\mathbf{P}\phi_{\mathrm{app}} = \mathbf{P}_I\phi_{\mathrm{app}} + \varepsilon^+\mu^+\mathbf{P}_{II}\phi_{\mathrm{app}}. \tag{3.4}$$

We now analyze the two components of the right-hand-side separately. - *Analysis of $\mathbf{P}_I\phi_{\mathrm{app}}$.* From the relations

$$\nabla\sigma \cdot \nabla\partial_z\phi_{\mathrm{app}} = \nabla\sigma \cdot \nabla\mathrm{Op}(\partial_z\Xi)\psi$$
$$= \mathrm{Op}(\nabla\sigma \cdot \nabla\partial_z\Xi)\psi + \mathrm{Op}(i(\nabla\sigma \cdot \xi)\partial_z\Xi)\psi,$$
$$(1 + \partial_z\sigma)\Delta\phi_{\mathrm{app}} = -\mathrm{Op}((1 + \partial_z\sigma)\Xi|\xi|^2)\psi + (1 + \partial_z\sigma)[\Delta, \mathrm{Op}(\Xi)]\psi,$$

we deduce that

$$\mathbf{P}_I\phi_{\mathrm{app}} = \mathrm{Op}(\mathbf{P}_I(X, z, i\xi, \partial_z)\Xi)\psi - 2\mu\mathrm{Op}(\nabla\sigma \cdot \nabla\partial_z\Xi)\psi$$
$$+ \mu(1 + \partial_z\sigma)[\Delta, \mathrm{Op}(\Xi)]\psi.$$

From the definition of Ξ, it is also easy to check that

$$\mathbf{P}_I(X, z, i\xi, \partial_z)\Xi = \varepsilon^+\mu^+\frac{1 + \mu^+|\nabla\sigma|^2}{1 + \partial_z\sigma}\exp\left(-i\mu^+\int_z^0 b\right)$$
$$\times\left(\frac{\partial_z a}{\varepsilon^+\sqrt{\mu^+}}\frac{\sinh(\sqrt{\mu^+}\int_{-1}^z a)}{\cosh(\sqrt{\mu^+}\int_{-1}^0 a)} + i\frac{\partial_z b}{\varepsilon^+}\frac{\cosh(\sqrt{\mu^+}\int_{-1}^z a)}{\cosh(\sqrt{\mu^+}\int_{-1}^0 a)}\right).$$

Since, moreover,

$$(1 + \partial_z\sigma)[\Delta, \mathrm{Op}(\Xi)] = \mathrm{Op}((1 + \partial_z\sigma)\Delta\Xi)) + 2\mathrm{Op}(i(1 + \partial_z\sigma)\nabla\Xi \cdot \xi),$$

one easily gets that

$$\mathbf{P}_I\phi_{\mathrm{app}} = \varepsilon^+\mu^+\mathrm{Op}(R_1 + R_2)\psi + \varepsilon^+\mu^+(1 + \partial_z\sigma)\mathrm{Op}(\frac{\Delta\Xi}{\varepsilon^+})\psi, \tag{3.5}$$

with

$$\varepsilon^+R_1(X, z, \xi) = \frac{1 + \mu^+|\nabla\sigma|^2}{1 + \partial_z\sigma}\exp\left(i\mu^+\int_z^0 b\right)$$
$$\times\left(\frac{\partial_z a}{\sqrt{\mu^+}}\frac{\sinh(\sqrt{\mu^+}\int_{-1}^z a)}{\cosh(\sqrt{\mu^+}\int_{-1}^0 a)} + i\partial_z b\frac{\cosh(\sqrt{\mu^+}\int_{-1}^z a)}{\cosh(\sqrt{\mu^+}\int_{-1}^0 a)}\right),$$
$$\varepsilon^+R_2(X, z, \xi) = 2i(1 + \partial_z\sigma)\nabla\Xi \cdot \xi - 2\nabla\sigma \cdot \nabla\partial_z\Xi.$$

We now turn to controlling all the components of (3.5):
– *Control of $\mathrm{Op}(R_j)\psi$, ($j = 1, 2$).* Let us remark that

$$\mathrm{Op}(R_j)\psi = \mathrm{Op}(\widetilde{\mathbf{R}}_j) \cdot \nabla\psi^\dagger \quad (j = 1, 2, 3),$$

with $\psi^\dagger = \exp(c_0\sqrt{\mu^+}z|D|)\psi$ and

$$\widetilde{\mathbf{R}}_j(X, z, \xi) = -i\frac{R_j(X, z, \xi)}{|\xi|}\frac{\xi}{|\xi|}\exp(-c_0\sqrt{\mu^+}z|\xi|),$$

and where $c_0 > 0$ is such that $\frac{1+\partial_z\sigma}{1+\mu^+|\nabla\sigma|^2} \geqq 2c_0$ for all $(X, z) \in S$ (and thus $a(X, z, \xi) \geqq 2c_0|\xi|$). It follows from the definition of R_j and c_0 that $\widetilde{\mathbf{R}}_j$ is a pseudo-differential operator of order zero, whose coefficients depend on second and lower order derivatives of ζ (through σ). Using Theorem 1 of [40], which gives precise estimates of the operator norm of pseudo-differential estimates, we then get that

$$\forall 0 \leqq s \leqq t_0, \quad \forall z \in [-1, 0], \qquad |\widetilde{\mathbf{R}}_j(X, z, D)f|_{H^s} \leqq M|f|_{H^s}$$

(it is easy to check that the possibly singular terms $1/\varepsilon^+$ in the definition of R_j are compensated by the ε^+ contained in $\sigma = \varepsilon^+(z+1)\zeta$). We thus deduce that

$$\|\Lambda^s \mathrm{Op}(R_1 + R_2)\psi\|_2 \leqq M\|\Lambda^s \nabla\psi^\dagger\|_2,$$
$$\leqq M|\psi|_{\dot{H}_\mu^{s+1/2}}, \tag{3.6}$$

the last inequality following from an easy computation (recall that $z \leqq 0$), as in Proposition 2.2 of [4] for instance.

  – *One can prove along the same lines that*

$$\left\|\Lambda^s \mathrm{Op}\left((1 + \partial_z\sigma)\frac{\Delta\Xi}{\varepsilon^+}\right)\psi\right\|_2 \leqq M(t_0 + 3)|\psi|_{\dot{H}_\mu^{s+1/2}} \tag{3.7}$$

(we have $M(t_0 + 3)$ rather than M in the right-hand side because $\Delta\Xi$ involves third order derivatives of ζ).

  From (3.5), (3.6) and (3.7), we get that

$$\mathbf{P}_I\phi_{\mathrm{app}} = \varepsilon^+\mu^+ h_1 \quad \text{with} \quad \|\Lambda^s h_1\|_2 \leqq M(t_0 + 3)|\psi|_{\dot{H}_\mu^{s+1/2}}. \tag{3.8}$$

  – *Analysis of* $\mathbf{P}_{II}\phi_{\mathrm{app}}$. From the definition of $\mathbf{P}_{II}$ and ϕ_{app}, one gets easily

$$\mathbf{P}_{II}\phi_{\mathrm{app}} = \mathrm{Op}(R_3)\psi, \quad \text{with} \quad R_3(X, z, \xi) = \frac{1}{\varepsilon^+}\left(\frac{1}{1+\partial_z\sigma}\partial_z|\nabla\sigma|^2 - \Delta\sigma\right)\partial_z\Xi.$$

Proceeding as in the previous point, we get, for all $0 \leqq s \leqq t_0$,

$$\|\mathbf{P}_{II}\phi_{\mathrm{app}}\|_2 \leqq M|\psi|_{\dot{H}_\mu^{s+1/2}}. \tag{3.9}$$

  – *End of the proof.* The result follows directly from (3.4), (3.8) and (3.9), with $h = h_1 + \mathbf{P}_{II}\phi_{\mathrm{app}}$ (it is straightforward to check that ϕ_{app} satisfies the boundary conditions). $\square$

  It follows from the lemma that the difference $u = \phi - \phi_{\mathrm{app}}$ solves

$$\begin{cases} \nabla^{\mu^+} \cdot P(\Sigma^\pm)\nabla^{\mu^+}u = -\varepsilon^+\mu^+ h \\ u_{|z=0} = 0, \qquad \partial_n u_{|z=-1} = 0. \end{cases} \tag{3.10}$$

The following lemma allows some control on u.

Lemma 4. *Let $u \in \dot{H}^1(\mathcal{S})$ and $\tilde{h} \in L^2(\mathcal{S})$ be such that*

$$\begin{cases} \nabla^{\mu^+} \cdot P(\Sigma^+) \nabla^{\mu^+} u = \tilde{h}, \\ u_{|z=0} = 0, \quad \partial_n u_{|z=-1} = 0. \end{cases}$$

Then, for all $0 \leqq s \leqq t_0$,

$$\|\Lambda^{s+1} \nabla^{\mu^+} u\|_2 \leqq \frac{1}{\sqrt{\mu}} M \|\Lambda^s \tilde{h}\|_2.$$

Proof. Let us first remark that $\tilde{h} = \nabla^{\mu^+} \cdot \mathbf{g}$, with $\mathbf{g} = \left(\int_{-1}^z h\right) \mathbf{e}_z$, and we thus deduce from Lemma 13 that

$$\|\Lambda^s \nabla^{\mu^+} u\|_2 \leqq M \|\Lambda^s \mathbf{g}\|_2 \leqq M \|\Lambda^s \tilde{h}\|_2. \tag{3.11}$$

For $1 \leqq j \leqq d$, one remarks further that $v = \partial_j u$ solves

$$\begin{cases} \nabla^{\mu^+} \cdot P(\Sigma^+) \nabla^{\mu^+} v = \nabla^{\mu^+} \cdot \mathbf{g}, \\ u_{|z=0} = 0, \quad \partial_n u_{|z=-1} = -\mathbf{e}_z \cdot \mathbf{g}_{|z=-1}, \end{cases}$$

with $\mathbf{g} = -\partial_j P(\Sigma^+) \nabla^{\mu^+} u + \frac{1}{\sqrt{\mu^+}} \tilde{h} \mathbf{e}_j$. We thus deduce from (13) that

$$\|\Lambda^s \partial_j \nabla^{\mu^+} u\|_2 \leqq M \|\Lambda^s \mathbf{g}\|_2$$
$$\leqq M(1 + \frac{1}{\sqrt{\mu}}) \|\Lambda^s \tilde{h}\|, \tag{3.12}$$

where we used (3.11) and (B.2) for the second inequality. The lemma is thus proved. □

Applying Lemma 4 to (3.10) with $\tilde{h} = -\varepsilon^+ \mu^+ h$, we get (replacing ε^+ and μ^+ by ε and μ since the ratios H^+/H^- and H^-/H^+ are controlled by M),

$$\|\Lambda^{s+1} \nabla^{\mu^+} u\|_2 \leqq \varepsilon \sqrt{\mu} M \|\Lambda^s h\|_2$$
$$\leqq \varepsilon \sqrt{\mu} M(t_0 + 3) |\psi|_{\dot{H}_\mu^{s+1/2}}, \tag{3.13}$$

where the last inequality follows from Lemma 3.

In order to prove the theorem, let us now remark that

$$\mathcal{G}^+ \psi - \partial_n \phi_{\text{app}}{}_{|z=0} = \partial_n u_{|z=0} = \mathbf{e}_z \cdot P(\Sigma^+) \nabla^{\mu^+} u_{|z=0}.$$

By Green's formula, we get, therefore, for all $\varphi \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \Lambda^{s+1/2} \left(\mathcal{G}^+ \psi - \partial_n \phi_{\text{app}}{}_{|z=0} \right) \varphi$$
$$= \int_{\mathcal{S}} P(\Sigma^+) \nabla^{\mu^+} u \cdot \nabla^{\mu^+} \Lambda^{s+1/2} \varphi^\dagger - \varepsilon^+ \mu^+ \int_{\mathcal{S}} h \Lambda^{s+1/2} \varphi^\dagger$$
$$= \int_{\mathcal{S}} \Lambda^{s+1} P(\Sigma^+) \nabla^{\mu^+} u \cdot \Lambda^{-1/2} \nabla^{\mu^+} \varphi^\dagger - \varepsilon^+ \mu^+ \int_{\mathcal{S}} \Lambda^s h \Lambda^{1/2} \varphi^\dagger,$$

where $\varphi^\dagger$ is defined as

$$\varphi^\dagger(\cdot, z) = \chi(\sqrt{\mu^+}z|D|)\varphi,$$

(χ being a smooth compactly supported function equal to 1 in a neighborhood of the origin). Since $\|\Lambda^{-1/2}\nabla^{\mu^+}\varphi^\dagger\|_2 \lesssim (\mu^+)^{1/4}|\varphi|_2$ and $\|\Lambda^{1/2}\varphi^\dagger\|_2 \lesssim (\mu^+)^{-1/4}|\varphi|_2$ (see Prop. 2.2 of [4]), we get, therefore,

$$\int_{\mathbb{R}^d} \Lambda^{s+1/2}\big(G^+\psi - \partial_n\phi_{\text{app}\,|_{z=0}}\big)\varphi \leqq M\big(\mu^{1/4}\|\Lambda^{s+1}\nabla^{\mu^+}u\|_2 + \varepsilon\mu^{3/4}\|\Lambda^s h\|_2\big)|\varphi|_2$$
$$\leqq \varepsilon\mu^{3/4}M(t_0 + 3)|\psi|_{\dot{H}_\mu^{s+1/2}}|\varphi|_2,$$

the last line being a consequence of (3.13) and Lemma 3. Moreover, since one computes easily that

$$\partial_n\phi_{\text{app}\,|_{z=0}} = \mathbf{e}_z \cdot P(\Sigma^+)\nabla^{\mu^+}\phi_{\text{app}\,|_{z=0}}$$
$$= \sqrt{\mu^+}\sqrt{|\xi|^2 + \mu^+(|\nabla\zeta|^2|\xi|^2 - (\nabla\zeta \cdot \xi)^2)}\tanh\Big(\int_{-1}^0 a\Big),$$

the first estimate of the theorem follows from a standard duality argument.

For the second estimate, we proceed as above to get

$$\int_{\mathbb{R}^d} \Lambda^s\big(G^+\psi - \partial_n\phi_{\text{app}\,|_{z=0}}\big)\varphi$$
$$= \int_{\mathcal{S}} \Lambda^{s+1}P(\Sigma^+)\nabla^{\mu^+}u \cdot \Lambda^{-1}\nabla^{\mu^+}\varphi^\dagger - \varepsilon^+\mu^+\int_{\mathcal{S}} \Lambda^s h\varphi^\dagger.$$

Since $\|\Lambda^{-1}\nabla^{\mu^+}\varphi^\dagger\|_2 \lesssim \sqrt{\mu}|\varphi|_2$ and $\|\varphi^\dagger\|_2 \lesssim |\varphi|_2$, we deduce as above by a duality argument that

$$|G^+\psi - \partial_n\phi_{\text{app}\,|_{z=0}}|_{H^s} \lesssim \sqrt{\mu}M\|\Lambda^{s+1}\nabla^{\mu^+}u\|_2 + \varepsilon\mu\|\Lambda^s h\|_2$$
$$\leqq \varepsilon\mu M(t_0 + 3)|\psi|_{\dot{H}_\mu^{s+1/2}},$$

where we used Lemma 4 (with $\widetilde{h} = -\varepsilon^+\mu^+h$) and Lemma 3 to get the second inequality. The end of the proof is then exactly as for the first estimate.

3.3. Symbolic Analysis of $(G^-)^{-1}G^+$

Theorem 4 shows that $\text{Op}(S^\pm)$, with $S^\pm$ as given by (3.3), provides a good description of $G^\pm$. It is thus natural to expect that a good symbolic description of the operator $(G^-)^{-1}G^+$ (which exists by Proposition 1) is given by

$$(G^-)^{-1}G^+ \sim \text{Op}\Big(\frac{S^+}{S^-}\Big) = \text{Op}\Big(-\frac{H^+}{H^-}\frac{\tanh(\sqrt{\mu^+}t^+)}{\tanh(\sqrt{\mu^-}t^-)}\Big),$$

with $t^\pm(X, \xi)$ as in Theorem 4. The following corollary shows that this is indeed the case.

Corollary 3. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+3}(\mathbb{R}^d)$ be such that* (2.1) *is satisfied. Then for all $0 \leqq s \leqq t_0$ and $\psi \in H^{s-1/2}(\mathbb{R}^d)$, one has, with $k = 0, 1$,*

$$\left|\left[(\mathcal{G}^-)^{-1}\mathcal{G}^+ - \mathrm{Op}(\frac{S^+}{S^-})\right]\psi\right|_{\dot{H}_\mu^{s+1/2}} \leqq \varepsilon\mu^{-k/4}M(t_0+3)|\psi|_{H^{s-k/2}}.$$

Remark 19. As for the symbolic analysis of $\mathcal{G}^+$, it is crucial to take into account the nonhomogeneous tail of the symbol. A simple look at the flat case shows, indeed, that one cannot expect an estimate better than $O(1/\mu^{3/4})$ if the principal symbol (constant equal to $-H^+/H^-$ here) is used instead of $S+/S^-$. In the case $k = 1$, we have the same $O(\varepsilon\sqrt{\mu})$ improvement as in Theorem 4. Note also that the $\mu^{-1/4}$ in the right-hand side of the estimate given in the corollary is not singular since it corresponds to the fact that the $\dot{H}_\mu^{s+1/2}$-norm (rather than the $H^{s+1/2}$-norm) is considered in the left-hand side. In the case $k = 0$, there is a further gain of $\mu^{1/4}$ in the estimate, but a loss of half-a-derivative, as in Theorem 4.

Remark 20. Contrary to Proposition 1, for instance, Corollary 3 requires that ψ belongs to a standard Sobolev space rather than a homogeneous one.

Proof. Denoting by $S^\pm(X, \xi)$ the symbol of the DN operator $\mathcal{G}^\pm$ as given by Theorem 4, we define $\widetilde{\psi}^- = \mathrm{Op}(\frac{S^+}{S^-})\psi$. We also denote by $\widetilde{\phi}^-$ the solution of (2.17)$_-$ with Dirichlet condition $\widetilde{\psi}^-$ at $z = 0$. Then, by definition of $\mathcal{G}^-$, one has $\partial_n\widetilde{\phi}^-_{|z=0} = \mathcal{G}^-\widetilde{\psi}^-$. If ϕ^- is as in (2.27), the quantity $u = \phi^- - \widetilde{\phi}^-$ therefore solves

$$\begin{cases} \nabla^{\mu^-} \cdot P(\Sigma^-)\nabla^{\mu^-}u = 0, \\ \partial_n u_{|z=0} = \mathcal{G}^+\psi - \mathcal{G}^-\widetilde{\psi}^-, \qquad \partial_n u_{|z=-1} = 0. \end{cases}$$

Proceeding as in the proof of Proposition 1, we get that

$$\int_{\mathcal{S}^-} P(\Sigma^-)\nabla^{\mu^-}\Lambda^s u \cdot \nabla^{\mu^-}\Lambda^s u = -\int_{\mathbb{R}^d} \Lambda^s\left(\mathcal{G}^+\psi - \mathcal{G}^-\widetilde{\psi}^-\right)\Lambda^s u_0$$
$$+ \int_{\mathcal{S}^-} [\Lambda^s, P(\Sigma^-)]\nabla^{\mu^-}u \cdot \nabla^{\mu^-}\Lambda^s u,$$

where u_0 stands for $u_{|z=0}$. Let us now state the following lemma.

Lemma 5. *One has, with $k = 0, 1$,*

$$\left|\left(\Lambda^s\left(\mathcal{G}^+\psi - \mathcal{G}^-\widetilde{\psi}^-\right), \Lambda^s u_0\right)\right| \leqq \varepsilon\mu^{1-k/4}M(t_0+3)|\psi|_{H^{s-k/2}}|u_0|_{\dot{H}_\mu^{s+1/2}}.$$

Proof of the lemma. We first consider the case $k = 1$. Let us remark[7] that

$$\left(\Lambda^s\mathcal{G}^+\psi, \Lambda^s u_0\right) = \left(\mathrm{Op}(\frac{S^+}{\mathfrak{P}^2})^*\Lambda^s\psi, \Lambda^s\mathfrak{P}^2 u_0\right) + R_1 + R_2,$$

---

[7] We use the same notation $\mathfrak{P}$ for the operator $\frac{|D|}{(1+\sqrt{\mu}|D|)^{1/2}}$ and its symbol $\frac{|\xi|}{(1+\sqrt{\mu}|\xi|)^{1/2}}$.

with

$$R_1 = \big([\Lambda^s, \mathcal{G}^+]\psi, \Lambda^s u_0\big)$$
$$R_2 = \big(\Lambda^{s-1/2}\psi, \Lambda^{1/2}(\mathcal{G}^+ - \mathrm{Op}(S^+))\Lambda^s u_0\big).$$

Remarking that $\mu^{1/4}|\mathfrak{P}\psi|_{H^{s-1}} \lesssim |\psi|_{H^{s-1/2}}$, we can use (2.24) and Theorem 4 to get

$$|R_j| \leqq \mu M \frac{\varepsilon}{\mu^{1/4}} |\psi|_{H^{s-1/2}} |u_0|_{\dot{H}^{s+1/2}_\mu} \qquad (j = 1, 2).$$

Recalling that $\widetilde{\psi}^- = \mathrm{Op}(S^+/S^-)\psi$, we write similarly

$$\big(\Lambda^s \mathcal{G}^- \widetilde{\psi}^-, \Lambda^s u_0\big) = \Big(\mathrm{Op}\Big(\frac{S^-}{\mathfrak{P}^2}\Big)^* \mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\Lambda^s \psi, \Lambda^s \mathfrak{P}^2 u_0\Big) + R_1' + R_2',$$

with

$$R_1' = \big([\Lambda^s, \mathcal{G}^-]\widetilde{\psi}^-, \Lambda^s u_0\big)$$
$$R_2' = \big(\Lambda^{s-1/2}\widetilde{\psi}^-, \Lambda^{1/2}(\mathcal{G}^- - \mathrm{Op}(S^-))\Lambda^s u_0\big),$$
$$R_3' = \Big(\Lambda^{1/2}\Big[\Lambda^s, \mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\Big]\psi, \Lambda^{-1/2}\mathrm{Op}\Big(\frac{S^-}{\mathfrak{P}^2}\Big)\Lambda^s \mathfrak{P}^2 u_0\Big).$$

Remarking (this is a consequence of Theorem 1 and Corollary 30 of [40]) that $\|\mathrm{Op}(S^+/S^-)\|_{H^{s-1/2} \to H^{s-1/2}} \leqq M$ $(0 \leqq s \leqq t_0)$, we get as above that R_1' and R_2' satisfy the same estimates as R_1 and R_2, respectively. Using Theorems 1 and 8 of [40], we also get that

$$|[\Lambda^s, \mathrm{Op}(S^+/S^-)]\psi|_{H^{1/2}} \leqq \varepsilon M |\psi|_{H^{s-1/2}}$$
$$|\mathrm{Op}(S^-/\mathfrak{P}^2)\Lambda^s \mathfrak{P}^2 u_0|_{H^{-1/2}} \leqq \mu M |\mathfrak{P}^2 u_0|_{H^{s-1/2}}.$$

Since $\mu|\mathfrak{P}^2 u_0|_{H^{s-1/2}} \lesssim \mu^{3/4}|u_0|_{\dot{H}^{s+1/2}_\mu}$, this allows us to conclude that R_3' satisfies the same estimate as R_j and R_j' $(j = 1, 2)$.

We thus have

$$\big(\Lambda^s(\mathcal{G}^+\psi - \mathcal{G}^-\widetilde{\psi}^-), \Lambda^s u_0\big) = R_1 + R_2 + R_1' + R_2' + R_3'$$
$$+ \Big(\Lambda^{1/2}\Big(\mathrm{Op}\Big(\frac{S^+}{\mathfrak{P}^2}\Big)^* - \mathrm{Op}\Big(\frac{S^-}{\mathfrak{P}^2}\Big)\mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\Big)\Lambda^s \psi, \Lambda^{s-1/2}\mathfrak{P}^2 u_0\Big). \quad (3.14)$$

We then deduce from the above estimates on the residual terms that

$$\big|\big(\Lambda^s(\mathcal{G}^+\psi - \mathcal{G}^-\widetilde{\psi}^-), \Lambda^s u_0\big)\big| \leqq |\psi|_{H^{s-1/2}}|\mathfrak{P}u_0|_{H^s}$$
$$\times \Big(\varepsilon\mu^{3/4}M + \mu^{-1/4}\Big\|\mathrm{Op}\Big(\frac{S^+}{\mathfrak{P}^2}\Big)^* - \mathrm{Op}\Big(\frac{S^-}{\mathfrak{P}^2}\Big)\mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\Big\|_{H^{-1/2} \to H^{1/2}}\Big).$$

The result follows, therefore, from the observation that

$$\Big\|\mathrm{Op}\Big(\frac{S^+}{\mathfrak{P}^2}\Big)^* - \mathrm{Op}\Big(\frac{S^-}{\mathfrak{P}^2}\Big)\mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\Big\|_{H^{-1/2} \to H^{1/2}} \leqq \varepsilon\mu M,$$

which is a consequence of the composition estimate[8] of Theorem 8 of [39] and of the estimate[9] $\|\mathrm{Op}(\frac{S^+}{\mathfrak{P}^2})^* - \mathrm{Op}(\frac{S^+}{\mathfrak{P}^2})\|_{H^{-1/2} \to H^{1/2}} \leqq \varepsilon \mu M$.

We now briefly indicate the modifications to be performed in the case $k = 0$. For the estimate of R_1, we just replace the control $\mu^{1/4}|\mathfrak{P}\psi|_{H^{s-1}} \lesssim |\psi|_{H^{s-1/2}}$ by $|\mathfrak{P}\psi|_{H^{s-1}} \leqq |\psi|_{H^s}$. For R_2, we write, rather,

$$R_2 = \big(\Lambda^s \psi, (\mathcal{G}^+ - \mathrm{Op}(S^+))\Lambda^s u_0\big),$$

and use the Cauchy–Schwarz inequality and the second estimate of Theorem 4. The same modifications must be done for R_1' and R_2', respectively. For R_3', we write

$$R_3' = \Big(\Lambda\Big[\Lambda^s, \mathrm{Op}\Big(\frac{S^+}{S^-}\Big)\Big]\psi, \Lambda^{-1}\mathrm{Op}\Big(\frac{S^-}{\mathfrak{P}^2}\Big)\Lambda^s \mathfrak{P}^2 u_0\Big),$$

and use the estimates

$$|[\Lambda^s, \mathrm{Op}(S^+/S^-)]\psi|_{H^1} \leqq \varepsilon M |\psi|_{H^s}$$
$$|\mathrm{Op}(S^-/\mathfrak{P}^2)\Lambda^s \mathfrak{P}^2 u_0|_{H^{-1}} \leqq \mu M |\mathfrak{P}^2 u_0|_{H^{s-1}} \leqq \mu M |u_0|_{\dot{H}_\mu^{s+1/2}}.$$

The control of the last term in (3.14) is modified along the same lines as R_3' and the result follows. □

Thanks to the lemma, one can proceed as in the proof of Proposition 1 to get (in the case $k = 1$, the modifications for the case $k = 0$ are straightforward),

$$\|\Lambda^s \nabla^{\mu^-} u\|_2 \leqq \varepsilon M (t_0 + 3)\big(\mu^{1/4}|\psi|_{H^{s-1/2}} + \|\Lambda^{s-1} \nabla^{\mu^-} \phi^-\|_2\big),$$

and thus, after a continuous induction on s,

$$\|\Lambda^s \nabla^{\mu^-} u\|_2 \leqq \varepsilon \mu^{1/4} M (t_0 + 3)|\psi|_{H^{s-1/2}}.$$

The result follows, therefore, from (2.19) and the fact that $u_0 = \big((\mathcal{G}^-)^{-1}\mathcal{G}^+ - \mathrm{Op}(S^+/S^-)\big)\psi$. □

3.4. Symbolic Analysis of $(\mathcal{G}^-)^{-1}\mathcal{G}_\mu[\varepsilon \zeta]$

The following corollary shows that the asymptotic description of $(\mathcal{G}^-)^{-1}\mathcal{G}_\mu[\varepsilon \zeta]$ that one naturally expects from the definition of $\mathcal{G}_\mu[\varepsilon \zeta]$ in Corollary 2 and Theorem 4 is correct. In order to state the corollary, it is convenient to introduce the zero-th order symbol

$$S_J(X, \xi) = \underline{\rho}^+ - \underline{\rho}^- \frac{H^-}{H^+} \frac{S^+(X, \xi)}{S^-(X, \xi)}. \qquad (3.15)$$

---

[8] Theorem 8 of [40] deals with commutator rather than composition estimates. However, the commutator estimates of Theorem 8 follow from a composition estimate, exactly in the same way as Theorem 7(iii) follows from Theorem 7(i).

[9] This estimate on the adjoint is not stated in [39]. However it can classically be derived with the same techniques as the commutator estimates of Theorem 8 of that reference. See also Proposition 1.8 of [27] and Chapter 13, §9 of [58].

Corollary 4. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+3}(\mathbb{R}^d)$ be such that* (2.1) *is satisfied. Then for all $0 \leqq s \leqq t_0$ and $\psi \in H^{s-1/2}(\mathbb{R}^d)$, one has, with $k = 0, 1$,*

$$\left|\left[(\mathcal{G}^-)^{-1}\mathcal{G}_\mu[\varepsilon\zeta] - \frac{1}{\underline{H}^+}\mathrm{Op}\Big(\frac{S^+}{S^-S_J}\Big)\right]\psi\right|_{\dot{H}_\mu^{s+1/2}} \leqq \varepsilon\mu^{-k/4}M(t_0+3)|\psi|_{H^{s-k/2}}.$$

Proof. Thanks to Remark 14, we can write

$$(\mathcal{G}^-)^{-1}\mathcal{G}_\mu[\varepsilon\zeta] = \frac{1}{\underline{H}^+}(\mathcal{G}^-)^{-1}\mathcal{G}^+J[\zeta]^{-1}.$$

It is a direct consequence of Corollary 3 that $\mathrm{Op}(S_J)$ provides a good approximation of $J[\zeta]$. The following lemma shows a corresponding property for its inverse. Throughout this proof, we consider only the case $k = 1$; the modifications that must be performed in the case $k = 0$ are absolutely similar to those made in the proof of Corollary 3.

Lemma 6. *Under the assumptions of the corollary, one has, for all $0 \leqq s \leqq t_0$ and $k = 0, 1$,*

$$\left|\Big(J[\zeta]^{-1} - \mathrm{Op}\Big(\frac{1}{S_J}\Big)\Big)\psi\right|_{\dot{H}_\mu^{s+1/2}} \leqq \varepsilon\mu^{-k/4}M(t_0+3)|\psi|_{H^{s-k/2}}.$$

Proof of the lemma. From Lemma 2, we know that

$$\left|(J[\zeta]^{-1} - \mathrm{Op}(\frac{1}{S_J})\psi\right|_{\dot{H}_\mu^{s+1/2}} \leqq M\left|(1 - J[\zeta]\mathrm{Op}(\frac{1}{S_J}))\psi\right|_{\dot{H}_\mu^{s+1/2}}$$

$$\leqq M\Big(\left|(1 - \mathrm{Op}(S_J)\mathrm{Op}(\frac{1}{S_J}))\psi\right|_{\dot{H}_\mu^{s+1/2}} + \left|(J[\zeta] - \mathrm{Op}(S_J))\mathrm{Op}(\frac{1}{S_J})\psi\right|_{\dot{H}_\mu^{s+1/2}}\Big)$$

$$\leqq M(t_0+3)\Big(\frac{1}{\mu^{1/4}}\left|(1 - \mathrm{Op}(S_J)\mathrm{Op}(\frac{1}{S_J}))\psi\right|_{H^{s+1/2}} + \frac{\varepsilon}{\mu^{1/4}}\left|\mathrm{Op}(\frac{1}{S_J})\psi\right|_{H^{s-1/2}}\Big),$$

where we used Corollary 3 to derive the last inequality. Since the pseudodifferential estimates of [39] show that $\|1 - \mathrm{Op}(S_J)\mathrm{Op}(1/S_J)\|_{H^{s-1/2}\to H^{s+1/2}} \leqq \varepsilon M$ and $\|\mathrm{Op}(1/S_J)\|_{H^{s-1/2}\to H^{s-1/2}} \leqq M$, the result follows easily. □

To conclude the proof of the corollary, let us remark that $\left|[(\mathcal{G}^-)^{-1}\mathcal{G}_\mu[\varepsilon\zeta] - \frac{1}{\underline{H}^+}\mathrm{Op}\big(\frac{S^+}{S^-S_J}\big)]\psi\right|_{\dot{H}_\mu^{s+1/2}}$ is bounded from above by

$$\frac{1}{\underline{H}^+}\left|[(\mathcal{G}^-)^{-1}\mathcal{G}^+\mathrm{Op}(\frac{1}{S_J}) - \mathrm{Op}(\frac{S^+}{S^-S_J})]\psi\right|_{\dot{H}_\mu^{s+1/2}}$$

$$+\frac{1}{\underline{H}^+}\left|(\mathcal{G}^-)^{-1}\mathcal{G}^+\big(J[\zeta]^{-1} - \mathrm{Op}(\frac{1}{S_J})\big)\right|_{\dot{H}_\mu^{s+1/2}}$$

$$\leqq \frac{1}{\underline{H}^+}\left|[(\mathcal{G}^-)^{-1}\mathcal{G}^+\mathrm{Op}(\frac{1}{S_J}) - \mathrm{Op}(\frac{S^+}{S^-S_J})]\psi\right|_{\dot{H}_\mu^{s+1/2}}$$

$$+\frac{\varepsilon}{\mu^{1/4}}M(t_0+3)|\psi|_{H^{s-1/2}},$$

where the second inequality comes from Proposition 1 and Lemma 6. It is then an easy consequence of Corollary 3 and of the pseudodifferential estimates used

many times in this section that the first component in the right-hand side of the last equality is also bounded from above by $\frac{\varepsilon}{\mu^{1/4}} M(t_0 + 3)|\psi|_{H^{s-1/2}}$, which concludes the proof. $\square$

3.5. Symbolic Analysis of $(\underline{\rho}^+ \frac{1}{H^-} G^- - \underline{\rho}^- \frac{1}{H^+} G^+)^{-1} \partial_j$

For the sake of clarity, we write

$$\widetilde{G} = -\left(\underline{\rho}^+ \frac{1}{H^-} G^- - \underline{\rho}^- \frac{1}{H^+} G^+\right) \quad \text{and} \quad \widetilde{S} = -\left(\underline{\rho}^+ \frac{1}{H^-} S^- - \underline{\rho}^- \frac{1}{H^+} S^+\right). \quad (3.16)$$

The operator $\mathfrak{P}^2 \circ \widetilde{G}^{-1} \partial_j$ ($1 \leq j \leq d$) is well defined thanks to Remark 8. It will be useful to describe it using a pseudodifferential operator, as in the following corollary.

Corollary 5. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+3}(\mathbb{R}^d)$ be such that (2.1) is satisfied. Then for all $0 \leq s \leq t_0, k = 0, 1$, and $f \in H^{s+1/2}(\mathbb{R}^d)$, one has*

$$|\mathfrak{P}^2 \circ \widetilde{G}^{-1} \partial_j f - \mathrm{Op}\left(\frac{\mathfrak{P}^2}{\widetilde{S}}\right) \partial_j f|_{H^{s+k/2}} \leq \frac{\varepsilon}{\mu^{1+k/4}} M(t_0 + 3)|(1 + \sqrt{\mu}|D|)^{1/2} f|_{H^s}.$$

Proof. We can write

$$\mathfrak{P}^2 \circ \widetilde{G}^{-1} \partial_j f = \mathrm{Op}\left(\frac{\mathfrak{P}^2}{\widetilde{S}}\right) \partial_j f + R_1 + R_2,$$

where, using the notation $\widetilde{f} = \widetilde{G}^{-1} \partial_j f$,

$$R_1 = \left[1 - \mathrm{Op}\left(\frac{\mathfrak{P}^2}{\widetilde{S}}\right) \circ \mathrm{Op}\left(\frac{\widetilde{S}}{\mathfrak{P}^2}\right)\right] \mathfrak{P}^2 \widetilde{f}, \qquad R_2 = \mathrm{Op}\left(\frac{\mathfrak{P}^2}{\widetilde{S}}\right)[\mathrm{Op}(\widetilde{S}) - \widetilde{G}]\widetilde{f}.$$

From the composition estimates of [40], we get, with $k = 0, 1$,

$$|R_1|_{H^{s+k/2}} \leq \varepsilon M(t_0 + 3)|\mathfrak{P}^2 \widetilde{f}|_{H^{s-1+k/2}} \leq \varepsilon \mu^{-k/4} M(t_0 + 3)|\mathfrak{P}\widetilde{f}|_{H^s},$$

the last inequality following from the observation that $|\mathfrak{P}f|_{H^{1/2}} \lesssim \mu^{-1/4}|f|_{H^1}$ and $|\mathfrak{P}f|_2 \lesssim |f|_{H^1}$. Since Remark 8 implies that

$$|\mathfrak{P}\widetilde{f}|_{H^s} \leq \mu^{-1} M|(1 + \sqrt{\mu}|D|)^{1/2} f|_{H^s},$$

we deduce that R_1 satisfies the bound given in the statement of the corollary.

For R_2, we get

$$|R_2|_{H^{s+k/2}} \leq \frac{1}{\mu} M(t_0 + 3)|[\mathrm{Op}(\widetilde{S}) - \widetilde{G}]\widetilde{f}|_{H^{s+k/2}}$$

$$\leq \varepsilon \mu^{-k/4} M(t_0 + 3)|\mathfrak{P}\widetilde{f}|_{H^s},$$

the second inequality being a consequence of the first ($k = 1$) and second ($k = 0$) points of Theorem 4. One then shows, as for R_1, that R_2 satisfies the desired estimate. $\square$

4. Quasilinearization of the Equations

The goal of this section is to derive a quasilinear system from the two-fluid equations (1.19). Throughout this section, the operators $G_\mu[\varepsilon\zeta]$ and $G_{\mu^\pm}[\varepsilon^\pm\zeta, 1]$ are denoted G and $G^\pm$, respectively, when no confusion is possible.

For all $j \in \mathbb{N}^*$, we also denote by $d^j G(\mathbf{h})\psi$ and $d^j G^\pm(\mathbf{h})\psi$ ($\mathbf{h} = (h_1, \ldots, h_j)$) the jth order derivative of the mappings $\zeta \mapsto G_\mu[\varepsilon\zeta]\psi$ and $\zeta \mapsto G_{\mu^\pm}^\pm[\varepsilon^\pm\zeta, 1]\psi$, respectively, in the direction $\mathbf{h}$.

Let us briefly justify, here, the introduction in the following sections of several new operators by considering a formal linear analysis. More precisely, the linearization of the two-fluid equations (1.8)–(1.11) around the constant flow

$$\mathbf{U}^\pm = \begin{pmatrix} c^\pm \\ 0 \end{pmatrix}, \qquad \zeta = 0$$

yields the following system

$$\begin{cases} \partial_t \zeta + c^\pm \cdot \nabla\zeta - \underline{w}^\pm = 0, \\ \partial_t \psi + g'\zeta + [\![\underline{\rho}^\pm c^\pm \cdot \nabla\psi^\pm]\!] = -\dfrac{\sigma}{\rho^+ + \rho^-} \mathfrak{k}(\zeta), \end{cases}$$

with $\psi = \rho^+ \psi^+ - \rho^- \psi^-$. The quantities $\psi^\pm$ (and therefore $\underline{w}^\pm$ since $\underline{w}^\pm = \pm|D|\tanh(\overline{H}^\pm|D|)\psi^\pm$) can be expressed in terms of ζ ad ψ by solving a linearized version of the transmission problem studied in Section 2.3. This leads to

$$-\underline{w}^\pm = \frac{\underline{\rho}^- \tanh(H^+|D|)}{\underline{\rho}^- \tanh(H^+|D|) + \underline{\rho}^+ \tanh(H^-|D|)} [\![c^\pm]\!] \cdot \nabla\zeta + G[0]\psi,$$

where $G[0]$ is the dimensional form of the operator $G_\mu[0]$ given in Remark 15. For the term $[\![\underline{\rho}^\pm c^\pm \cdot \nabla\psi^\pm]\!]$ that appears in the second equation, we write

$$[\![\underline{\rho}^\pm c^\pm \cdot \nabla\psi^\pm]\!] = \langle c^\pm\rangle \cdot \nabla\psi + [\![c^\pm]\!] \cdot \nabla\langle\underline{\rho}^\pm\psi^\pm\rangle.$$

The dependence on ζ is specific to the two-fluid system (that is, it disappears if $\rho^- = 0$), and is responsible for the Kelvin–Helmholtz instabilities through the operator $e(D)$ in the resulting set of equations in ζ, ψ,

$$\begin{cases} \partial_t \zeta + T(D)\zeta - G[0]\psi = 0, \\ \partial_t \psi + T(D)\psi + (g' - \cdot e(D) - \dfrac{\sigma}{\rho^+ - \rho^-}\Delta)\zeta = 0, \end{cases} \tag{4.1}$$

with

$$T(D) = \frac{c^+\underline{\rho}^+ \tanh(H^+|D|) + c^-\underline{\rho}^- \tanh(H^+|D|)}{\underline{\rho}^- \tanh(H^+|D|) + \underline{\rho}^+ \tanh(H^-|D|)} \cdot \nabla,$$

$$e(D) = \frac{\rho^+\rho^-}{\underline{\rho}^- \tanh(H^+|D|) + \underline{\rho}^+ \tanh(H^-|D|)} \frac{([\![c^\pm]\!] \cdot D)^2}{|D|}.$$

The diagonal terms in (4.1) are governed by the operator $T(D)$, which can be seen as a nonlocal transport term; consequently, they play no role in the stability analysis. Since $\mathcal{G}[0]$ is a positive operator, the linear stability of (4.1) depends on the sign of the stability operator

$$\text{Ins}(D) = g' - e(D) - \frac{\sigma}{\rho^+ - \rho^-}\Delta,$$

in the sense that stability for all frequencies is obtained if $\text{Ins}(\xi) > 0$ for all $\xi \in \mathbb{R}^d$. Conversely, Kelvin–Helmholtz instability is the mechanism that amplifies the frequencies for which $\text{Ins}(\xi) < 0$.

   In the following sections, we quasilinearize the full nonlinear equations, and a structure similar to (4.1) appears. In particular, the operators $\mathcal{T}[U]$, $\mathcal{E}$ and $\textit{Ins}$ that are introduced in Section 4.1 below are the natural generalization (in dimensionless form) of the operators $T(D)$, $e(D)$ and Ins introduced in the linear analysis above. In order to quasilinearize the equations, we differentiate them several times; in Section 4.2, we study the action of differential operators on $\mathcal{G}$. The quasilinear system is then derived in Section 4.3; as for the water-wave equations, the "good unknown" plays a crucial role.

4.1. Definition and Basic Properties of Some New Operators

   We define, first, a linear operator $\mathcal{T}[U]$ of order one,

$$\mathcal{T}[U]f = \nabla \cdot (f\underline{V}^+) + \underline{\rho}^- \underline{H}^- \mathcal{G} \circ (\mathcal{G}^-)^{-1}\big(\nabla \cdot (f[\![\underline{V}^\pm]\!])\big), \qquad (4.2)$$

where $\underline{V}^\pm$ and $\underline{w}^\pm$ are defined using the mappings $w^\pm[\varepsilon\zeta]$ and $\mathcal{V}_{/\!/}^\pm[\varepsilon\zeta]$ of Corollary 1 as

$$\underline{V}^\pm = \mathcal{V}_{/\!/}^\pm[\varepsilon\zeta]\psi - \varepsilon\underline{w}^\pm\nabla\zeta, \qquad \underline{w}^\pm = w^\pm[\varepsilon\zeta]\psi. \qquad (4.3)$$

In the water-wave case ($\rho^- = 0$), this operator simplifies considerably into $\mathcal{T}[U]f = \nabla \cdot (f\underline{V}^+)$. The symbolic analysis of Section 3 allows us to prove that, in the general case, it behaves roughly the same way in the sense that it is a first order operator with a skew symmetric principal symbol. Note that it is necessary here to use a symbolic analysis "with tail" as in the previous section in order to have a nonsingular dependence on the parameters in the estimates of the proposition.

Proposition 4. *Let $t_0 > d/2$ and $U = (\zeta, \psi)$, with $\zeta \in H^{t_0+3}(\mathbb{R}^d)$ satisfying (2.1) and $\psi \in \dot{H}^{t_0+3}(\mathbb{R}^d)$. Then*

 1. *For all $0 \leqq s \leqq t_0$ and $f \in H^{s+1/2}(\mathbb{R}^d)$, one has*

$$|\mathcal{T}[U]f|_{H^{s-1/2}} \leqq M|f|_{H^{s+1/2}}|\nabla\psi|_{H^{t_0+1/2}}.$$

 2. *For all $\mathfrak{a} = 1 + \mathfrak{b}$, with $\mathfrak{b} \in H^{t_0+1}(\mathbb{R}^d)$, and all $f \in L^2(\mathbb{R}^d)$,*

$$\big(\mathfrak{a}\mathcal{T}[U]f, f\big) \leqq M(t_0+3)(1 + |\mathfrak{b}|_{H^{t_0+1}})|\nabla\psi|_{H^{t_0+1}}|f|_2^2.$$

3. *Let K be a $d \times d$ symmetric matrix with entries in $H^{t_0+1}(\mathbb{R}^d)$; then for all $f \in H^1(\mathbb{R}^d)$,*

$$\left(\nabla \cdot K \nabla \mathcal{T}[U]f, f\right) \leqq M(t_0 + 3)|K|_{H^{t_0+1}}|\nabla\psi|_{H^{t_0+1}}|f|_{H^1}^2.$$

4. *For all $f \in \dot{H}^{1/2}(\mathbb{R}^d)$, $g \in H^{1/2}(\mathbb{R}^d)$, one has*

$$\left(\mathcal{T}[U]^* f, g\right) \leqq M|\nabla\psi|_{H^{t_0+2}}|\mathfrak{P}f|_2\left(|g|_2 + \mu^{1/4}|g|_{H^{1/2}}\right).$$

Proof. In the proofs below, we reduce $\mathcal{T}[U]$ to its second component in (4.2), since the estimates are much easier to prove for the first component and can therefore be omitted.

For the first estimate, we write $\mathcal{G} \circ (\mathcal{G}^-)^{-1}\nabla \cdot (f[\![\underline{V}^\pm]\!]) = A + B$ with

$$A = \mathcal{G} \circ (\mathcal{G}^-)^{-1}\nabla \cdot \left(\frac{\mu^{1/4}|D|^{1/2}}{1 + \mu^{1/4}|D|^{1/2}}(f[\![\underline{V}^\pm]\!])\right)$$

$$B = \mathcal{G} \circ (\mathcal{G}^-)^{-1}\nabla \cdot \left(\frac{1}{1 + \mu^{1/4}|D|^{1/2}}(f[\![\underline{V}^\pm]\!])\right).$$

Recalling that $\mathcal{G} = \frac{1}{\underline{H}}^+ \mathcal{G}^+ \circ J[\zeta]^{-1}$, we use (2.20), Lemma 2 and Remark 8 to get $|A|_{H^{s-1/2}} \leqq M|f[\![\underline{V}^\pm]\!]|_{H^{s+1/2}}$. Proceeding in the same way, but using (2.21) instead of (2.20), one can check that B can be controlled similarly. Since with $0 \leqq s \leqq t_0$ one has $|f[\![\underline{V}^\pm]\!]|_{H^{s+1/2}} \leqq M|f|_{H^{s+1/2}}|\nabla\psi|_{H^{t_0+1/2}}$ (from the definition of $\underline{V}^\pm$ and Corollary 1), the result follows.

For the second point of the proposition, we write

$$\left(\mathfrak{a}\mathcal{G} \circ (\mathcal{G}^-)^{-1}\nabla \cdot (f[\![\underline{V}^\pm]\!]), f\right) = \left(\nabla \cdot (f[\![\underline{V}^\pm]\!]), \mathrm{Op}\left(\frac{S^+}{S^- S_J}\right)(\mathfrak{a}f)\right)$$

$$+ \left(f[\![\underline{V}^\pm]\!], \nabla\left[(\mathcal{G}^-)^{-1} \circ \mathcal{G} - \mathrm{Op}\left(\frac{S^+}{S^- S_J}\right)\right](\mathfrak{a}f)\right).$$

Since the operator $f \mapsto \mathfrak{a}\mathrm{Op}(\frac{S^+}{S^- S_J})^*\nabla \cdot (f[\![\underline{V}^\pm]\!])$ is skew symmetric at leading order, the result for the first term of the right-hand side follows from the pseudo-differential estimates of [40]. In order to control the second term, we decompose it into $C_1 + C_2$ with

$$C_1 = \left(f[\![\underline{V}^\pm]\!], \frac{\mu^{1/4}|D|^{1/2}\nabla}{1 + \mu^{1/4}|D|^{1/2}}\left((\mathcal{G}^-)^{-1} \circ \mathcal{G} - \mathrm{Op}\left(\frac{S^+}{S^- S_J}\right)\right)(\mathfrak{a}f)\right),$$

$$C_2 = \left(f[\![\underline{V}^\pm]\!], \frac{\nabla}{1 + \mu^{1/4}|D|^{1/2}}\left((\mathcal{G}^-)^{-1} \circ \mathcal{G} - \mathrm{Op}\left(\frac{S^+}{S^- S_J}\right)\right)(\mathfrak{a}f)\right).$$

From the Cauchy–Schwarz inequality and Corollary 4 (with $k = 1$), we get $|C_1| \leqq M(t_0+3)|f[\![\underline{V}^\pm]\!]|_2|\mathfrak{a}f|_2$. For C_2, we proceed similarly but with $k = 0$ in Corollary 4 to get that C_2 can be controlled as C_1, and the result follows.

The proof of the third point being very similar to the second one, we omit it. For the last point, the term to control is

$$A = \left([\![\underline{V}^\pm]\!] \cdot \nabla(\mathcal{G}^-)^{-1}\mathcal{G}f, g\right)$$

$$= -\frac{1}{\underline{H}^+}\left(\mathfrak{P}(\mathcal{G}^-)^{-1}\mathcal{G}^+ J[\zeta]^{-1}f, (1 + \sqrt{\mu}|D|)^{1/2}\frac{\nabla}{|D|}(g[\![\underline{V}^\pm]\!])\right).$$

It is therefore a consequence of the Cauchy–Schwarz inequality, Proposition 1 and Lemma 2 that $|A| \leqq M |\mathfrak{P} f|_2| (1 + \sqrt{\mu}|D|)^{1/2} (g|\underline{V}^{\pm}|)|_2$, from which one deduces the result by a standard commutator estimate and Corollary 1. □

The linearization formula for $\partial^{\alpha}(G_{\mu}[\varepsilon \zeta]\psi)$ (see Proposition 6 below) is quite simple if given in terms of the "good unknowns" $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$, where $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$ are defined as

$$\zeta_{(\alpha)} = \partial^{\alpha}\zeta, \tag{4.4}$$

$$\psi_{(\alpha)} = \partial^{\alpha}\psi - \varepsilon \underline{w} \partial^{\alpha}\zeta, \quad \text{with} \quad \underline{w} = \underline{\rho}^{+} \underline{w}^{+} - \underline{\rho}^{-} \underline{w}^{-}, \tag{4.5}$$

or equivalently, if $\psi^{\pm} = \phi^{\pm}_{|z=0}$ ($\phi^{\pm}$ solving (2.37)),

$$\psi_{(\alpha)} = \underline{\rho}^{+} \psi^{+}_{(\alpha)} - \underline{\rho}^{-} \psi^{-}_{(\alpha)}, \quad \text{with} \quad \psi^{\pm}_{(\alpha)} = \partial^{\alpha}\psi^{\pm} - \varepsilon \underline{w}^{\pm} \partial^{\alpha}\zeta.$$

The principal part of the linearization formula for $\partial^{\alpha}(G_{\mu}[\varepsilon \zeta]\psi)$ is given in terms of $G\psi_{(\alpha)}$ and $\mathcal{T}[U]\partial^{\alpha}\zeta$, but to handle the surface tension, it is also necessary to take into account the subprincipal part (with respect to ψ). In order to describe this subprincipal part, it is convenient to introduce the following notation,

$$\zeta_{\langle \check{\alpha} \rangle} = (\zeta_{(\check{\alpha}^1)}, \ldots, \zeta_{(\check{\alpha}^{d+1})}), \qquad \psi_{\langle \check{\alpha} \rangle} = (\psi_{(\check{\alpha}^1)}, \ldots, \psi_{(\check{\alpha}^{d+1})}), \tag{4.6}$$

where $\check{\alpha}^j \in \mathbb{N}^{d+1}$ is such that $\check{\alpha}^j + \mathbf{e}_j = \alpha$.

We can now define the operator $G_{(\alpha)}\psi_{\langle \check{\alpha} \rangle}$ that arises in the description of the subprincipal part of $\partial^{\alpha}(G_{\mu}[\varepsilon \zeta]\psi)$,

$$G_{(\alpha)}\psi_{\langle \check{\alpha} \rangle} = \sum_{j=1}^{d+1} \alpha_j d\, G(\partial_j \zeta) \psi_{(\check{\alpha}^j)}. \tag{4.7}$$

Next, the linearization of the surface tension term shows that

$$\frac{1}{\varepsilon \sqrt{\mu}} \partial^{\alpha} k(\varepsilon \sqrt{\mu}\zeta) = -\nabla \cdot \mathcal{K}[\varepsilon \sqrt{\mu}\nabla \zeta]\nabla \partial^{\alpha}\zeta + \mathcal{K}_{(\alpha)}[\varepsilon \sqrt{\mu}\nabla \zeta]\zeta_{\langle \check{\alpha} \rangle} + \cdots$$

where the dots stand for derivatives of ζ of order lower or equal to $|\alpha|$ and $\zeta_{(\alpha)}$ is as in (4.6). The positive definite $d \times d$ matrix $\mathcal{K}[\nabla \zeta]$ is defined as

$$\mathcal{K}[\nabla \zeta] = \frac{(1 + |\nabla \zeta|^2)\mathrm{Id} - \nabla \zeta \otimes \nabla \zeta}{(1 + |\nabla \zeta|^2)^{3/2}}, \tag{4.8}$$

while the second order operator $\mathcal{K}_{(\alpha)}[\nabla \zeta]$ is given for all $F = (f_1, \ldots, f_{d+1})^T$ by

$$\mathcal{K}_{(\alpha)}[\nabla \zeta]F = -\nabla \cdot \left[\sum_{j=1}^{d+1} \left(d\, \mathcal{K}(\nabla \partial_j \zeta) \nabla f_j + d\, \mathcal{K}(\nabla f_j) \nabla \partial_j \zeta \right) \right]. \tag{4.9}$$

Finally, we introduce the first order operator $\mathcal{E}[\zeta] : \dot{H}^{1/2}(\mathbb{R}^d)^d \to H^{-1/2}(\mathbb{R}^d)^d$, which plays an important role for Kelvin–Helmholtz instabilities,

$$\mathcal{E}[\zeta] = \nabla \circ (\underline{\rho}^{+} \frac{1}{\underline{H}^{-}} G^{-} - \underline{\rho}^{-} \frac{1}{\underline{H}^{+}} G^{+})^{-1} \circ \nabla^{T}. \tag{4.10}$$

The following proposition gathers some of its main properties.

Proposition 5. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ be such that (2.1) is satisfied.*

1. *There exists a constant $c \leqq M$ such that for all $V \in \dot{H}^{1/2}(\mathbb{R}^d)$,*

$$0 \leqq \mu(\mathcal{E}[\zeta]V, V) \leqq c \,|(1 + \sqrt{\mu}|D|)^{1/2}V|_2^2;$$

   we denote by $\mathfrak{e}(\zeta)$ the smallest such constant.

2. *Moreover, if ζ is time dependent and satisfies (2.1) uniformly in time, then, for all $F \in H^{1/2}(\mathbb{R}^d)^d$*

$$\left|([\partial_t, \mathcal{E}[\zeta]]F, F)\right| \leqq \frac{\varepsilon}{\mu} M |\partial_t \zeta|_\infty |(1 + \sqrt{\mu}|D|)^{1/2}F|_2^2.$$

Proof. With $\widetilde{G}$ as in (3.16), we can write $(\mathcal{E}[\zeta]V, V) = (\widetilde{G}w, w)$, with $w = \widetilde{G}^{-1}\nabla \cdot V$ and the positiveness of $\mathcal{E}[\zeta]$ follows from the positiveness of $\widetilde{G}$. Since $\widetilde{G} = -\frac{1}{H^-}G^- \circ J[\zeta]$, the upper bound follows from Lemma 2 and Remark 8.

For the second point of the proof, we remark that $[\partial_t, \widetilde{G}^{-1}] = -\widetilde{G}^{-1}[\partial_t, \widetilde{G}]\widetilde{G}^{-1}$, so that

$$\left([\partial_t, \mathcal{E}[\zeta]]F, F\right) = -\left([\partial_t, \widetilde{G}]\widetilde{G}^{-1}\nabla \cdot F, \widetilde{G}^{-1}\nabla \cdot F\right).$$

Using Proposition 3.6 of [4], we deduce that

$$\left|([\partial_t, \mathcal{E}[\zeta]]F, F)\right| \leqq \mu M |\mathfrak{P}\widetilde{G}^{-1}\nabla \cdot F|_2,$$

and the result follows from Remark 8. ☐

4.2. Linearization Formulas

We now establish linearization formulas for the operator G. These formulas play an important role in the quasilinearization of the equations presented in Section 4.3. Proposition 6 below gives a linearization formula at leading order; in order to measure the size of the neglected terms, it is necessary to introduce at this point the energy $\mathcal{E}_\sigma^N(U)$ defined for all $N \in \mathbb{N}$, $\sigma \geqq 0$, and $U = (\zeta, \psi)$ as

$$\mathcal{E}_\sigma^N(U) = |\nabla \psi|_{H^{t_0+2}}^2 + \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| \leqq N} |\partial^\alpha \zeta|_{H_\sigma^1}^2 + |\psi_{(\alpha)}|_{\dot{H}_\mu^{1/2}}^2. \qquad (4.11)$$

It is convenient to introduce a specific notation for the terms involving only ζ in this energy, namely,

$$|\zeta|_{H_\sigma^{1,N}} = \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| \leqq N} |\partial^\alpha \zeta|_{H_\sigma^1} \qquad (4.12)$$

(the difference between $H_\sigma^{1,N}$ and H_σ^{1+N} is that the former allows for time derivatives of ζ). Another quantity that will be related to the energy through the stability criterion derived in Section 5 is $|\zeta|_{<N+1/2>}$, defined as

$$|\zeta|_{<N+1/2>} = \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha|=N} \left||D|^{1/2}\partial^\alpha \zeta\right|_2. \qquad (4.13)$$

For all $T > 0$, we also denote by $E^N_{\sigma,T}$ the functional space

$$E^N_{\sigma,T} = \{U \in C([0, T]; H^N \times \dot{H}^{t_0+3}(\mathbb{R}^d)), \quad \sup_{0 \leqq t \leqq T} \mathcal{E}^N_\sigma(U(t)) < \infty\}. \quad (4.14)$$

We finally denote by $\mathfrak{m}^N(U)$ any constant of the form

$$\mathfrak{m}^N(U) = C\Big(M, \mathcal{E}^N_\sigma(U), \frac{1}{\mathrm{Bo}}\Big), \quad (4.15)$$

with M as in (2.2). We can now give an important linearization formula for $\mathcal{G}$ (note that for $d = 1, 2$, the condition (4.16) is satisfied with $t_0 = 3/2$ and $N \geqq 5$).

Proposition 6. *Let $T > 0$, $t_0 > d/2$ and $N \in \mathbb{N}$ be such that*

$$[(N + 1)/2] \geqq 1 \vee t_0 + 1/2 \quad \text{and} \quad N \geqq t_0 + 7/2. \quad (4.16)$$

Furthermore, let $U = (\zeta, \psi) \in E^N_{\sigma,T}$ be such that (2.1) is uniformly satisfied on $[0, T]$. Then for all $\alpha \in \mathbb{N}^{d+1}$ with $1 \leqq |\alpha| \leqq N$, one has

$$\frac{1}{\mu}\partial^\alpha(\mathcal{G}\psi) = \frac{1}{\mu}\mathcal{G}\psi_{(\alpha)} + \varepsilon R_\alpha, \ (|\alpha| \leqq N - 1)$$

$$\frac{1}{\mu}\partial^\alpha(\mathcal{G}\psi) = \frac{1}{\mu}\mathcal{G}\psi_{(\alpha)} - \varepsilon\mathcal{T}[U]\partial^\alpha\zeta + \frac{1}{\mu}\mathcal{G}_{(\alpha)}\psi_{\langle\check{\alpha}\rangle} + \varepsilon R_\alpha, \ (|\alpha| = N),$$

where the linear operators $\mathcal{T}[U]$ and $\mathcal{G}_{(\alpha)}$ are defined in (4.2) and (4.7), while R_α satisfies the estimate

$$\forall 0 \leqq t \leqq T, \quad |R_\alpha(t)|_{H^1_\sigma} \leqq \mathfrak{m}^N(U(t)).$$

Proof. Let us denote by $\phi^\pm$ the solution of the transmission problem (2.37) provided by Proposition 2, and let us write $\psi^\pm = \phi^\pm_{|z=0}$ (or equivalently define $\psi^\pm$ as in (2.39)). For the sake of notational simplicity, we also omit to write the dependence on the time variable when this does not raise any confusion. Consistently with the notations of Section 2.2.4, we also denote by $d\mathcal{G}(h)\psi$ the derivative of the mapping $\zeta \mapsto \mathcal{G}_\mu[\varepsilon\zeta]\psi$ in the direction h.

We focus on the most difficult case, namely $|\alpha| = N$ (when $|\alpha| \leqq N - 1$, it is a consequence of the first estimate of Proposition 4 that $\varepsilon\mathcal{T}[U]\partial^\alpha\zeta$ can be put in the residual term). The proof relies heavily on the following lemma (note that the last estimate of the third point is not used in the present proof but will be necessary to establish Proposition 7).

Lemma 7. *Let $t_0 > d/2$ and $\zeta \in H^{t_0+2}(\mathbb{R}^d)$ be such that (2.1) is satisfied.*

1. *For all $\psi \in \dot{H}^{1/2}(\mathbb{R}^d)$ and $h \in H^{t_0+2}(\mathbb{R}^d)$, one has*

$$d\mathcal{G}(h)\psi = -\varepsilon\mathcal{G}(h[\![\underline{\rho}^\pm\underline{w}^\pm]\!]) - \varepsilon\mu\mathcal{T}[U]h.$$

2. *One can replace $\mathcal{G}^\pm$ by $\mathcal{G}$ or $w^\pm[\varepsilon\zeta]$ in (2.31)–(2.34).*

3. *Let $r \geqq 0$ and $\delta \in \mathbb{N}^{d+1}$ be such that $r + |\delta| \leqq N - 1$, with N as in the Proposition. Then one has*

$$|\mathfrak{P}\partial^\delta \psi|_{H_\sigma^{r+1}} \leqq \mathfrak{m}^N(U) \quad \text{and} \quad |\mathfrak{P}\partial^\delta \psi|_{H^{r+1}} \leqq \mathfrak{m}^N(U)\big(1 + \varepsilon\mu^{1/4}|\zeta|_{<N+1/2>}\big).$$

Proof of the lemma. 1. Recalling that $G = \frac{1}{\underline{H}^+} G^+ \circ J[\zeta]^{-1}$, where $J[\zeta]$ is as in Lemma 2, one gets

$$dG(h) = \frac{1}{\underline{H}^+} dG^+(h) J[\zeta]^{-1} - G \circ dJ[\zeta](h) \circ J[\zeta]^{-1}.$$

Since one can compute

$$dJ[\zeta](h) = \underline{\rho}^- \frac{H^-}{H^+}(G^-)^{-1} \circ dG^-(h) \circ (G^-)^{-1} \circ G^+ - \underline{\rho}^- \frac{H^-}{H^+}(G^-)^{-1} \circ dG^+(h),$$

we get, observing that $\psi^+ = J[\zeta]^{-1}\psi$ and $\psi^- = \frac{H^-}{H^+}(G^-)^{-1} \circ G^+ \psi^+$, that

$$dG(h)\psi = \frac{1}{\underline{H}^+} dG^+(h)\psi^+ - \underline{\rho}^- G \circ (G^-)^{-1}\big(dG^-(h)\psi^- - \frac{H^-}{H^+} dG^+(h)\psi^+\big).$$

We can thus use (2.30) to get

$$dG(h)\psi = -\frac{\varepsilon}{\underline{H}^+}(1 + \underline{\rho}^- \underline{H}^- G \circ (G^-)^{-1})G^+(h\underline{w}^+) + \varepsilon\underline{\rho}^- G(h\underline{w}^-) - \varepsilon\mu \mathcal{T}[U]h.$$

The result is then a consequence of Remark 14.

2. Proceeding as in the first point, it is possible to show that $d^j G(\mathbf{h})$ and $dw^\pm(\mathbf{h})$ can be written in terms of G^+, $J[\zeta]^{-1}$ and their shape derivatives, so that one can deduce the result from (2.31)–(2.34), Remark 12 and Lemma 2.

3. Let us first prove the estimate on the H_σ^{r+1}-norm of $\mathfrak{P}\partial^\delta \psi$. Thanks to the assumptions made on r and δ, one has

$$|\mathfrak{P}\partial^\delta \psi|_{H_\sigma^{r+1}} \leqq \sum_{\beta \in \mathbb{N}^{d+1}, |\beta| \leqq N-1} |\mathfrak{P}\partial^\beta \psi|_{H_\sigma^1}.$$

Since $\partial^\beta \psi = \psi_{(\beta)} + \varepsilon\underline{w}\partial^\beta \zeta$, we get

$$|\mathfrak{P}\partial^\delta \psi|_{H_\sigma^{r+1}} \leqq \sum_{\beta \in \mathbb{N}^{d+1}, |\beta| \leqq N-1} \big(|\mathfrak{P}\psi_{(\beta)}|_{H_\sigma^1} + \varepsilon|\mathfrak{P}(\underline{w}\partial^\beta \zeta)|_{H_\sigma^1}\big)$$

$$\leqq \max\left\{1, \frac{1}{\sqrt{\text{Bo}}}\right\} \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| \leqq N} \big(|\mathfrak{P}\psi_{(\alpha)}|_2 + \varepsilon|\underline{w}|_{H^{t_0+2}}|\zeta|_{H_\sigma^{1,N}}\big),$$

where we used the identity $|\mathfrak{P}f|_2 \leqq |\nabla f|_2$ to derive the second inequality. Since $\underline{w} = \underline{\rho}^+ w^+[\zeta]\psi - \underline{\rho}^- w^-[\zeta]\psi$, the result follows easily from Corollary 1.

For the estimate in H^{r+1}-norm, one derives along the same lines the identity

$$|\mathfrak{P}\partial^\delta \psi|_{H^{r+1}} \leqq \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| = N} \big(|\mathfrak{P}\psi_{(\alpha)}|_2 + \varepsilon|\mathfrak{P}(\underline{w}\partial^\alpha \zeta)|_2\big).$$

From the product estimate (which follows directly from Lemma 5.1 of [34] for instance),

$$|fg|_{\dot{H}^{1/2}_\mu} \lesssim |\nabla f|_{H^{t_0}}|g|_2 + |f|_\infty |g|_{\dot{H}^{1/2}_\mu}, \tag{4.17}$$

and the observation that $|\underline{w}|_\infty \leq \sqrt{\mu}\mathfrak{m}^N(U)$ (Corollary 1), we get $|\mathfrak{P}(\underline{w}\partial^\alpha\zeta)|_2 \leq \mathfrak{m}^N(U)(1 + \varepsilon\sqrt{\mu}|\mathfrak{P}\partial^\alpha\zeta|_2)$. The result then follows easily. □

Denoting by $\{\partial^\alpha, \mathcal{G}\}\psi$ the Poisson bracket (with $\check{\alpha}^j$ as in (4.7))

$$\{\partial^\alpha, \mathcal{G}\}\psi = \sum_{j=1}^{d+1} \alpha_j d\,\mathcal{G}(\partial_j\zeta)\partial^{\check{\alpha}^j}\psi,$$

we can write

$$\partial^\alpha \mathcal{G}\psi = \mathcal{G}\partial^\alpha\psi + d\,\mathcal{G}(\partial^\alpha\zeta)\psi + \mathcal{G}_{(\alpha)}\psi_{\langle\check{\alpha}\rangle} + \varepsilon\mu R_\alpha, \tag{4.18}$$

with $\varepsilon\mu R_\alpha = (\{\partial^\alpha, \mathcal{G}\}\psi - \mathcal{G}_{(\alpha)}\psi_{\langle\check{\alpha}\rangle}) + \varepsilon\mu R'_\alpha$, and where $\varepsilon\mu R'_\alpha$ is a sum of terms of the form

$$d^j\mathcal{G}(\partial^{\iota^1}\zeta, \ldots, \partial^{\iota^j}\zeta)\partial^\delta\psi =: A_{j,\delta,\iota}, \tag{4.19}$$

where $j \in \mathbb{N}$, $\iota = (\iota^1, \ldots, \iota^j) \in \mathbb{N}^{(d+1)j}$ and $\delta \in \mathbb{N}^{d+1}$ satisfy

$$\sum_{i=1}^{j} |\iota^i| + |\delta| = N, \qquad 0 \leq |\delta| \leq N - 2 \quad \text{and} \quad |\iota^i| < N. \tag{4.20}$$

We first give some control on the $A_{j,\delta,\iota}$ (and thus on R'_α). Let l be such that $|\iota^l| = \max_{1\leq m\leq j} |\iota^m|$. Let us distinguish four cases:

(i) *The case $|\delta| = j = 1$ and $\iota = \iota^1$ (and thus $|\iota| = N - 1$).* One then has $A_{j,\delta,\iota} = d\,\mathcal{G}(\partial^\iota\zeta)\partial^\delta\psi$, and we can therefore use the first point of Lemma 7 to get

$$\frac{1}{\mu}A_{j,\delta,\iota} = -\varepsilon\frac{1}{\sqrt{\mu}}\mathcal{G}(\partial^\iota\zeta[\![\underline{\varrho}^\pm + \frac{1}{\sqrt{\mu}}(\underline{w}^\pm[\zeta]\partial^\delta\psi)]\!]) - \varepsilon\mathcal{T}[\zeta, \partial^\delta\psi]\partial^\iota\zeta.$$

We can thus deduce from Corollaries 1 and 2, and Proposition 4 that

$$\frac{1}{\mu}|A_{j,\delta,\iota}|_{H^1_\sigma} \leq \varepsilon M(t_0 + 7/2)|\zeta|_{H^{1,N}_\sigma}|\mathfrak{P}\partial^\delta\psi|_{H^{t_0+5/2}_\sigma};$$

since $t_0 + 7/2 \leq N$, these two estimates, together with the third point of Lemma 7, yield that

$$|A_{j,\delta,\iota}|_{H^1_\sigma} \leq \varepsilon\mu\mathfrak{m}^N(U). \tag{4.21}$$

(ii) *The case $|\delta| \leq N - 1 \vee t_0 - 3/2$ and $|\iota^l| < N - 1$.* It follows from (2.34)[10] that

$$\frac{1}{\mu}|A_{j,\delta,\iota}|_{H^1_\sigma} \leq \varepsilon M|\partial^{\iota^l}\zeta|_{H^{5/2}_\sigma}\prod_{m\neq l}|\partial^{\iota^m}\zeta|_{H^{1\vee t_0+5/2}_\sigma}|\mathfrak{P}\partial^\delta\psi|_{H^{1\vee t_0+3/2}_\sigma}.$$

---

[10] We still refer to (2.34) if $\mathcal{G}^\pm$ is replaced by $\mathcal{G}$, thanks to Lemma 7.ii; we do the same for all the identities involved in Lemma 7.ii

From the assumption that $|\iota^l| < N - 1$ and the definition of l, we also get that for all $m \neq l$, $|\iota^m| \leq [(N-2)/2]$, and thus, from the assumption made on N, we have $|\iota^m| + 1 \vee t_0 + 3/2 \leq N$. Using the assumption on δ, we deduce from the last point of Lemma 7 that (4.21) holds.

(iii) *The case $|\delta| = 0$ and $|\iota^l| = N - 1$ (we take for simplicity $l = 1$).* Then one has $A_{j,\delta,\iota} = d^2 G(\partial^{\iota^1} \zeta, \partial^{\iota^2} \zeta) \psi$. Inequality (2.33) provides a good control in terms of regularity, but not with respect to its dependence on μ. We can remark, however, after differentiating the formula stated in Lemma 7.i, that

$$
d^2 G(h_1, h_2)\psi = -\varepsilon\sqrt{\mu}\, dG(h_2)(h_1[\![\underline{\rho}^{\pm} \frac{1}{\sqrt{\mu}}\underline{w}^{\pm}]\!])
$$
$$
-\varepsilon\sqrt{\mu}\, G(h_1[\![\underline{\rho}^{\pm} \frac{1}{\sqrt{\mu}}dw^{\pm}(h_2)\psi]\!]) - \varepsilon\mu\, d\mathcal{T}(h_2)h_1.
$$

With $h_j = \partial^{\iota^j}\zeta$, one can use (2.31) to control the first term of the right-hand side, Corollary 2 and Lemma 7.ii for the second one, and Lemma 7.ii for the last one. Inequality (4.21) is then satisfied.

(iv) *The case $|\delta| \geq N - 1 \vee t_0 - 1/2$.* Using (2.32) and the second point of the Lemma, we now get

$$
|A_{j,\delta,\iota}|_{H^1_\sigma} \leq M\varepsilon^j \mu \prod_{m=1}^{j} |\partial^{\iota^m}\zeta|_{H^{1\vee t_0+2}_\sigma} |\mathfrak{P}\partial^\delta\psi|_{H^2_\sigma}.
$$

Since, by (4.20), $|\delta| + 2 \leq N$ and $|\iota^m| \leq t_0 + 2$, one can conclude as in the previous case that (4.21) holds for N as in the statement of the proposition.

We have therefore proved that the residual R'_α satisfies the estimate stated in the proposition. Remark now that $\frac{1}{\mu}(\{\partial^\alpha, G\}\psi - G_{(\alpha)}\psi_{(\check{\alpha})})$ is a sum of terms of the form (with $\underline{w}$ as in (4.5))

$$
\frac{1}{\sqrt{\mu}} dG(\partial^\iota\zeta)(\frac{1}{\sqrt{\mu}}\underline{w}\partial^\delta\zeta) \qquad (|\iota| = 1, \quad \iota + \delta = \alpha)
$$

that are bounded in H^1_σ-norm by $\varepsilon\mathfrak{m}^N(U)$ thanks to (2.31) and Corollary 1. We deduce that R_α satisfies the estimate of the proposition. The fact that (4.18) coincides with the identity given in the proposition is a consequence of the first point of Lemma 7. □

We end this section with another linearization formula that describes the way $1/\underline{H}^{\pm}G^{\pm}\psi^{\pm}_{(\alpha)}$ departs from $G\psi_{(\alpha)}$ (when $\alpha = 0$, both terms are equal). This proposition will be used to express $\psi^{\pm}_{(\alpha)}$ in terms of the "good unknowns" ζ and $\psi_{(\alpha)}$ in the proof of Proposition 8. In this sense, it can be viewed as a generalization of the transmission problem solved in Proposition 2. The additional term with respect to the case $\alpha = 0$ is responsible for the Kelvin–Helmholtz instabilities, as shown in the next section—of course, in the water-wave case ($\rho^- = 0$), one has $G^+ = G$ and $\psi^+_{(\alpha)} = \psi_{(\alpha)}$ and this destabilizing term disappears.

Proposition 7. *Under the same assumptions as in Proposition* 6, *one has*

$$\frac{1}{\underline{H}^{\pm}} \mathcal{G}^{\pm} \psi^{\pm}_{(\alpha)} = r^{\pm}_{\alpha}, \qquad (|\alpha| \leqq N - 1),$$

and, when $|\alpha| = N$,

$$\frac{1}{\underline{H}^{\pm}} \mathcal{G}^{\pm} \psi^{\pm}_{(\alpha)} = \mathcal{G}\psi_{(\alpha)} - \underline{\rho}^{\mp} \varepsilon \mu \underline{H}^{\mp} \mathcal{G} \circ (\mathcal{G}^{\mp})^{-1} \nabla \cdot (\partial^{\alpha} \zeta \llbracket \underline{V}^{\pm} \rrbracket) + r^{\pm}_{\alpha};$$

the residual $r^{\pm}_{\alpha}$ *satisfies the estimate, with* $|\zeta|_{<N+1/2>}$ *as in* (4.13),

$$\underline{\rho}^{\pm} |(\mathcal{G}^{\pm})^{-1} r^{\pm}_{\alpha}|_{\dot{H}^{3/2}_{\mu}} \leqq \mathfrak{m}^{N}(U)\big(1 + \varepsilon \mu^{1/4} \underline{\rho}^{+} \underline{\rho}^{-} |\zeta|_{<N+1/2>}\big).$$

Remark 21. A consequence of this proposition is that for all $|\alpha| \leqq N$,

$$\underline{\rho}^{\pm} |\mathfrak{P}\psi^{\pm}_{(\alpha)}|_{2} \leqq \mathfrak{m}^{N}(U)\big(1 + \varepsilon \mu^{1/4} \underline{\rho}^{+} \underline{\rho}^{-} |\zeta|_{<N+1/2>}\big).$$

Proof. As for the proof of Proposition 6, we focus on that case $|\alpha| = N$. Applying ∂^{α} to the identity $\frac{1}{\underline{H}^{\mp}} \mathcal{G}^{\pm} \psi^{\pm} = \mathcal{G}\psi$ and using (2.30) and Lemma 7.i we obtain

$$\frac{1}{\underline{H}^{\pm}} \mathcal{G}^{\pm} \psi^{\pm}_{(\alpha)} - \varepsilon \mu \nabla \cdot (\partial^{\alpha} \zeta \underline{V}^{\pm}) = \mathcal{G}\psi_{(\alpha)} - \varepsilon \mu \mathcal{T}[U]\partial^{\alpha}\zeta + r^{\pm}_{\alpha},$$

where $r^{\pm}_{\alpha}$ is a sum of terms of the form,

$$B^{\pm}_{j,\delta,\iota} = d^{j} \mathcal{G}(\partial^{\iota^{1}}\zeta, \ldots, \partial^{\iota^{j}}\zeta)\partial^{\delta}\psi - \frac{1}{\underline{H}^{\pm}} d^{j} \mathcal{G}^{\pm}(\partial^{\iota^{1}}\zeta, \ldots, \partial^{\iota^{j}}\zeta)\partial^{\delta}\psi^{\pm}$$

with $\sum_{i=1}^{j} |\iota^{i}| + |\delta| = N, 0 \leqq |\delta| \leqq N - 1$ and $|\iota^{i}| < N$. The formula of the proposition is obtained after replacing $\mathcal{T}[U]$ by its explicit expression (4.2).

We now must control $\underline{\rho}^{\pm}(\mathcal{G}^{\pm})^{-1} B^{\pm}_{j,\delta,\iota}$ in $\dot{H}^{3/2}_{\mu}(\mathbb{R}^{d})$. The $+$ case is more difficult than the $-$ case because $\underline{\rho}^{-}$ is possibly very small while we always have $\underline{\rho}^{+} \geqq 1/2$. Therefore, we only show how to control $B^{+}_{j,\delta,\iota}$; moreover, only three endpoint cases deserve a proof, namely:

(i) *The case* $j = |\iota^{1}| = 1$ *and* $|\delta| = N - 1$. One then has

$$B^{+}_{j,\delta,\iota} = d\mathcal{G}(\partial^{\iota^{1}}\zeta)\partial^{\delta}\psi - \frac{1}{\underline{H}^{\pm}} d\mathcal{G}^{+}(\partial^{\iota^{1}}\zeta)\partial^{\delta}\psi^{+},$$

$$= \frac{1}{\underline{H}^{+}} d\mathcal{G}^{+}(\partial^{\iota^{1}}\zeta)(J^{-1}\partial^{\delta}\psi - \partial^{\delta}\psi^{+}) - \frac{1}{\underline{H}^{+}}\mathcal{G}^{+} J^{-1} dJ(\partial^{\iota^{1}}\zeta)J^{-1}\partial^{\delta}\psi,$$

where $J = J[\zeta]$ and where we used the identity $\mathcal{G} = \frac{1}{\underline{H}^{+}} \mathcal{G}^{+} J[\zeta]^{-1}$. Since by (2.39), $\psi^{+} = J[\zeta]^{-1}\psi$, we thus get

$$B^{+}_{j,\delta,\iota} = -\frac{1}{\underline{H}^{+}} d\mathcal{G}^{+}(\partial^{\iota^{1}}\zeta)J^{-1}[J, \partial^{\delta}]J^{-1}\psi - \frac{1}{\underline{H}^{+}}\mathcal{G}^{+} J^{-1} dJ(\partial^{\iota^{1}}\zeta)J^{-1}\partial^{\delta}\psi$$

$$= B_{1} + B_{2}.$$

We now need the following lemma.

Lemma 8. *For all $\delta \in \mathbb{N}^{d+1}$, $|\delta| \leqq N - 1$, one has*

$$|[J, \partial^\delta]u|_{\dot{H}_\mu^{3/2}} \leqq \varepsilon \underline{\rho}^- \mathfrak{m}^N(U)(1 + \mu^{1/4}|\zeta|_{<N+1/2>}).$$

Proof of the lemma. Recalling that $J[\zeta] = \underline{\rho}^+ - \underline{\rho}^- \frac{H^-}{H^+}(\mathcal{G}^-)^{-1}\mathcal{G}^+$, and writing $v = (\mathcal{G}^-)^{-1}\mathcal{G}^+ u$, we get $[\partial^\delta, J]u = \underline{\rho}^- \frac{H^-}{H^+}[\partial^\delta, (\mathcal{G}^-)^{-1}\mathcal{G}^+]$ and we observe moreover that

$$[\partial^\delta, (\mathcal{G}^-)^{-1}\mathcal{G}^+] = -(\mathcal{G}^-)^{-1}[\partial^\delta, \mathcal{G}^-]v + (\mathcal{G}^-)^{-1}[\partial^\delta, \mathcal{G}^+]u.$$

Since both terms in the right-hand side can be handled similarly, we just give the details for the first one. It is a sum of terms of the form

$$C_{j,\delta',\iota} = (\mathcal{G}^-)^{-1}d^j\mathcal{G}^-(\partial^{\iota^1}\zeta, \ldots, \partial^{\iota^j}\zeta)\partial^{\delta'}v,$$

with $\sum_{i=1}^j |\iota^i| + |\delta'| = |\delta|$ and $|\delta'| < |\delta|$. We can control all these terms thanks to (2.35), (2.36) and Remark 12, except the most singular one (with respect to ζ), which corresponds to $j = 1$, $\iota^1 = \delta$ and $\delta' = 0$. For this term, we write, using (2.30),

$$C_{1,0,\delta} = -\varepsilon\partial^\delta\zeta(w^-[\varepsilon\zeta]v) - \varepsilon^-\mu^-(\mathcal{G}^-)^{-1}\nabla \cdot (\partial^\delta\zeta(\mathcal{V}_{/\!/}^-[\varepsilon\zeta]v)).$$

Using the inequality $|\partial^\delta\zeta|_{\dot{H}_\mu^{3/2}} \leqq \mu^{-1/4}|\zeta|_{<N+1/2>}$ to control the first term and Remark 8 to control the second one, we get

$$|C_{1,0,\delta}|_{\dot{H}_\mu^{3/2}} \leqq \varepsilon\mu^{-1/4}|\zeta|_{<N+1/2>}|w^-[\varepsilon\zeta]v|_{H^{3/2\vee t_0}}$$
$$+\varepsilon(|\partial^\delta\zeta|_{H^1} + \mu^{1/4}|\zeta|_{<N+1/2>})|\mathcal{V}_{/\!/}^-[\varepsilon\zeta]v|_{H^{3/2\vee t_0}}$$
$$\leqq \varepsilon\mathfrak{m}^N(U)(1 + \mu^{1/4}|\zeta|_{<N+1/2>}),$$

where the second inequality is a consequence of Corollary 1. The lemma then follows easily. □

Using (12) and the lemma, we get the desired control on B_1; for B_2 there is no particular difficulty and we omit the proof.

  (ii) *The case $j = |\delta| = 1$ and $|\iota^1| = N - 1$.* One proceeds exactly as in the proof of Lemma 8.

  (iii) *The case $|\delta| = 0$, $j = 2$ and $|\iota^1| = N - 1$.* One then has

$$B_{j,\delta,\iota}^+ = d^2\mathcal{G}(\partial^{\iota^1}\zeta, \partial^{\iota^2}\zeta)\psi - \frac{1}{\underline{H^\pm}}d^2\mathcal{G}^+(\partial^{\iota^1}\zeta, \partial^{\iota^2}\zeta)\psi^+,$$

and the scenario is the same as in case (iii) in the proof of Proposition 6. □

4.3. The Quasilinear System

We show in Proposition 8 below that the two-fluid equations (1.19) can be "quasilinearized". In these quasilinearized equations, an operator plays a central role with respect to the formation of Rayleigh–Taylor/Kelvin–Helmholtz instabilities. We thus call it the *instability operator*.

Definition 3. (*Instability operator*) Let $t_0 > d/2$, $T > 0$ and $(\zeta, \psi) \in E_{\sigma,T}^N$ ($N \geqq t_0 + 2$) be such that (2.1) is uniformly satisfied on $[0, T]$. The instability operator $\mathit{Ins}[U]$ is defined as

$$\mathit{Ins}[U]f = \mathfrak{a}f - \varepsilon^2 \mu \underline{\rho}^+ \underline{\rho}^- [\![\underline{V}^\pm]\!] \cdot \mathcal{E}[\zeta](f[\![\underline{V}^\pm]\!]) - \frac{1}{\mathrm{Bo}} \nabla \cdot \mathcal{K}[\varepsilon \sqrt{\mu} \nabla \zeta] \nabla f,$$

with

$$\mathfrak{a} = 1 + \varepsilon [\![\underline{\rho}^\pm (\partial_t + \varepsilon \underline{V}^\pm \cdot \nabla) \underline{w}^\pm]\!].$$

For notational convenience, we also introduce the matricial operators

$$\mathcal{A}[U] = \begin{pmatrix} 0 & -\frac{1}{\mu}\mathcal{G} \\ \mathit{Ins}[U] & 0 \end{pmatrix}, \qquad \mathcal{B}[U] = \begin{pmatrix} \varepsilon\mathcal{T}[U] & 0 \\ 0 & -\varepsilon\mathcal{T}[U]^* \end{pmatrix}, \qquad (4.22)$$

and we also define $\mathcal{C}_\alpha[U]$ as

$$\mathcal{C}_\alpha[U] = \begin{pmatrix} 0 & -\frac{1}{\mu}\mathcal{G}_{(\alpha)} \\ \frac{1}{\mathrm{Bo}}\mathcal{K}_{(\alpha)}[\varepsilon\sqrt{\mu}\nabla\zeta] & 0 \end{pmatrix}. \qquad (4.23)$$

We can now state the main result of this section.

Proposition 8. *Let $T > 0$, $t_0 > d/2$ and N be as in Proposition 6. If $U = (\zeta, \psi) \in E_{\sigma,T}^N$ satisfies (2.1) uniformly on $[0, T]$ and solves (1.19), then for all $\alpha \in \mathbb{N}^{d+1}$ with $1 \leqq |\alpha| \leqq N$, the couple $U_{(\alpha)} = (\zeta_{(\alpha)}, \psi_{(\alpha)})$ solves*

$$\partial_t U_{(\alpha)} + \mathcal{A}[U]U_{(\alpha)} = \varepsilon(R_\alpha, S_\alpha)^T, \quad (|\alpha| < N),$$

$$\partial_t U_{(\alpha)} + \mathcal{A}[U]U_{(\alpha)} + \mathcal{B}[U]U_{(\alpha)} + \mathcal{C}_\alpha[U]U_{(\check{\alpha})} = \varepsilon(R_\alpha, S_\alpha)^T, \quad (|\alpha| = N),$$

where $U_{(\check{\alpha})} = (\zeta_{(\alpha)}, \psi_{(\check{\alpha})})^T$ and the residuals R_α and S_α satisfy the estimates

$$|R_\alpha|_{H_\sigma^1} + |S_\alpha|_{\dot{H}_\mu^{1/2}} \lesssim \mathfrak{m}^N(U)\big(1 + \varepsilon\mu^{1/4}\underline{\rho}^+\underline{\rho}^-|[\![\underline{V}^\pm]\!]|_\infty|\zeta|_{<N+1/2>}\big),$$

uniformly on $[0, T]$.

Remark 22. The "quasilinear" system we refer to is the system of evolution equations on $(U_{(\alpha)})_{0\leqq|\alpha|\leqq N}$ (with $U_{(0)} = U$) formed by (1.19) for $\alpha = 0$ and the equations of the proposition for $1 \leqq |\alpha| \leqq N$.

Proof. Throughout this proof, we denote by $\phi^\pm$ the solution of the transmission problem (2.37) provided by Proposition 2, and write $\psi^\pm = \phi^\pm|_{z=0}$ (or equivalently, we define $\psi^\pm$ by (2.39)).

The first equation is obtained directly by applying ∂^α to the first equation of (1.19) and using Proposition 6.

For the second equation, we focus on the most difficult case, namely, $|\alpha| = N$. We decompose first $\alpha = \beta + \gamma \in \mathbb{N}^{d+1}$, with $|\gamma| = 1$. Inspired by [34], we apply first ∂^γ to the second equation of (1.19) and use the definition (4.3) of $\underline{V}^\pm$ and $\underline{w}^\pm$ to get

$$\partial_t \partial^\gamma \psi + \partial^\gamma \zeta + \varepsilon [\![\underline{\rho}^\pm \underline{V}^\pm \cdot (\nabla \partial^\gamma \psi^\pm - \varepsilon \underline{w}^\pm \nabla \partial^\gamma \zeta)]\!]$$
$$- \frac{\varepsilon}{\mu} [\![\underline{\rho}^\pm \underline{w}^\pm \partial^\gamma (G\psi)]\!] = -\frac{1}{\mathrm{Bo}} \frac{1}{\varepsilon \sqrt{\mu}} \partial^\gamma \mathit{k}(\varepsilon \sqrt{\mu} \zeta). \qquad (4.24)$$

We are now going to apply ∂^β to this equation. We will thus find many terms of the form $\partial^\beta(fg)$; in order to check that most of them can be put in the residual term (that is that[11] $|\partial^\beta(fg)|_{\dot{H}^{1/2}_\mu} \lesssim \varepsilon \mathfrak{m}^N(U)$), we rely heavily on the product estimate (4.17). For the sake of clarity, we also introduce the notation

$$a \sim b \iff |a-b|_{\dot{H}^{1/2}_\mu} \lesssim \varepsilon \mathfrak{m}^N(U)\big(1 + \varepsilon \mu^{1/4} \underline{\rho}^+ \underline{\rho}^- |[\![\underline{V}^\pm]\!]|_\infty |\zeta|_{<N+1/2>}\big). \qquad (4.25)$$

Lemma 9. *Under the assumptions of the proposition, and with $\alpha = \beta + \gamma$ ($|\gamma| = 1$), the following identities hold:*

$$\varepsilon \partial^\beta \big(\underline{\rho}^\pm \underline{V}^\pm \cdot (\nabla \partial^\gamma \psi^\pm - \varepsilon \underline{w}^\pm \nabla \partial^\gamma \zeta)\big) \sim \varepsilon \underline{\rho}^\pm \underline{V}^\pm \cdot \nabla \psi^\pm_{(\alpha)}$$
$$+ \varepsilon^2 \underline{\rho}^\pm (\underline{V}^\pm \cdot \nabla \underline{w}^\pm) \partial^\alpha \zeta$$
$$- \varepsilon^2 \underline{\rho}^\pm \underline{V}^\pm \cdot \{\partial^\beta, \underline{w}^\pm\} \nabla \partial^\gamma \zeta,$$
$$\frac{\varepsilon}{\mu} \partial^\beta \big(\underline{\rho}^\pm \underline{w}^\pm \partial^\gamma (G\psi)\big) \sim \frac{\varepsilon}{\mu} \underline{\rho}^\pm \underline{w}^\pm \partial^\alpha (G\psi)$$
$$+ \frac{\varepsilon}{\mu} \underline{\rho}^\pm \{\partial^\beta, \underline{w}^\pm\} \partial^\gamma (G\psi),$$

where $\{\partial^\beta, \underline{w}^\pm\} = \sum_{j=1}^{d+1} \beta_j \partial_j \underline{w}^\pm \partial^{\check{\beta}^j}$, with $\check{\beta}^j + \mathbf{e}_j = \beta$, is the usual Poisson bracket.

Proof. Let us address the first assertion of the lemma. Since $\varepsilon \underline{\rho}^\pm \partial^\beta \big(\underline{V}^\pm \cdot (\nabla \partial^\gamma \psi^\pm - \varepsilon \underline{w}^\pm \nabla \partial^\gamma \zeta)\big)$ is a sum of terms of the form $A_{\beta',\beta''} = \varepsilon \underline{\rho}^\pm \partial^{\beta'} \underline{V}^\pm \cdot \partial^{\beta''}(\nabla \partial^\gamma \psi^\pm - \varepsilon \underline{w}^\pm \nabla \partial^\gamma \zeta)$, with $\beta' + \beta'' = \beta$, and remarking that

$$A_{0,\beta} \sim \varepsilon \underline{\rho}^\pm \underline{V}^\pm \cdot \nabla \psi^\pm_{(\alpha)} + \varepsilon^2 \underline{\rho}^\pm (\underline{V}^\pm \cdot \nabla \underline{w}^\pm) \partial^\alpha \zeta - \varepsilon^2 \underline{\rho}^\pm \underline{V}^\pm \cdot \{\partial^\beta, \underline{w}^\pm\} \nabla \partial^\gamma \zeta,$$

we are led to prove that $A_{\beta',\beta''} \sim 0$ if $|\beta''| < N - 1$. The most difficult configuration corresponds to $\beta'' = 0$ or $|\beta''| = N - 2$, and we thus omit the proof for the

---

[11] The notation $A \lesssim \mathfrak{m}^N(U)$ is used as a shortcut for $A(t) \lesssim \mathfrak{m}^N(U(t))$, for all $0 \leq t \leq T$.

intermediate cases. Moreover, since the proof follows the same lines for $\beta'' = 0$ and $|\beta''| = N - 2$, we give it only in the latter case. By (4.17), we get

$$|A_{\beta',\beta''}|_{\dot{H}_\mu^{1/2}} \leqq \varepsilon \underline{\rho}^\pm |\partial^{\beta'} \underline{V}^\pm|_{H^{t_0+1}} \big(|\partial^{\beta''} g|_2 + |\partial^{\beta''} g|_{\dot{H}_\mu^{1/2}}\big),$$

with $g = \nabla \partial^\gamma \psi^\pm - \varepsilon \underline{w}^\pm \nabla \partial^\gamma \zeta$. Since $|\beta'| = 1$, we deduce easily from Corollary 1 that $|\partial^{\beta'} \underline{V}^\pm|_{H^{t_0+1}} \leqq \mathfrak{m}^N(U)$; moreover, $\partial^{\beta''} g$ can be put under the form

$$\partial^{\beta''} g = \nabla \psi_{(\beta''+\gamma)}^\pm - \varepsilon [\partial^{\beta''}, \underline{w}^\pm] \nabla \partial^\gamma \zeta + \varepsilon (\partial^{\gamma+\beta''} \zeta) \nabla \underline{w}^\pm.$$

Since the last two components of the right-hand side are bounded from above in L^2 and $\dot{H}_\mu^{1/2}$ by $\mathfrak{m}^N(U)$ (this is a consequence of (4.17), Corollary 1 and the identity $|f|_{\dot{H}_\mu^{1/2}} \leqq |f|_{\dot{H}^1}$), we have obtained that

$$|A_{\beta',\beta''}|_{\dot{H}_\mu^{1/2}} \leqq \varepsilon \underline{\rho}^\pm \mathfrak{m}^N(U)(1 + |\nabla \psi_{(\beta''+\gamma)}^\pm|_2 + |\nabla \psi_{(\beta''+\gamma)}^\pm|_{\dot{H}_\mu^{1/2}})$$

$$\leqq \varepsilon \mathfrak{m}^N(U)(1 + \sum_{|\kappa| \leqq N} \underline{\rho}^\pm |\psi_{(\kappa)}^\pm|_{\dot{H}_\mu^{1/2}})$$

$$\leqq \varepsilon \mathfrak{m}^N(U)\big(1 + \varepsilon \mu^{1/4} \underline{\rho}^+ \underline{\rho}^- |[\![\underline{V}^\pm]\!]|_\infty |\zeta|_{<N+1/2>}\big),$$

where Remark 21 has been used to derive the last inequality.

In order to prove the second assertion of the lemma, we proceed as above to check that it is sufficient to prove that $\frac{\varepsilon}{\mu} \underline{\rho}^\pm \partial^{\beta'} \underline{w}^\pm \partial^{\beta''+\gamma} (\mathcal{G}\psi) \sim 0$ if $\beta' + \beta'' = \beta$ and $|\beta''| < N - 2$. We give the proof of the most difficult case, corresponding here to $\beta' = \beta$, $\beta'' = 0$. Thanks to (4.17), Corollaries 1 and 2 and Lemma 7, it is enough to prove that

$$\frac{\underline{\rho}^\pm}{\sqrt{\mu}} |\partial^\beta \underline{w}^\pm|_{\dot{H}_\mu^{1/2}} \leqq \mathfrak{m}^N(U)\big(1 + \varepsilon \mu^{1/4} \underline{\rho}^+ \underline{\rho}^- |[\![\underline{V}^\pm]\!]|_\infty |\zeta|_{<N+1/2>}\big). \qquad (4.26)$$

Recalling that $\underline{w}^\pm = w^\pm[\zeta]\psi$ and differentiating this relation with respect to ∂^β yields, after multiplication by ε (in order to use the convenient notation $\sim$),

$$(1 + \varepsilon^2 \mu |\nabla \zeta|^2) \frac{\varepsilon}{\sqrt{\mu}} \partial^\beta \underline{w}^\pm + 2\varepsilon^2 \sqrt{\mu} \nabla \partial^\beta \zeta \cdot (\varepsilon \underline{w}^\pm \nabla \zeta)$$

$$\sim \frac{\varepsilon}{\sqrt{\mu^+}} \partial^\beta \mathcal{G}^\pm \psi^\pm + \varepsilon^2 \sqrt{\mu} \nabla \zeta \cdot \partial^\beta \nabla \psi^\pm + \varepsilon^2 \sqrt{\mu} \nabla \partial^\beta \zeta \cdot \nabla \psi^\pm.$$

Since one can deduce from (2.30), (2.31) and (2.33) that

$$\frac{\varepsilon}{\sqrt{\mu^+}} \partial^\beta \mathcal{G}^\pm \psi^\pm \sim \frac{\varepsilon}{\sqrt{\mu^+}} \mathcal{G}^\pm \psi_{(\beta)}^\pm - \varepsilon^2 \sqrt{\mu} \nabla \cdot (\partial^\beta \zeta \underline{V}^\pm),$$

we get the formula (after multiplication by $\underline{\rho}^\pm$)

$$(1 + \varepsilon^2 \mu |\nabla \zeta|^2) \frac{\varepsilon \underline{\rho}^\pm}{\sqrt{\mu}} \partial^\beta \underline{w}^\pm \sim \frac{\varepsilon}{\sqrt{\mu^+}} \mathcal{G}^\pm (\underline{\rho}^\pm \psi_{(\beta)}^\pm) + \varepsilon^2 \sqrt{\mu} \nabla \zeta \cdot \nabla (\underline{\rho}^\pm \psi_{(\beta)}^\pm).$$

Dividing this formula by ε, it is then easy to deduce from Remark 21 (and (2.20) to handle the first term of the right-hand side) that (4.26) holds. $\square$

Lemma 9 allows one to put most of the terms of the second equation of ∂^β (4.24) in the residual. However, this is not the case of the Poisson brackets that appear in both identities of Lemma 9, because they are not *uniformly* controlled by the energy, since they require (when evaluated in $\dot{H}_\mu^{1/2}$-norm) a control of $(N + 1/2)$ derivatives of ζ. Using the control of the derivatives of order $(N + 1)$ provided by $|\zeta|_{H_\sigma^{1,N}}$ would induce a singularity when the surface tension goes to zero. Fortunately, this singularity disappears when one takes the difference of the expressions considered in Lemma 9, which is the quantity that appears in ∂^β (4.24). Indeed, one has

$$-\varepsilon^2 \underline{\rho}^\pm \underline{V}^\pm \cdot \{\partial^\beta, \underline{w}^\pm\} \nabla \partial^\gamma \zeta - \frac{\varepsilon}{\mu} \underline{\rho}^\pm \{\partial^\beta, \underline{w}^\pm\} \partial^\gamma (\mathcal{G}\psi)$$

$$\sim -\varepsilon \underline{\rho}^\pm \{\partial^\beta, \underline{w}^\pm\} \Big(\varepsilon \underline{V}^\pm \cdot \nabla \partial^\gamma \zeta + \frac{1}{\mu} \partial^\gamma (\mathcal{G}\psi)\Big)$$

$$\sim 0,$$

where (2.30) has been used to derive the second identity.

It follows from this analysis that ∂^β (4.24) can be written under the form

$$\partial_t \partial^\alpha \psi + \partial^\alpha \zeta + \varepsilon [\![\underline{\rho}^\pm \underline{V}^\pm \cdot (\nabla \psi_{(\alpha)}^\pm + \varepsilon \partial^\alpha \zeta \nabla \underline{w}^\pm)]\!]$$

$$- \frac{\varepsilon}{\mu} [\![\underline{\rho}^\pm \underline{w}^\pm \partial^\alpha (\mathcal{G}\psi)]\!] \sim -\frac{1}{\text{Bo}} \frac{1}{\varepsilon \sqrt{\mu}} \partial^\alpha \mathit{k}(\varepsilon \sqrt{\mu} \zeta).$$

Using the first equation, one can replace $\frac{1}{\mu} \mathcal{G}\psi$ by $\partial_t \zeta$ in this expression,

$$\partial_t \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta + \varepsilon [\![\underline{\rho}^\pm \underline{V}^\pm \cdot \nabla \psi_{(\alpha)}^\pm]\!] \sim -\frac{1}{\text{Bo}} \frac{1}{\varepsilon \sqrt{\mu}} \partial^\alpha \mathit{k}(\varepsilon \sqrt{\mu} \zeta),$$

with $\mathfrak{a}$ as in Definition 3. We can then use the identity

$$[\![f^\pm g^\pm]\!] = \langle f^\pm \rangle [\![g^\pm]\!] + [\![f^\pm]\!] \langle g^\pm \rangle$$

to obtain

$$\partial_t \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta + \varepsilon \langle \underline{V}^\pm \rangle \cdot \nabla \psi_{(\alpha)} + \varepsilon [\![V^\pm]\!] \cdot \nabla \langle \underline{\rho}^\pm \psi_{(\alpha)}^\pm \rangle \sim -\frac{1}{\text{Bo}} \frac{\partial^\alpha \mathit{k}(\varepsilon \sqrt{\mu} \zeta)}{\varepsilon \sqrt{\mu}}.$$

$$(4.27)$$

The quantity $\nabla \langle \underline{\rho}^\pm \psi_{(\alpha)}^\pm \rangle$ depends on $\psi_{(\alpha)}$ and $\partial^\alpha \zeta$ and on their first order derivatives. This dependence is trivial in the water-wave case since $\langle \underline{\rho}^\pm \psi_{(\alpha)}^\pm \rangle = \frac{1}{2} \psi_{(\alpha)}$; when $\rho^- \neq 0$, it is responsible for the Kelvin–Helmholtz instabilities. The following lemma makes this dependence precise.

Lemma 10. *One has, with $\mathcal{E}[\zeta]$ as defined in* (4.10)*,*

$$\varepsilon [\![\underline{V}^\pm]\!] \cdot \nabla \langle \underline{\rho}^\pm \psi_{(\alpha)}^\pm \rangle \sim \frac{\varepsilon}{2} [\![\underline{V}^\pm]\!] \cdot \nabla \big[(1 + 2\underline{\rho}^- \underline{H}^- (\mathcal{G}^-)^{-1} \circ \mathcal{G}) \psi_{(\alpha)} \big]$$

$$- \varepsilon^2 \mu \underline{\rho}^+ \underline{\rho}^- [\![\underline{V}^\pm]\!] \cdot \mathcal{E}[\zeta]([\![\underline{V}^\pm]\!] \partial^\alpha \zeta).$$

Proof. From Proposition 7, we get

$$\frac{1}{\underline{H}^{\pm}}\psi^{\pm}_{(\alpha)} = (\mathcal{G}^{\pm})^{-1}\mathcal{G}\psi_{(\alpha)}$$

$$-\underline{\rho}^{\mp}\varepsilon\mu\underline{H}^{\mp}(\mathcal{G}^{\pm})^{-1}\mathcal{G}(\mathcal{G}^{\mp})^{-1}\nabla\cdot(\partial^{\alpha}\zeta[\![\underline{V}^{\pm}]\!]) + (\mathcal{G}^{\pm})^{-1}r^{\pm}_{\alpha},$$

We deduce from this expression that

$$\varepsilon[\![\underline{V}^{\pm}]\!]\cdot\nabla\langle\underline{\rho}^{\pm}\psi^{\pm}_{(\alpha)}\rangle = \frac{\varepsilon}{2}[\![\underline{V}^{\pm}]\!]\cdot\nabla\big[(1+2\underline{\rho}^{-}\underline{H}^{-}(\mathcal{G}^{-})^{-1}\circ\mathcal{G})\psi_{(\alpha)}\big]$$

$$-\varepsilon^{2}\mu\underline{\rho}^{+}\underline{\rho}^{-}[\![\underline{V}^{\pm}]\!]\cdot\mathcal{E}[\zeta]([\![\underline{V}^{\pm}]\!]\partial^{\alpha}\zeta) + \varepsilon[\![\underline{V}^{\pm}]\!]$$

$$\times\nabla\langle\underline{\rho}^{\pm}\underline{H}^{\pm}(\mathcal{G}^{\pm})^{-1}r^{\pm}_{\alpha}\rangle.$$

The lemma follows from the fact that $\varepsilon[\![\underline{V}^{\pm}]\!]\cdot\nabla\langle\underline{\rho}^{\pm}\underline{H}^{\pm}(\mathcal{G}^{\pm})^{-1}r^{\pm}_{\alpha}\rangle \sim 0$, which is itself a direct consequence of the bounds on $r^{\pm}_{\alpha}$ given in Proposition 7. □

Thanks to the lemma, we can write (4.27) under the form

$$\partial_{t}\psi_{(\alpha)} - \varepsilon\mathcal{I}[U]^{*}\psi_{(\alpha)} + \mathfrak{a}\zeta_{(\alpha)} - \varepsilon^{2}\mu\underline{\rho}^{+}\underline{\rho}^{-}[\![\underline{V}^{\pm}]\!]\cdot\mathcal{E}[\zeta](\zeta_{(\alpha)}[\![\underline{V}^{\pm}]\!])$$

$$\sim -\frac{1}{\mathrm{Bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial^{\alpha}\mathcal{K}(\varepsilon\sqrt{\mu}\zeta).$$

The proposition is thus a direct consequence of the observation that

$$\frac{1}{\mathrm{Bo}}\frac{1}{\varepsilon\sqrt{\mu}}\partial^{\alpha}\mathcal{K}(\varepsilon\sqrt{\mu}\zeta) \sim -\frac{1}{\mathrm{Bo}}\nabla\cdot\mathcal{K}[\varepsilon\sqrt{\mu}\nabla\zeta]\nabla\partial^{\alpha}\zeta + \frac{1}{\mathrm{Bo}}\mathcal{K}_{(\alpha)}[\varepsilon\sqrt{\mu}\nabla\zeta]\zeta_{(\check{\alpha})}.$$

□

5. Main Results

5.1. The Stability Criterion

5.1.1. A First Criterion We state and comment in this section the stability criterion that ensures the existence of a "stable" solution, that is, of a solution that exists on a time scale consistent with physical observations.

Before stating this criterion, let us recall that $\mathfrak{e}(\zeta)$ is defined in Proposition 5, as

$$\mathfrak{e}(\zeta) = \sup_{V\in H^{1/2}(\mathbb{R}^{d})^{d}, V\neq 0} \mu\frac{(\widetilde{\mathcal{G}}^{-1}\nabla\cdot V, \nabla\cdot V)}{|(1+\sqrt{\mu}|D|)^{1/2}V|^{2}_{2}},$$

with $\widetilde{\mathcal{G}} = \underline{\rho}^{-}\frac{1}{\underline{H}^{+}}\mathcal{G}^{+} - \underline{\rho}^{+}\frac{1}{\underline{H}^{-}}\mathcal{G}^{-}$. We also define $\mathfrak{c}(\zeta)$ as

$$\mathfrak{c}(\zeta) = \mathfrak{e}(\zeta)^{2}(1 + \varepsilon^{2}\mu|\nabla\zeta|^{2}_{\infty})^{3/2}$$

and recall that $\Upsilon = (\underline{\rho}^{+}\underline{\rho}^{-})^{2}\frac{a^{4}}{H^{2}}\frac{(\rho^{+}+\rho^{-})g'}{4\sigma}$.

The (nondimensionalized) stability criterion can then be stated as

$$\Upsilon \mathfrak{c}(\zeta) \max_{|\alpha| \leqq 1} \left| \partial^\alpha \llbracket \underline{V}^\pm \rrbracket \right|_\infty^4 < \inf_{\mathbb{R}^d} \mathfrak{a}, \tag{5.1}$$

with $\mathfrak{a} = 1 + \varepsilon \llbracket \underline{\rho}^\pm (\partial_t + \varepsilon \underline{V}^\pm \cdot \nabla) \underline{w}^\pm \rrbracket$ and where we recall that ∂^α can be either a time or space derivative.

In order to measure the extent to which this criterion is satisfied, we introduce the function $\mathfrak{d}(U)$ defined as

$$\mathfrak{d}(U) = \inf_{\mathbb{R}^d} \mathfrak{a} - \Upsilon \mathfrak{c}(\zeta) \max_{|\alpha| \leqq 1} \left| \partial^\alpha \llbracket \underline{V}^\pm \rrbracket \right|_\infty^4 ; \tag{5.2}$$

quite intuitively, we expect the solutions to be more "stable" for large values of $\mathfrak{d}(U)$ than for smaller ones.

Remark 23. In the flat case $\zeta = 0$, one can check that

$$\mathfrak{c}(0) = \sup_{x \geq 0} \frac{x}{(1 + x)(\underline{\rho}^- \tanh(\underline{H}^+ x) + \underline{\rho}^+ \tanh(\underline{H}^- x))}.$$

In the general case, it is possible to obtain an upper bound on $\mathfrak{c}(\zeta)$ by tracking the constants in Proposition 5. The most important information, however, is that $\mathfrak{c}(\zeta) \sim 1$ for all the possible values of ε and μ.

5.1.2. An Alternative Version   We show in Theorem 5 below that (5.1) ensures the existence of a "stable" solution if the fluid depths do not vanish. If we assume, moreover, that the jump of the horizontal velocity is not identically zero at $t = 0$ (which is the general configuration), it is possible to give a more elegant version of the stability criterion, namely,

$$\Upsilon \mathfrak{c}(\zeta) \left| \llbracket \underline{V}^\pm \rrbracket \right|_\infty^4 < \inf_{\mathbb{R}^d} \mathfrak{a}. \tag{5.1'}$$

Consequently, one replaces $\mathfrak{d}(U)$ in (5.2) by

$$\mathfrak{d}'(U) = \inf_{\mathbb{R}^d} \mathfrak{a} - \Upsilon \mathfrak{c}(\zeta) \left| \llbracket \underline{V}^\pm \rrbracket \right|_\infty^4 . \tag{5.2'}$$

Remark 24. It is, of course, possible to give a version with dimensions of the stability criterion (5.1'). Remarking that $(\rho^+ + \rho^-) g' \mathfrak{a} = \llbracket \partial_z P^\pm {}_{|z=\zeta} \rrbracket$, where $P^\pm$ is the pressure in the Euler equations (1.4), we obtain

$$\llbracket -\partial_z P^\pm {}_{|z=\zeta} \rrbracket > \frac{1}{4} \frac{(\rho^+ \rho^-)^2}{\sigma (\rho^+ + \rho^-)^2} \mathfrak{c}(\zeta) \left| \omega \right|_\infty^4,$$

where $\omega = \llbracket V^\pm {}_{|z=\zeta} \rrbracket$. This is the criterion stated in (1.1) in the introduction.

Remark 25. In the water-wave case ($\rho^- = 0$), (5.1) and (5.1') reduce to the condition $\inf_{\mathbb{R}^d} \mathfrak{a} > 0$ (or to the Taylor condition $-\partial_z P_{|z=\zeta} > 0$ in physical variables). This condition is the same with or without surface tension.[12] This condition is usually not imposed in the literature when surface tension is present;[13] the reason it appears here is that we are interested only in "stable" solutions. In the water-wave case, this amounts to discarding configurations where the water is *above* the air; though there exists a local solution thanks to surface tension, it is obviously not "stable" in the intuitive meaning of this word.

5.1.3. A Practical Criterion   We end this section with the derivation of a "practical criterion" that can be used in (shallow water) applications to have an *a priori* idea of the stability of an interfacial wave if we are given its physical characteristics (amplitude and wavelength).

The nondimensionalization performed in Appendix 6.3 (and supported by the experimental data of [28], for instance) is such that $\underline{V}^\pm$ is roughly of size one (this remains true as long as the dynamics does not depart too much from the linear analysis, which is the case for "stable" waves until their breaking point). It follows that $\mathfrak{a}$ is roughly of size one and that a practical criterion can be stated as

$$\Upsilon \ll 1 : \text{ Stable configuration;} \quad \Upsilon \gg 1 : \text{ Unstable configuration.} \tag{5.3}$$

When $\Upsilon \sim 1$, a refined analysis is of course needed and the full criterion (5.1) should be used.

Remark 26. It seems that the wavelength does not play any role in (5.3). Its contribution is, however, hidden in the assumption that $\underline{V}^\pm$ is of size one, which is true only in shallow water, that is, when $H^2/\lambda^2 \ll 1$.

5.2. Initial Conditions

The main step in the proof of the existence of stable solutions is to prove that the energy $\mathcal{E}_\sigma^N(U)$ defined in (4.11), namely,

$$\mathcal{E}_\sigma^N(U) = |\nabla\psi|_{H^{t_0+2}}^2 + \sum_{\alpha\in\mathbb{N}^{d+1},|\alpha|\leq N} |\zeta_{(\alpha)}|_{H_\sigma^1}^2 + |\psi_{(\alpha)}|_{\dot{H}_\mu^{1/2}}^2,$$

is controlled for positive times by the energy at $t = 0$. Since $\mathcal{E}_\sigma^N(U)$ involves time derivatives, we have to specify which sense we give to the initial energy $\mathcal{E}_\sigma^N(U^0)$. If we denote, for all $\alpha \in \mathbb{N}^{d+1}, \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$ so that $\partial_t^{\alpha_0}$ corresponds to the time derivatives of ∂^α, the problem is to choose initial values $U_{(\alpha)}^0$ for $(U_{(\alpha)})_{|t=0}$ (with $U_{(\alpha)} = (\zeta_{(\alpha)}, \psi_{(\alpha)})$) when $\alpha_0 > 0$, in terms of U^0 and its spacial derivatives.

---

[12] With the convention that $(\underline{\rho}^-)^2/\sigma = 0$ if $\underline{\rho}^- = 0$ and $\sigma = 0$.

[13] But it is of course imposed in the references considering the zero surface tension limit, for example [6,7,44,50].

This can be done by a finite induction. When $\alpha_0 = 0$ (no time derivative), we take, of course,

$$U_{(\alpha)}^0 = \left(\partial^\alpha \zeta^0, \partial^\alpha \psi^0 - \varepsilon \underline{w}^0 \partial^\alpha \zeta^0\right)^T \quad \text{with} \quad \underline{w}^0 = [\![\underline{\rho}^\pm w^\pm [\varepsilon \zeta^0] \psi^0]\!].$$

Now, let $1 \leqq n \leqq N$ and assume that $U_{(\beta)\,|_{t=0}} = U_{(\beta)}^0$ has been chosen for all $\beta \in \mathbb{N}^{d+1}$ with $\beta_0 < n$. We remark that for all α with $\alpha_0 = n$ we have

$$U_{(\alpha)\,|_{t=0}} = \left(\partial_t \zeta_{(\alpha')}, \partial_t \psi_{(\alpha')} + \varepsilon \partial_t \underline{w} \partial^{\alpha'} \zeta\right)_{|_{t=0}},$$

with $\alpha' = (\alpha_0 - 1, \alpha_1, \ldots, \alpha_d)$, and we are therefore led to set initial conditions for $\partial_t U_{(\alpha')}$, which is achieved by using Proposition 8.

The initial energy, which we denote slightly abusively by $\mathcal{E}_\sigma^N(U^0)$, is therefore defined as

$$\mathcal{E}_\sigma^N(U^0) = |\nabla \psi^0|_{H^{t_0+2}}^2 + \sum_{\alpha \in \mathbb{N}^{d+1}, |\alpha| \leqq N} |\zeta_{(\alpha)}^0|_{H_\sigma^1}^2 + |\psi_{(\alpha)}^0|_{\dot{H}_\mu^{1/2}}^2, \qquad (5.4)$$

with $U_{(\alpha)}^0$ as constructed above.

Remark 27. One proceeds in the same way to give a formulation of the stability criterion (5.1) or (5.1') at time $t = 0$ in terms of U^0 and its spacial derivatives.

5.3. Local Existence of "stable" Solutions

We state here a theorem ensuring that the interfacial waves equations (1.19) admit "stable" solutions if the depth of both fluid layers does not vanish for the initial condition, and if the stability criterion (5.1) or (5.1') is satisfied. By "stable" solution, we mean two things.

Firstly, the existence time must not shrink to 0 as $\sigma \to 0$. More precisely, the existence time is not measured by the size of the surface tension coefficient σ but by the size of $\mathfrak{d}(U)$.

Secondly, the existence time must not vanish for "acceptable" values of the physical parameters ε and μ, namely $0 \leqq \mu \leqq 1$ and $0 \leqq \varepsilon \leqq 1$.

Recalling that the quantity $\mathcal{E}^N(U^0)$ has been defined in (5.4) and $E_{\sigma,T}^N$ in (4.14), we can now state the following theorem (whose proof is given in Section 5.5).

Theorem 5. *Let* $t_0 = 3/2$, $N \geq 5$ *and* $U^0 = (\zeta^0, \psi^0)^T \in L^2(\mathbb{R}^d) \times \dot{H}^{1/2}(\mathbb{R}^d)$ *be such that* $\mathcal{E}_\sigma^N(U^0) < \infty$ *and satisfies the nonvanishing depth condition*

$$\exists h_{\min}^\pm > 0, \qquad \inf_{X \in \mathbb{R}^d} (1 \pm \varepsilon^\pm \zeta^0(X)) \geqq h_{\min}^\pm.$$

If U^0 *satisfies the stability criterion* (5.1), *then there exists* $T > 0$ *and a unique solution* $U \in E_{\sigma,T}^N$ *to* (1.19) *with initial condition* U^0. *Moreover,*

$$\forall t \in [0, T], \qquad \mathcal{E}_\sigma^N(U(t)) \leqq C(T)$$

and the dependence of T on the initial data if of the form

$$\frac{1}{T} = C\Big(\mathfrak{m}^N(U^0), \frac{1}{\mathfrak{d}(U^0)}\Big),$$

with $\mathfrak{m}^N(U^0)$ and $\mathfrak{d}(U^0)$ as defined in (4.15) and (5.2), respectively.

If the jump of the horizontal component of the initial velocity is nonzero, one can replace the stability criterion (5.1) by its alternative version (5.1'), and get the following theorem.

Theorem. Let $t_0 = 3/2$, $N \geq 5$ and $U^0 = (\zeta^0, \psi^0)^T \in L^2(\mathbb{R}^d) \times \dot{H}^{1/2}(\mathbb{R}^d)$ be such that $\mathcal{E}^N_\sigma(U^0) < \infty$ and satisfies the nonvanishing depth condition

$$\exists h^\pm_{\min} > 0, \qquad \inf_{X \in \mathbb{R}^d}(1 \pm \varepsilon^\pm \zeta^0(X)) \geqq h^\pm_{\min},$$

together with the condition $\omega^0 = [\![\underline{V}^{0,\pm}]\!] \neq 0$. If U^0 satisfies the stability criterion (5.1'), then the conclusion of Theorem 5 still holds, with T depending on the initial data through

$$\frac{1}{T} = C\Big(\mathfrak{m}^N(U^0), \frac{1}{\mathfrak{d}'(U^0)}, \frac{|\partial_t\omega^0|_\infty + |\nabla\omega^0|_\infty}{|\omega^0|_\infty}\Big).$$

Remark 28. 1. The existence time also depends implicitly on $\frac{H^\pm}{H^\mp}$; since we address here physical configurations where H^+ and H^- are of the same order,[14] this dependence is harmless.

2. There is also an implicit dependence of T on $1/\mathrm{Bo}$. Since we are not interested in situations where capillary forces dominate the effect of gravity (for example droplets), the Bond number is always larger than one and the dependence on $1/\mathrm{Bo}$ is harmless.

3. It is straightforward to deduce from Theorem 5 some convergence results as ρ^- and/or σ go to zero. For instance, convergence to the water-wave equations without surface tension is obtained when ρ^- and σ go to zero with $\Upsilon \to 0$—and in particular if $(\rho^-)^2/\sigma \to 0$, a result established in [50] for a liquid droplet with rotational effects under the slightly stronger condition $(\rho^-)^2 \lesssim \sigma^{7/3}$. Theorem 5 shows that it is possible to take the shallow water limit at the same time, which justifies the use of the standard (one fluid) shallow water models for the air/water interface (see Section 6.1 for more details).

5.4. Persistence of "stable" Solutions Over Large Times

We have seen in Remark 25 that in the water-wave case without surface tension, the stability criterion (5.1) reduces to the Rayleigh–Taylor criterion $\inf_{\mathbb{R}^d} \mathfrak{a} > 0$. If this condition is satisfied, we know from [4] that the solution U exists for a time

---

[14] As said in the introduction, this assumption is made for the sake of clarity and the method presented here could be adapted to other configurations.

scale $O(1/\varepsilon)$. When $\varepsilon \sim 1$, this results coincides with Theorem 5, but when $\varepsilon \ll 1$, Theorem 5 is weaker than [4] since it provides an existence time of order $O(1)$ only.

We propose in this section a stronger version of the stability criterion (5.1) (one can adapt (5.1') in the same way) that allows us to show that the solution furnished by Theorem 5 persists over larger times when $\varepsilon \ll 1$. This stronger criterion can be stated as

$$\varepsilon^{-2\gamma} \Upsilon^2 \mathfrak{c}(\zeta) \max_{|\alpha| \leqq 1} \left| \partial^\alpha [\![\underline{V}^\pm]\!] \right|_\infty^4 < \inf_{\mathbb{R}^d} \mathfrak{a}, \quad \text{with} \quad 0 \leqq \gamma \leqq 1 \tag{5.5}$$

(in the case $\gamma = 0$, (5.5) coincides of course with (5.1)). This leads us to introduce the following generalization of (5.2)

$$\mathfrak{d}_\gamma(U) = \inf_{\mathbb{R}^d} \mathfrak{a} - \Upsilon \mathfrak{c}(\zeta) \max_{|\alpha| \leqq 1} \left| \partial^\alpha [\![\underline{V}^\pm]\!] \right|_\infty^4, \tag{5.6}$$

which is used in the statement below.

Theorem 6. *Let $t_0 = 3/2$, $N \geqq 5$ and $U^0 = (\zeta^0, \psi^0)^T \in L^2(\mathbb{R}^d) \times \dot{H}^{1/2}(\mathbb{R}^d)$ be such that $\mathcal{E}_\sigma^N(U^0) < \infty$ and satisfies the nonvanishing depth condition*

$$\exists h_{\min}^\pm > 0, \quad \inf_{X \in \mathbb{R}^d} (1 \pm \varepsilon^\pm \zeta^0(X)) \geqq h_{\min}^\pm.$$

If moreover U^0 satisfies the strong stability criterion (5.5) for some $\gamma \in [0, 1]$, then there exists $T > 0$ and a unique solution $U \in E_{\sigma, \varepsilon^{-\gamma} T}^N$ to (1.19) with initial condition U^0. Moreover,

$$\forall t \in [0, \varepsilon^{-\gamma} T], \quad \mathcal{E}_\sigma^N(U(t)) \leqq C(T)$$

and the dependence of T on the initial data is of the form

$$\frac{1}{T} = C(\mathfrak{m}^N(U^0), \frac{1}{\mathfrak{d}_\gamma(U^0)}).$$

Remark 29. If (5.5) is satisfied with $\gamma = 1$, we recover the same time scale as the one provided by [4] for water waves. We refer to Section 6 for further discussion.

Remark 30. The practical criterion corresponding to (5.5) is obtained by replacing Υ by $\varepsilon^{-2\gamma} \Upsilon$ in (5.3).

5.5. Proof of Theorems 5, 5' and 6

We give here the proof of Theorems 5 and 6. The proof of Theorem 5' requires only minor adaptations with respect to the proof of Theorem 5; we briefly indicate them in footnotes.

5.5.1. Preliminary Results   We construct below a symmetrizer for the quasilinear system (5.9). It is of crucial importance that the energy norm associated to this symmetrizer (see (5.10) below) be (uniformly) equivalent to the energy $\mathcal{E}_\sigma^N(U)$ previously introduced in (4.11). A first ingredient is the coercivity property of $\frac{1}{\mu}\mathcal{G}$ established in Proposition 3. The second ingredient is the following lemma[15] which shows that, up to lower order terms that can be controlled and under the stability condition (5.1), $(\mathit{Ins}[U]u, u)$ is uniformly equivalent to $|u|^2_{H^1_\sigma}$.

Lemma 11. *Let $T > 0, t_0 > d/2$ and $U = (\zeta, \psi) \in E^N_{\sigma,T}$ be such that (2.1) and (5.1) are uniformly satisfied on $[0, T]$. Then the following identities hold uniformly on $[0, T]$,*

1. *For all $u \in H^1_\sigma(\mathbb{R}^d)$, and with $\mathfrak{m}^1(U)$ as defined in (4.15), one has*

$$(\mathit{Ins}[U]u, u) \lesseqgtr \mathfrak{m}^1(U)|u|^2_{H^1_\sigma}.$$

2. *There exists a numerical constant c_2 such that*

$$\frac{1}{2}\mathfrak{d}(U)|u|_{H^1_\sigma} \leqq (\mathit{Ins}[U]u, u) + a(U)|u|^2_2 + b(U)|u|^2_{H^{-1/2}},$$

   with $a(U) = \varepsilon^2\underline{\rho}^+\underline{\rho}^-\mathfrak{e}(\zeta)\big|[\![\underline{V}^\pm]\!]\big|^2_\infty$ and $b(U) = \varepsilon^2\sqrt{\mu}\underline{\rho}^+\underline{\rho}^-c_2\mathfrak{e}(\zeta)\big|[\![\underline{V}^\pm]\!]\big|^2_{H^{t_0+1}}$, and $\mathfrak{d}(U)$ as in (5.2).
3. *With $|\cdot|_{\langle1/2\rangle}$ as defined in (4.13), one has*

$$\varepsilon^2\sqrt{\mu}\underline{\rho}^+\underline{\rho}^-\max_{|\alpha|\leqq 1}\big|\partial^\alpha[\![\underline{V}^\pm]\!]\big|^2_\infty|u|^2_{\langle1/2\rangle} \leqq \mathfrak{m}^1(U)|u|_{H^1_\sigma}.$$

Proof. By definition of $\mathit{Ins}[U]$ (see Definition 3), one has

$$(\mathit{Ins}[U]u, u) = (\mathfrak{a}u, u) - \varepsilon^2\mu\underline{\rho}^+\underline{\rho}^-(\mathcal{E}(u[\![\underline{V}^\pm]\!]), u[\![\underline{V}^\pm]\!]) + \frac{1}{\mathrm{Bo}}(\mathcal{K}\nabla u, \nabla u).$$

In order to give some control on the three components of this expression, we need the following straightforward estimates,

$$
\begin{array}{ccccc}
(\inf_{\mathbb{R}^d}\mathfrak{a})|u|^2_2 & & \leqq & (\mathfrak{a}u, u) & \leqq \mathfrak{m}^1(U)|u|^2_2 \\
\dfrac{1}{(1+\varepsilon^2\mu|\nabla\zeta|^2_\infty)^{3/2}}|\nabla u|^2_2 & \leqq & (\mathcal{K}[\varepsilon\sqrt{\mu}\zeta]\nabla u, \nabla u) & \leqq & |\nabla u|^2_2,
\end{array}
$$

---

[15] If one replaces (5.1) by (5.1') then the estimate of the third point holds only for $\alpha = 0$. In the case $|\alpha| = 1$, we must therefore replace the estimate of the lemma by

$$\varepsilon^2\sqrt{\mu}\underline{\rho}^+\underline{\rho}^-\mathfrak{e}(\zeta)\big|\partial^\alpha[\![\underline{V}^\pm]\!]\big|^2_\infty|u|^2_{\langle1/2\rangle} \leqq \Big(\frac{|\partial^\alpha[\![\underline{V}^\pm]\!]\|_\infty}{|[\![\underline{V}^\pm]\!]\|_\infty}\Big)^2\mathfrak{m}^1(U)|u|_{H^1_\sigma}.$$

and, using Proposition 5,

$$
0 \leq (\mathcal{E}(u[\![\underline{V}^\pm]\!]), u[\![\underline{V}^\pm]\!]) \leq \frac{\mathfrak{e}(\zeta)}{\mu} |(1 + \sqrt{\mu}|D|)^{1/2} (u[\![\underline{V}^\pm]\!])|_2^2
$$

$$
\leq \frac{\mathfrak{e}(\zeta)}{\sqrt{\mu}} |[\![\underline{V}^\pm]\!]|_\infty^2 ||D|^{1/2} u|_2^2 + \frac{\mathfrak{e}(\zeta)}{\mu} |[\![\underline{V}^\pm]\!]|_\infty^2 |u|_2^2 + \frac{c_2 \mathfrak{e}(\zeta)}{\sqrt{\mu}} |[\![\underline{V}^\pm]\!]|_{H^{t_0+1}}^2 |u|_{H^{-1/2}}^2,
$$

where we used the estimate

$$
|(1 + \mu^{1/4}|D|^{1/2})(u[\![\underline{V}^\pm]\!])|_2 \leq \sqrt{c_2} \mu^{1/4} |[\![\underline{V}^\pm]\!]|_{H^{t_0+1}} |u|_{H^{-1/2}} + |[\![\underline{V}^\pm]\!]|_\infty |u|_2
$$
$$
+ \mu^{1/4} |[\![\underline{V}^\pm]\!]|_\infty ||D|^{1/2} u|_2 \tag{5.7}
$$

to derive the last inequality (here c_2 is a numerical constant).

The upper bound on $(Ins[U]u, u)$ given in the lemma is a direct consequence of these inequalities. For the lower bound, note that they also yield

$$
(Ins[U]u, u) \geq A - a(U)|u|_2^2 - b(U)|u|_{H^{-1/2}}^2,
$$

with

$$
A = (\inf_{\mathbb{R}^d} \mathfrak{a})|u|_2^2 - \sqrt{\mu} a(U)|u|_{(1/2)}^2 + \frac{1}{\mathrm{Bo}} \frac{|\nabla u|_2^2}{(1 + \varepsilon^2 \mu |\nabla \zeta|_\infty^2)^{3/2}}
$$

$$
= \int_{\mathbb{R}^d} \left((\inf_{\mathbb{R}^d} \mathfrak{a}) - \sqrt{\mu} a(U)|\xi| + \frac{1}{\mathrm{Bo}} \frac{|\xi|^2}{(1 + \varepsilon^2 \mu |\nabla \zeta|_\infty^2)^{3/2}} \right) |\widehat{u}(\xi)|^2 \mathrm{d}\xi.
$$

Basic calculus shows that under the stability condition (5.1), one has $2A \geq \mathfrak{d}(U)|u|_{H_\sigma^1}^2$, and the result follows. Noting that A remains positive if one replaces $|[\![\underline{V}^\pm]\!]|_\infty$ by $\max_{|\alpha|\leq 1} |\partial^\alpha [\![\underline{V}^\pm]\!]|_\infty$ in $a(U)$, one gets the last point of lemma as the first one. □

5.5.2. The Mollified Quasilinear System

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth, compactly supported function equal to one in a neighborhood of the origin. For all $\iota > 0$, we denote by J^ι the mollifier $J^\iota = \chi(\iota|D|)$. Let us consider the following mollified version of the two-fluid wave equations (1.19),

$$
\begin{cases}
\partial_t \zeta - \dfrac{1}{\mu} J^\iota G\psi = 0, \\[2mm]
\partial_t \psi + J^\iota \zeta + \varepsilon \dfrac{1}{2} J^\iota \left([\![\rho^\pm |\nabla \psi^\pm|^2]\!] - \dfrac{1}{\mu}(1 + \varepsilon^2 \mu |\nabla \zeta|^2)[\![\rho^\pm (\underline{w}^\pm)^2]\!] \right) \\[2mm]
\quad = -\dfrac{1}{\mathrm{Bo}} \dfrac{1}{\varepsilon \sqrt{\mu}} J^\iota \mathit{k}(\varepsilon \sqrt{\mu} \zeta).
\end{cases} \tag{5.8}
$$

From standard results on ODEs, (5.8) has a unique maximal solution $U^\iota = (\zeta^\iota, \psi^\iota)$ with initial data (ζ^0, ψ^0) on a time interval $[0, T_{max}^\iota)$. Proceeding exactly as for the proof of Proposition 8, one can check that for all $\alpha \in \mathbb{N}^{d+1}, 1 \leq |\alpha| \leq N$,

the quantity $U^\iota_{(\alpha)} = (\zeta^\iota_{(\alpha)}, \psi^\iota_{(\alpha)})$ (with $\zeta^\iota_{(\alpha)} = \partial^\alpha \zeta^\iota$ and $\psi^\iota_{(\alpha)} = \partial^\alpha \psi^\iota - \varepsilon[\![\underline{\rho}^\pm \underline{w}^\pm [\varepsilon \zeta^\iota] \psi^\iota]\!] \partial^\alpha \zeta^\iota)$ solves[16]

$$\partial_t U_{(\alpha)} + J^\iota \mathcal{A}[U] U_{(\alpha)} + J^\iota \mathcal{B}[U] U_{(\alpha)} + J^\iota \mathcal{C}_\alpha[U] U_{(\check\alpha)} = \varepsilon (J^\iota R_\alpha, J^\iota S_\alpha + S'_\alpha)^T \quad (5.9)$$

(as in Proposition 8, the operators $\mathcal{B}[U]$ and $\mathcal{C}_\alpha[U]$ must be removed if $|\alpha| < N$), where R_α and S_α satisfy the same estimates as in Proposition 8, and where S'_α is given by

$$S'_\alpha = -(1 - J^\iota)[\![\underline{\rho}^\pm \partial_t \underline{w}^\pm]\!] \zeta_{(\alpha)}].$$

Our strategy is to derive energy estimates for the system of evolution equations on $(U^\iota_{(\alpha)})_{0 \leq |\alpha| \leq N}$ formed by (5.8) and (5.9). After showing that these energy estimates do not depend on the mollifying parameter ι, we will construct by compactness a solution $(U_{(\alpha)})_{0 \leq |\alpha| \leq N}$ to the limit system corresponding to $\iota = 0$ (that is, $J^\iota = 1$) on a time interval $[0, T_\sigma]$. The component corresponding to $\alpha = 0$ of this solution furnishes a solution to (1.19). Without further analysis, one has *a priori* $T_\sigma \to 0$ as $\sigma \to 0$. An energy estimate on the exact equations—as opposed to the mollified ones—allows us to prove that the existence time can be taken as in the statement of Theorem 5. We then prove that the solution persists over larger time if the stronger criterion (5.5) is satisfied.

5.5.3. Symmetrizer and Energy   *Since $d = 1, 2$, we can take $t_0 = 3/2 > d/2$ and any $N \geq 5$ satisfies the condition* (4.16). A symmetrizer for the "quasilinear" system $(5.9)_\alpha$ is given by

$$\mathcal{S}[U] = \mathcal{S}^1[U] + \mathcal{S}^2_\alpha[U] + \mathcal{S}^3[U],$$

where $\mathcal{S}^j[U]$ $(1 \leq j \leq 3)$ are diagonal operators defined as

$$\mathcal{S}^1[U] = \mathrm{diag}(\mathit{Ins}[U], \frac{1}{\mu}\mathcal{G}),$$

$$\mathcal{S}^2_\alpha[U] = \mathrm{diag}(\frac{1}{\mathrm{Bo}} \mathcal{K}_{(\alpha)}[\varepsilon \sqrt{\mu} \nabla \zeta], \frac{1}{\mu}\mathcal{G}_{(\alpha)}), \quad (\text{or } \mathcal{S}^2_\alpha[U] = 0 \quad \text{if} \quad |\alpha| < N),$$

$$\mathcal{S}^3[U] = \mathrm{diag}(a(U) + b(U)\Lambda^{-1}, 0),$$

where the constants $a(U)$ and $b(U)$ are as in Lemma 11.

Multiplying (5.9) by $\mathcal{S}^1[U]$ symmetrizes the highest order terms, but since $\mathcal{S}^1[U]$ is a second order operator, the subprincipal terms need to be symmetrized also (otherwise the commutator term is not controlled by the energy) and this explains the presence of $\mathcal{S}^2_\alpha[U]$. The $\mathcal{S}^3$ component contains two terms of order 0 and -1, respectively; its role is to ensure that $(\mathcal{S}[U]U, U)$ controls the energy $\mathcal{E}^N_\sigma(U)$.

---

[16] For the sake of clarity, we write $\zeta_{(\alpha)}$ and $\psi_{(\alpha)}$ instead of $\zeta^\iota_{(\alpha)}$ and $\psi^\iota_{(\alpha)}$ when no confusion is possible.

This discussion motivates the introduction of the energies $\mathcal{F}_\sigma^l(U)$, for all $1 \leqq l \leqq N$,

$$\mathcal{F}_\sigma^l(U) = \sum_{0 \leqq |\alpha| \leqq l} \mathcal{F}_\sigma^\alpha(U), \tag{5.10}$$

with

$$\begin{aligned}
\mathcal{F}_\sigma^\alpha(U) &= \frac{1}{2}\big([S^1[U] + S^3[U]]U_{(\alpha)}, U_{(\alpha)}\big) \quad (\alpha \neq 0), \\
\mathcal{F}_\sigma^0(U) &= |\zeta|_{H_\sigma^1}^2 + \frac{1}{\mu}(\psi, \mathcal{G}_\mu[0]\psi) \qquad (\alpha = 0),
\end{aligned} \tag{5.11}$$

where $U_{(\alpha)} = (\zeta_{(\alpha)}, \psi_{(\alpha)})$. This definition raises at least two important remarks:

1. The component $S^2[U]$ does not appear in the energy; the reason is that $(S^2[U]\partial_t U_{(\alpha)}, U_{(\alpha)})$ can be controlled by the energy without using the equation to replace the time derivative by space derivatives. This is made possible by the fact that the energy controls both space *and* time derivatives. This method was introduced to handle surface tension in the water-wave case in [51] (see also [11]) and can be adapted here with small modifications.

2. The presence of $S^3(U)$ induces nonsymmetric terms. Since these terms are of lower order, they do not raise any difficulty as far as local existence is concerned. However, they may destroy the stability criterion since they may interfere with the zero order component (namely, $\mathfrak{a}$) of $Ins[U]$—or at least restrict its range of validity (very small values of $\underline{\rho}^-$ for instance). This explains why the zero order term of $S^3(U)$ is treated with special care in the computations below.

Let us finally remark that, owing to Proposition 3, Lemma 7 and Lemma 11, the energy $\mathcal{F}_\sigma^N(U)$ is equivalent to $\mathcal{E}_\sigma^N(U)$, as defined in (4.11),

$$\mathcal{E}_\sigma^N(U) \leqq (M + \frac{2}{\mathfrak{d}(U)})\mathcal{F}_\sigma^N(U) \quad \text{and} \quad \mathcal{F}_\sigma^N(U) \leqq \mathfrak{m}^1(U)\mathcal{E}_\sigma^N(U), \tag{5.12}$$

with $\mathfrak{d}(U)$ as in (5.2).

5.5.4. Energy Estimates *For notational simplicity, we write $\mathcal{E} = \mathcal{E}[\zeta]$, $\mathcal{K} = \mathcal{K}[\varepsilon\sqrt{\mu}\zeta]$, $\mathcal{T} = \mathcal{T}[U]$ etc.* We first consider the case $\alpha \neq 0$. Taking the L^2-scalar product of (5.9) with $(S^1 + S^3)U_{(\alpha)} + S_\alpha^2 U_{(\check\alpha)}$ yields

$$\begin{aligned}
\big((S^1 &+ S^3)\partial_t U_{(\alpha)}, U_{(\alpha)}\big) + \big(\partial_t U_{(\alpha)}, S_\alpha^2 U_{(\check\alpha)}\big) \\
&+ \big(S^3 J^\iota \mathcal{A}U_{(\alpha)}, U_{(\alpha)}\big) + \big(S^3 J^\iota C_\alpha U_{(\check\alpha)}, U_{(\alpha)}\big) + \big(J^\iota C_\alpha U_{(\check\alpha)}, S_\alpha^2 U_{(\check\alpha)}\big) \\
&+ \big((S^1 + S^3)J^\iota \mathcal{B}U_{(\alpha)}, U_{(\alpha)}\big) + \big(J^\iota \mathcal{B}U_{(\alpha)}, S_\alpha^2 U_{(\check\alpha)}\big) \\
&= \varepsilon\big(J^\iota(R_\alpha, S_\alpha + S_\alpha')^T, (S^1 + S^3)U_{(\alpha)} + S_\alpha^2 U_{(\check\alpha)}\big),
\end{aligned}$$

where we used the fact that $\big(J^\iota \mathcal{A}U_{(\alpha)}, S^1 U_{(\alpha)}\big) = 0$ and $\big(J^\iota \mathcal{A}U_{(\alpha)}, S_\alpha^2 U_{(\check\alpha)}\big) + \big(J^\iota C_\alpha U_{(\check\alpha)}, S^1 U_{(\alpha)}\big) = 0$. By definition of $\mathcal{F}_\sigma^\alpha(U)$ we get therefore

$$\frac{d}{dt}\Big(\mathcal{F}_\sigma^\alpha(U) + (U_{(\alpha)}, S_\alpha^2 U_{(\check\alpha)})\Big) = \sum_{j=1}^7 A_j + B_1 + B_2, \tag{5.13}$$

with

$$
\begin{aligned}
&A_1 = \frac{1}{2}\big((\partial_t \mathcal{S}^1 + \partial_t \mathcal{S}^3) U_{(\alpha)}, U_{(\alpha)}\big), \quad A_2 = \big(U_{(\alpha)}, \partial_t(\mathcal{S}^2_\alpha U_{(\check\alpha)})\big), \\
&A_3 = -\big(\mathcal{S}^3 J^\iota \mathcal{A} U_{(\alpha)}, U_{(\alpha)}\big), \qquad\qquad A_4 = -\big(\mathcal{S}^3 J^\iota \mathcal{C}_\alpha U_{(\alpha)}, U_{(\alpha)}\big) \\
&A_5 = -\big(J^\iota \mathcal{C}_\alpha U_{(\check\alpha)}, \mathcal{S}^2_\alpha U_{(\check\alpha)}\big), \qquad A_6 = -\big((\mathcal{S}^1 + \mathcal{S}^3) J^\iota \mathcal{B} U_{(\alpha)}, U_{(\alpha)}\big), \\
&A_7 = -\big(J^\iota \mathcal{B} U_{(\alpha)}, \mathcal{S}^2_\alpha U_{(\check\alpha)}\big)
\end{aligned}
$$

and

$$
\begin{aligned}
B_1 &= \varepsilon\big(J^\iota (R_\alpha, S_\alpha)^T, (\mathcal{S}^1 + \mathcal{S}^3) U_{(\alpha)} + \mathcal{S}^2_\alpha U_{(\check\alpha)}\big), \\
B_2 &= \varepsilon\big(J^\iota (0, S'_\alpha)^T, \mathcal{S}^1 U_{(\alpha)} + \mathcal{S}^2_\alpha U_{(\check\alpha)}\big).
\end{aligned}
$$

We now give some control on the different components of the right-hand side of (5.13).

- *Control of A_1.* We can decompose $A_1 = A_{11} + A_{12} + A_{13} + A_{14}$ with

$$
\begin{aligned}
&A_{11} = ([\partial_t, \mathit{Ins}[U]]\zeta_{(\alpha)}, \zeta_{(\alpha)}), \quad A_{12} = (\tfrac{1}{\mu}[\partial_t, \mathcal{G}]\psi_{(\alpha)}, \psi_{(\alpha)}), \\
&A_{13} = ((\partial_t a)\zeta_{(\alpha)}, \zeta_{(\alpha)}), \qquad\quad A_{14} = ((\partial_t b)\Lambda^{-1/2}\zeta_{(\alpha)}, \Lambda^{-1/2}\zeta_{(\alpha)}).
\end{aligned}
$$

One gets easily that $|A_{1j}| \leqq \varepsilon \mathfrak{m}^N(U)$ for $2 \leqq j \leqq 4$ (this is straightforward for A_{13} and A_{14}, and a consequence of Proposition 3.6 of [4] for A_{12}). We thus focus our attention on A_{11}. First, note that

$$
\begin{aligned}
A_{11} = {}&((\partial_t \mathfrak{a})\zeta_{(\alpha)}, \zeta_{(\alpha)}) + \frac{1}{\mathrm{Bo}}\big([\partial_t, \mathcal{K}]\nabla\zeta_{(\alpha)}, \nabla\zeta_{(\alpha)}\big) \\
&-2\varepsilon^2 \mu \underline{\rho}^+\underline{\rho}^- \big(\mathcal{E}(\zeta_{(\alpha)}\llbracket \underline{V}^\pm \rrbracket), \zeta_{(\alpha)}\partial_t \llbracket \underline{V}^\pm \rrbracket\big) \\
&-\varepsilon^2 \mu \underline{\rho}^+\underline{\rho}^- \big([\partial_t, \mathcal{E}]\zeta_{(\alpha)}\llbracket \underline{V}^\pm \rrbracket), \zeta_{(\alpha)}\llbracket \underline{V}^\pm \rrbracket\big).
\end{aligned}
$$

It is straightforward that the first two components are bounded from above by $\varepsilon \mathfrak{m}^N(U)$. To control the other two components of this identity, we use Proposition 5 and (5.7) to obtain

$$
\begin{aligned}
|A_{11}| \leqq {}&\varepsilon \mathfrak{m}^N(U)\big(1 + \varepsilon^2\sqrt{\mu}\,\underline{\rho}^+\underline{\rho}^-|\llbracket \underline{V}^\pm \rrbracket|^2_\infty ||D|^{1/2}\zeta_{(\alpha)}|^2_2\big) \\
&+\varepsilon^2\sqrt{\mu}\,\underline{\rho}^+\underline{\rho}^-|\llbracket \underline{V}^\pm \rrbracket|_\infty|\partial_t\llbracket \underline{V}^\pm \rrbracket|_\infty ||D|^{1/2}\zeta_{(\alpha)}|^2_2. \qquad (5.14)
\end{aligned}
$$

Thanks to the third point of Lemma 11, we get[17] $|A_{11}| \leqq \mathfrak{m}^N(U)$.

- *Control of A_2.* From the definition of $\mathcal{S}^2_\alpha$ and the fact that time derivatives are allowed in the energy (4.11), it is easy to get that $|A_2| \leqq \varepsilon \mathfrak{m}^N(U)$.

---

[17] If we assume (5.1') rather than (5.1), then one has to adapt this estimate as indicated in footnote (15).

– *Control of A_3.* From the definition of $\mathcal{S}^3$, it follows that

$$A_3 = a(U)\big(J^\iota \frac{1}{\mu} \mathcal{G}\psi_{(\alpha)}, \zeta_{(\alpha)}\big) + b(U)\big(\frac{1}{\mu}\mathcal{G}\psi_{(\alpha)}, J^\iota \Lambda^{-1}\zeta_{(\alpha)}\big).$$

Owing to (2.23), we get

$$|A_3| \leqq M |\mathfrak{P}\psi_{(\alpha)}|_2 \big(a(U)|\mathfrak{P}\zeta_{(\alpha)}|_2 + b(U)|\mathfrak{P}\zeta_{(\alpha)}|_{H^{-1}}\big)$$

and therefore, from the definition of $a(U)$ and $b(U)$ and Lemma 11, $|A_3| \leqq \varepsilon \mathfrak{m}^N(U)$.

– *Control of A_4.* Since one has

$$A_4 = a(U)\big(J^\iota \frac{1}{\mu} \mathcal{G}_{(\alpha)}\psi_{(\check{\alpha})}, \zeta_{(\alpha)}\big) + b(U)\big(\frac{1}{\mu}\mathcal{G}_{(\alpha)}\psi_{(\check{\alpha})}, J^\iota \Lambda^{-1}\zeta_{(\alpha)}\big).$$

and because $|\mathcal{G}_{(\alpha)}\psi_{(\check{\alpha})}|_2 \leqq \varepsilon \mathfrak{m}^N(U)$ (see (2.32) and Lemma 7), we easily get that $|A_4| \leqq \varepsilon \mathfrak{m}^N(U)$.

– *Control of A_5.* As for A_4, one gets without any difficulty that $|A_5| \leqq \varepsilon \mathfrak{m}^N(U)$.

– *Control of A_6.* We decompose $A_6 = A_{61} + A_{62}$ with

$$A_{61} = -\big(\mathcal{S}^1 J^\iota \mathcal{B}U_{(\alpha)}, U_{(\alpha)}\big) \quad \text{and} \quad A_{62} = -\big(\mathcal{S}^3 J^\iota \mathcal{B}U_{(\alpha)}, U_{(\alpha)}\big).$$

We can first remark that

$$A_{61} = -\varepsilon\big(\mathit{Ins}[U]J^\iota \mathcal{T}\zeta_{(\alpha)}, \zeta_{(\alpha)}\big) + \varepsilon\big(\frac{1}{\mu}\mathcal{G}J^\iota \mathcal{T}^*\psi_{(\alpha)}, \psi_{(\alpha)}\big).$$

We therefore deduce that $|A_{61}| \leqq \varepsilon \mathfrak{m}^N(U)$ from the third point of Lemma 11 and the following identities,

$$\big|\big(\mathit{Ins}[U]J^\iota \mathcal{T}\zeta_{(\alpha)}, \zeta_{(\alpha)}\big)\big| \\ \leqq \mathfrak{m}^N(U)\big(1 + \varepsilon^2 \underline{\rho}^+ \underline{\rho}^- \sqrt{\mu} \|[\![\underline{V}^\pm]\!]\|_{W^{1,\infty}}^2 |\zeta|_{\langle N+1/2\rangle}^2\big), \tag{5.15}$$

$$\big|\big(\frac{1}{\mu}\mathcal{G}J^\iota \mathcal{T}^*\psi_{(\alpha)}, \psi_{(\alpha)}\big)\big| \leqq \mathfrak{m}^N(U), \tag{5.16}$$

whose proofs are given in Appendix 6.3. The component A_{62} is much easier to handle and one gets readily that $|A_{62}| \leqq \varepsilon \mathfrak{m}^N(U)$.

– *Control of A_7.* One can write

$$A_7 = \varepsilon\big(\mathcal{T}\zeta_{(\alpha)}, \frac{1}{\mathrm{Bo}}\mathcal{K}_{(\alpha)}\zeta_{(\check{\alpha})}\big) - \varepsilon\big(\mathcal{T}^*\psi_{(\alpha)}, \frac{1}{\mu}\mathcal{G}_{(\alpha)}\psi_{(\check{\alpha})}\big).$$

Using the first point of Proposition 4 and the definition of $\mathcal{K}_{(\alpha)}$ to control the first term, and the last point of Proposition 4 and (2.31)–(2.32) to control the second one, we get that $|A_7| \leqq \varepsilon \mathfrak{m}^N(U)$.

– *Control of B_1.* Thanks to the bounds established on R_α and S_α in Proposition 8 and to the third point of Lemma 11, there is no pain in getting $|B_1| \leqq \varepsilon \mathfrak{m}^N(U)$.

– *Control of B_2* From the definition of S'_α, (2.23) and (2.32), we get that

$$|B_2| \lesssim \mathfrak{m}^N(U)\big(1 + |\zeta_{(\alpha)}|_{H^1}\big)$$
$$\lesssim (1 + \sqrt{\mathrm{Bo}})\mathfrak{m}^N(U).$$

Gathering all the information coming from the above estimates, we deduce from (5.13) that for all $1 \le |\alpha| \le N$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{F}^\alpha_\sigma(U^\iota) + (U^\iota_{(\alpha)}, \mathcal{S}^2_\alpha U^\iota_{(\check\alpha)})\big) \lesssim (1 + \sqrt{\mathrm{Bo}})\mathfrak{m}^N(U^\iota). \qquad (5.17)$$

For $\alpha = 0$, we first rewrite the equations (5.8) under the form

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}J^\iota\mathcal{G}_\mu[0]\psi + \varepsilon J^\iota\mathcal{N}_1(U) = 0, \\[2mm] \partial_t\psi + J^\iota\Big(1 - \dfrac{1}{\mathrm{Bo}}\nabla\cdot\Big(\dfrac{1}{\sqrt{1 + \varepsilon^2\mu|\nabla\zeta|^2}}\nabla\bullet\Big)\Big)\zeta + \varepsilon J^\iota\mathcal{N}_2(U) = 0, \end{cases} \qquad (5.18)$$

with

$$\mathcal{N}_1(U) = -\frac{1}{\varepsilon\mu}\big(\mathcal{G}_\mu[\varepsilon\zeta]\psi - \mathcal{G}_\mu[0]\psi\big),$$

$$\mathcal{N}_2(U) = \frac{1}{2}\Big(\llbracket\underline\rho^\pm|\nabla\psi^\pm|^2\rrbracket - \frac{1}{\mu}(1 + \varepsilon^2\mu|\nabla\zeta|^2)\llbracket\underline\rho^\pm(\underline{w}^\pm)^2\rrbracket\Big)$$
$$+ \frac{1}{\varepsilon\mathrm{Bo}}\nabla\cdot(1 - \frac{1}{\sqrt{1 + \varepsilon^2\mu|\nabla\zeta|^2}})\nabla\zeta.$$

We then take the L^2 product of (5.18) with $\big((1 - \frac{1}{\mathrm{Bo}}\Delta)\zeta, \frac{1}{\mu}\mathcal{G}_\mu[0]\psi\big)$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}^0_\sigma(U) \lesssim \varepsilon|\mathcal{N}_1(U)|_{H^1_\sigma}|\zeta|_{H^1_\sigma} + \varepsilon|\mathfrak{P}\mathcal{N}_2(U)|_2|\mathfrak{P}\psi|_2$$
$$\lesssim \varepsilon\mathfrak{m}^N(U), \qquad (5.19)$$

where the control of $\mathcal{N}_1(U)$ and $\mathcal{N}_2(U)$ does not raise any particular difficulty.

Summing (5.17) and (5.19) over all $\alpha \in \mathbb{N}^{d+1}$, first with $|\alpha| \le N - 1$ and then with $|\alpha| \le N$ (and recalling that $\mathcal{S}^2_\alpha = 0$ if $|\alpha| < N$), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{F}^{N-1}_\sigma(U^\iota)\big) \lesssim (1 + \sqrt{\mathrm{Bo}})\mathfrak{m}^N(U^\iota),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\mathcal{F}^N_\sigma(U^\iota) + \sum_{|\alpha|=N}(U^\iota_{(\alpha)}, \mathcal{S}^2_\alpha U^\iota_{(\check\alpha)})\big) \lesssim (1 + \sqrt{\mathrm{Bo}})\mathfrak{m}^N(U^\iota),$$

and therefore, for any constant $\mathfrak{C}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{F}}^N_\sigma(U^\iota) \lesssim (1 + \sqrt{\mathrm{Bo}})\mathfrak{m}^N(U^\iota), \qquad (5.20)$$

with $\widetilde{\mathcal{F}}^N_\sigma(U^\iota) = \mathcal{F}^N_\sigma(U^\iota) + \mathfrak{C}\mathcal{F}^{N-1}_\sigma(U^\iota) + \sum_{|\alpha|=N}(U^\iota_{(\alpha)}, \mathcal{S}^2_\alpha U^\iota_{(\check\alpha)})$.

Now, it follows from the definition of $\mathcal{S}_\alpha^2$ that there exists a constant $c_3(U^\iota) \leqq M$, such that

$$\sum_{|\alpha|=N} |(U_{(\alpha)}^\iota, \mathcal{S}_\alpha^2 U_{\langle\breve\alpha\rangle}^\iota)| \leqq \frac{1}{2}\mathcal{F}_\sigma^N(U^\iota) + c_3(U^\iota)\mathcal{F}_\sigma^{N-1}(U^\iota).$$

As long as $\mathfrak{C} \geqq c_3(U^\iota)$ (and $\mathfrak{C} \leqq M$), one has

$$\frac{1}{2}\mathcal{F}_\sigma^N(U^\iota) \leqq \widetilde{\mathcal{F}}_\sigma^N(U^\iota) \leqq M\mathcal{F}_\sigma^N(U^\iota).$$

With the help of (5.12), this in turn implies that

$$\mathcal{E}_\sigma^N(U^\iota) \leqq 2(M + \frac{2}{\mathfrak{d}(U^\iota)})\widetilde{\mathcal{F}}_\sigma^N(U^\iota); \tag{5.21}$$

it follows, therefore, from (5.20) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{F}}_\sigma^N(U^\iota) \leqq (1 + \sqrt{\mathrm{Bo}})\widetilde{\mathfrak{m}}^N(U^\iota), \tag{5.22}$$

where $\widetilde{\mathfrak{m}}^N(U^\iota) = C\big(M, \widetilde{\mathcal{F}}_\sigma^N(U^\iota), \frac{1}{\mathrm{Bo}}, \frac{1}{\mathfrak{d}(U^\iota)}\big).$

5.5.5. Construction of a Solution for Times of Order $O(1)$ We deduce classically from (5.22) that there exists $T > 0$ as in the statement of the theorem such that $\widetilde{\mathcal{F}}_\sigma^N(U^\iota(t)) \leqq C(T, \widetilde{\mathcal{F}}_\sigma^N(U^0))$ for all $t \in [0, T * (1 + \sqrt{\mathrm{Bo}})^{-1}]$ (we proceed as in Section 5.2 to define the initial energy $\widetilde{\mathcal{F}}_\sigma^N(U^0)$).

Since this time interval does not depend on the mollifying parameter ι, we can use a standard compactness argument (see, for example, [1] for details in a related context) to construct an exact solution $U \in E_{\sigma,T}^N$ to (1.19)—note that we use (5.21) to deduce a bound on $\mathcal{E}_\sigma^N(U)$ from the bound on $\widetilde{\mathcal{F}}_\sigma^N(U)$.

As said in the introduction, the Bond number Bo is generally very large for applications and the existence time of the solution constructed in the previous solution is much shorter than what can be physically observed. This is the reason we paid considerable attention to obtaining energy estimates independent of Bo. In fact, the presence of the prefactor $(1 + \sqrt{\mathrm{Bo}})$ in (5.22) comes only from the control of B_2 in Section 5.5.4. Since B_2 disappears if the energy estimates of Section 5.5.4 are performed on the exact equations (1.19) rather than the mollified ones, one can replace (5.22) for the solution U constructed above by

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\mathcal{F}}_\sigma^N(U) \leqq \widetilde{\mathfrak{m}}^N(U), \tag{5.23}$$

from which we deduce that it is possible to extend the time interval of existence to $[0, T]$. We omit the proof of the uniqueness, which is completely standard.

5.5.6. Proof of Theorem 6   In Section 5.5.4, all the A_j $(1 \leqq j \leqq 7)$ and B_k $(1 \leqq k \leqq 2)$ are bounded from above by $\varepsilon \mathfrak{m}^N(U)$, except A_1 and B_2. Since B_2 is a mollifying error that vanishes for the exact solution, we can focus our attention on A_1, and more specifically on A_{11}, since A_{1l} $(2 \leqq l \leqq 4)$ are bounded from above by $\varepsilon \mathfrak{m}^N(U)$. For A_{11}, we recall that we only have $|A_{11}| \leqq \mathfrak{m}^N(U)$. More precisely, (5.14) yields

$$|A_{11}| \leqq \varepsilon \mathfrak{m}^N(U) + \varepsilon^2 \sqrt{\mu} \underline{\rho}^+ \underline{\rho}^- \mathfrak{e}(\zeta) |[\![\underline{V}^{\pm}]\!]|_\infty |\partial_t [\![\underline{V}^{\pm}]\!]|_\infty \big||D|^{1/2} \zeta_{(\alpha)}\big|_2^2.$$

Using the third point of Lemma 11, the second component of the right-hand side can be bounded from above by $\mathfrak{m}^N(U)$ but not by $\varepsilon \mathfrak{m}^N(U)$. The stronger stability criterion (5.5) allows us to improve this estimate, as shown in the lemma below (whose proof is very similar to the proof of the third point of Lemma 11 and is therefore omitted).

Lemma 12. *Let* $T > 0, t_0 > d/2$ *and* $U = (\zeta, \psi) \in E^N_{\sigma, T}$ *be such that* (2.1) *and* (5.5) *are uniformly satisfied on* $[0, T]$. *Then one has*

$$\varepsilon^2 \sqrt{\mu} \underline{\rho}^+ \underline{\rho}^- \mathfrak{e}(\zeta) \max_{|\alpha| \leqq 1} \big|\partial^\alpha [\![\underline{V}^{\pm}]\!]\big|_\infty^2 |u|^2_{\langle 1/2 \rangle} \leqq \varepsilon^\gamma \mathfrak{m}^1(U) |u|_{H^1_\sigma}.$$

Owing to this lemma, we get $|A_{11}| \leqq \varepsilon^\gamma \mathfrak{m}^N(U)$ and (5.23) can therefore be improved to

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{\mathcal{F}}^N_\sigma(U) \leqq \varepsilon^\gamma \widetilde{\mathfrak{m}}^N(U),$$

from which the theorem follows easily.

6. Applications

Physical implications of the stability criterion (5.1) will be investigated in separate works, but in order to reenforce its relevance, we describe some of them briefly. We also show how Theorem 5 can be used to give a rigorous justification to two-fluid asymptotic models.

In the main, there are two different kinds of applications. The first is provided by air–water interfaces. In this case, stability of interfacial waves is a consequence of the smallness of ρ^- (as in the zero density/zero surface tension limit considered in [50], for instance). The second application concerns internal waves for which both densities are comparable and stability is a consequence of the smallness of the shear velocity at the interface.

6.1. Air–Water Interface in Coastal Oceanography

We are interested here in applying the above theoretical results to typical configurations in coastal oceanography (see ♯1 in the introduction). We want to investigate two things in particular:

1. Can we use Theorem 6 to confirm that the standard models used in coastal oceanography (for example Boussinesq) do not need any correction due to the presence of the air?
2. Can we still neglect the density of the air near the breaking point?

Before addressing these questions, let us give some numerical values for the physical parameters involved. The density of the air (at one atmosphere and around 20 Celsius) is $\rho^- = 1.2\,\text{kg m}^{-3}$. For sea water, the density is $\rho^+ = 1025\,\text{kg m}^{-3}$. These values lead to $\underline{\rho}^- = 0.001$ and $\underline{\rho}^+ = 0.999$. For the air/water interface, the surface tension coefficient is $\sigma = 0.073\,\text{N m}^{-1}$, and we take for the acceleration of gravity $g = 9.81\,\text{m s}^{-2}$.

We also assume here that the air layer has the same height $H^+ = H^- = H$ as the water layer (this is, of course, not realistic, but as said in the introduction, the techniques used here could certainly be generalized to, say, an air layer of infinite depth, and the qualitative behavior should remain the same[18]).

6.1.1. Validity of the One-Fluid Long Wave Models

We are interested in long waves models for which $\varepsilon^+ \sim \mu^+ \ll 1$. It is known [4,20] in the one-fluid case that these models correctly describe the solutions of the water-wave equations for times of order $O(1/\varepsilon)$.

Let us consider first a typical "long wave" over a depth $H^+ = H = 5\,\text{m}$, of wavelength $\lambda = 35\,\text{m}$, and amplitude 0.1 m. As said in Remark 29, in order to show that taking into account the density of the air does not yield significant modifications to the standard one-fluid theory, we have to check that the strong stability criterion (5.5) is satisfied,[19] with $\gamma = 1$. We rather check that its practical version (see Remark 30) is satisfied, which is the case since one computes easily that $\varepsilon^{-2}\Upsilon \sim 4.10^{-4} \ll 1$. The solutions of the one-fluid and two-fluid models exist therefore on the same time scale.

We also know from [4] (one-fluid) and [12] (two-fluid) that these solutions are correctly described by standard long wave models (Boussinesq, KdV, etc.). The influence of the air density and the surface tension on these models being negligible (for instance, surface tension induces a modification of the dispersive term of the KdV model of less than $10^{-4}\%$), this confirms the relevance of the one-fluid models (without surface tension) to describe such phenomena.

6.1.2. Kelvin–Helmholtz Instabilities for Breaking Waves and White Caps

A commonly observed value for ε^+ near the breaking point of the wave is $\varepsilon^+ \sim 0.4$, corresponding to an amplitude $a = 6\,\text{m}$ for a depth $H^+ = 15\,\text{m}$. One computes $\Upsilon \sim 0.27$, which is not small enough to give any definitive conclusion on the presence or absence of Kelvin–Helmholtz instabilities. We therefore have to use the

---

[18] Since $\underline{\rho}^-$ is very small, the influence of H^- on the typical length H is negligible. In our examples, only the third decimal of H is changed if H^- is taken equal to $+\infty$.

[19] For these values, the Bond number is Bo $\sim 1.7.10^8$. A local well-posedness result relying only on the extra control furnished by surface tension, see Section 5.5.5 would therefore give an existence about $\sqrt{\text{Bo}}\varepsilon^{-1} \sim 6.10^5$ times smaller than Theorem 6.

exact criterion (5.1). This requires further experimental work, but it is likely that *for some configurations, Kelvin–Helmholtz instabilities are at the origin of wave breaking*. This might be the case for instance for spilling breakers (as opposed to plunging breakers). It is not understood which kind of singularity of the water-wave equations creates such breakers; this might be because it is a *two-fluid* singularity similar to the "white caps" observed in presence of wind. The occurrence of these white caps is commonly explained by the *linear* Kelvin criterion stated in the introduction. It would be interesting to check whether a nonlinear version of our *nonlinear* stability criterion (5.1) with nonzero background explains part of the discrepancies of this theory (see [13]).

6.2. Internal Waves

Waves at the interface of two immiscible fluids of comparable density are called internal waves. Waves at the interface of two layers of water of different density also carry the same name and are often described using the two-fluid formalism (see the recent review [30]). The stability criterion (5.1) is discussed here in these two configurations.

6.2.1. The Koop–Butler Experiment We consider a classical experiment by KOOP and BUTLER [37]. The upper fluid is deionized water ($\rho^- = 998$ kg m^{-3}) and the lower fluid is Freon TF ($\rho^+ = 1563$ kg m^{-3}). The depths of both layers in this experiment are $H^+ = 1.366$ cm and $H^- = 6.948$ cm. One gets therefore $\underline{\rho}^+ = 0.610$, $\underline{\rho}^- = 0.390$ and $H = 1.989$ cm.

The interfacial tension coefficient σ is not provided in [37], but the chemistry literature suggests that $\sigma = 0.005$ N m^{-1} is a reasonable value. Koop and Butler observed different interfacial waves of amplitude ranging from $a = 0.034$ cm to $a = 0.68$ cm. The dimensionless parameter Υ, therefore, satisfies $5.39 \times 10^{-7} \leqq \Upsilon \leqq 0.086$. According to the practical criterion (5.3), these correspond to stable configurations. Though the practical version of the strong stability criterion (5.5) (see Remark 30) is satisfied, we do not discuss large time existence here because the viscosity of Freon TF should then be taken into account.

6.2.2. The Grue et al. Experiment and Oceanic Internal Waves The configuration studied in [28] is slightly different from the experiment mentioned in the previous section. Indeed, the authors do not consider two immiscible fluids, but brine $\rho^+ = 1022$ kg m^{-3} and water $\rho^- = 999$ kg m^{-3} ($\underline{\rho}^+ = 0.506$, $\underline{\rho}^- = 0.494$). This configuration is thus closer to what is observed for oceanic internal waves.

The difference with the case of two immiscible fluids is that there is a thin zone (pycnocline) in which the density is allowed to vary continuously between ρ^+ and ρ^-. This continuous stratification is known to have a stabilizing role, which is important for small amplitude interfacial waves.

It seems, therefore, that the physical framework studied in the present article differs from [28]. However, water-brine interfaces have been described with great success with models based the two-fluid equations (1.3)–(1.8) (see [30] and references therein). It follows from our analysis that surface tension must be included

in these equations if we want to use them to describe water-brine interfaces. This surface tension is somehow artificial, so a natural question is to know which value it should be given.

We propose here to use our analysis and the experiments of [28] to sketch a method to find a realistic value for σ. The main idea is that *its value should be such that the Kelvin–Helmholtz instabilities predicted by the theoretical analysis should coincide with those observed in the experiments.*

In [28], the depth of the two layers are respectively $H^+ = 0.62$ m and $H^- = 0.15$ m, so that $H = 0.243$ m.

The authors report a set of measurements (Fig. 7 in [28]) for interfacial waves of amplitude ranging from $a = 0.033$ m to $a = 0.226$ m. Kelvin–Helmholtz instabilities are not observed up to $a = 0.184$ m but are present for the experiment with largest amplitude $a = 0.226$ m. It is therefore reasonable to assume that the critical amplitude is $a = 0.2$ m. According to the practical criterion (5.3), this critical value should correspond to $\Upsilon \sim 1$. We can then use this relation to get $\sigma = 0.095$ N m^{-1}; this value is comparable to the air–water surface tension, but slightly higher, which is not surprising since the stabilizing effects of the continuous stratification are taken into account with such a choice. Further investigation is required to check whether this approach can prove useful in the study of oceanic internal waves.

6.2.3. Formula for the Maximal Amplitude of Interfacial Waves   The experiment of [28] cited above suggests that there is a critical amplitude for progressive waves above which their propagation is made impossible by the appearance of Kelvin–Helmholtz instability. We propose here a simple formula for this maximal amplitude. Assuming that the appearance of Kelvin–Helmholtz instabilities corresponds to $\Upsilon \sim 1$ (which is supported by our mathematical analysis and the above experimental comparisons), we find

$$a^4 = \frac{4\sigma H^2}{(\underline{\rho}^+ \underline{\rho}^-)^2 (\rho^+ + \rho^-) g'}. \tag{6.1}$$

For the Koop–Butler experiment, this formula gives $a \sim 1$cm, which is larger than the amplitude of the wave observed. This is in accordance with the fact that these waves do not exhibit Kelvin–Helmholtz instabilities.

For a wave (air water interface) propagating over a depth $H \sim 10$m, we find $a \sim 7.5m$; wave breaking is therefore likely to occur before Kelvin–Helmholtz instabilities.

6.3. Full Justification of Two-Fluid Asymptotic Models

We can use Theorem 6 to provide a rigorous justification of asymptotic models for internal waves, on the relevant time scale (to our knowledge, the only rigorous result so far is a justification of the Benjamin–Ono equations on the too-short time scale $O(\text{Bo}^{-1/2})$ that can be found in [47]):

Many asymptotic regimes and models exist in the literature ([16,21,46] and references therein). Theorem 6 can be used to justify rigorously most of them along the procedure used in [4] for water-wave models.

- (Consistency) One proves that any family of smooth enough, uniformly bounded, solutions to (1.19) existing on the relevant time scale solves the asymptotic model up to a small residual. This is done for a wide class of regimes in [12].
- (Convergence) One proves that such exact solutions to the two-fluid equations (1.19) remain close to exact solutions of the asymptotic model. From the previous step, this only requires energy estimates on the asymptotic model (generally much easier than for the full two-fluid equations).
- (Existence) One proves that smooth, uniformly bounded family of solutions to (1.19) whose existence is assumed in the previous steps exist indeed. As in the water-wave case, this is the most difficult step. It is here ensured by Theorem 6 if the stability criterion (5.5) is satisfied (which is the case for the stable waves of the above experimental observations).

Let us for instance implement this procedure in the "shallow-water/shallow-water" regime characterized by $\varepsilon \sim 1$, $\mu \ll 1$. In this regime, the interface elevation ζ and $v = \nabla \psi$ are commonly described [16,21] by the solution (ζ^a, v^a) of the following system (for notational simplicity, we give it for $d = 1$, but there is no additional difficulty to take into account the nonlocal generalization of this system to the case $d = 2$ derived in [12] and studied in [29]),

$$
\begin{cases}
\partial_t \zeta^a + \partial_x \left[\dfrac{h^-(\zeta^a)h^+(\zeta^a)}{\underline{\rho}^+ h^-(\zeta^a) + \underline{\rho}^- h^+(\zeta^a)} v^a \right] = 0 \\
\partial_t v^a + \partial_x \zeta^a + \varepsilon \dfrac{1}{2} \partial_x \left[\dfrac{\underline{\rho}^+ h^-(\zeta^a)^2 - \underline{\rho}^- h^+(\zeta^a)^2}{[\underline{\rho}^+ h^-(\zeta^a) + \underline{\rho}^- h^+(\zeta^a)]^2} (v^a)^2 \right] = 0
\end{cases}
\tag{6.2}
$$

with $h^-(\zeta^a) = \underline{H}^-(1 - \varepsilon^- \zeta^a)$ and $h^+(\zeta^a) = \underline{H}^+(1 + \varepsilon^+ \zeta^a)$. We justify the approximation furnished by the solutions of this system in the following sense.

Theorem 7. *Let $U^0 = (\zeta^0, \psi^0)^T$ be as in the assumptions of Theorem 5, and assume that (5.1) is satisfied uniformly with respect to $\mu \in (0, 1)$. Let also $v^0 = \partial_x \psi^0$ and assume that*

$$
\inf_{\mathbb{R}^d} \left(1 - \varepsilon^2 \underline{\rho}^+ \underline{\rho}^- \frac{(\underline{H}^+ + \underline{H}^-)^2}{[\rho^+ h^-(\zeta^0) + \rho^- h^+(\zeta^0)]^3} (v^0)^2 \right) > 0
\tag{6.3}
$$

Then there exists $T > 0$ such that for all $\mu \in (0, 1)$:

(i) *there exists a unique solution $(\zeta^a, v^a) \in C([0, T]; H^{N-1/2}(\mathbb{R}^2))$ to (6.2) with initial condition (ζ^0, v^0);*

(ii) *the solution U provided by Theorem 5 exists on $[0, T]$. Moreover, with $v = \partial_x \psi$, one has*

$$
|(\zeta, v) - (\zeta^a, v^a)|_{L^\infty([0,T] \times \mathbb{R}^d)} \leqq \mu C(U^0).
$$

Remark 31. Condition (6.3) ensures the hyperbolicity of (6.2); it is not present for the standard one-fluid shallow-water equations. An asymptotic expansion of the instability operator $Ins[U]$ shows that the term responsible for the Kelvin–Helmholtz instability (the one involving $\mathcal{E}[\zeta]$) contributes to (6.3), which was suspected

but not rigorously established so far. The shallow-water/shallow-water model (6.2), however, is certainly too unstable and the analysis performed here on the full Euler equations shows that the surface tension term should be kept to control high frequency instabilities.

Proof. For the first point, we remark that $|\zeta^0|_{H^{N-1/2}} + |v^0|_{H^{N-1/2}} \leqq \mathfrak{m}^N(U)$; the existence of a solution $(\zeta^a, v^a) \in C([0, T]; H^{N-1/2}(\mathbb{R})^2)$ to (6.2) with initial condition (ζ^0, v^0) (with $T = T(\mathfrak{m}^N(U)))$ is straightforward because the assumption made in the statement of the theorem insures the hyperbolicity of (6.2), as shown in Theorem 1 of [29].

Taking a smaller T if necessary, it is possible to assume that the solution of Theorem 5 exists also on $[0, T]$; this theorem ensures also that $\sup_{[0,T]} \mathcal{E}_\sigma^N(U)$ is uniformly bounded with respect to $\mu \in (0, 1)$. Thanks to the consistency result provided by Theorem 4 of [12], this ensures that (ζ, v) solves (6.2) up to a residual $\mu(r_1^\mu, r_2^\mu)$, with (r_1^μ, r_2^μ) uniformly bounded with respect to $\mu \in (0, 1)$ in $C([0, T]; H^1(\mathbb{R})^2)$. A standard error estimate on the hyperbolic system (6.2) yields therefore

$$|(\zeta, v) - (\zeta^a, v^a)|_{L^\infty([0,T]; H^1(\mathbb{R})^2} \leqq \mu C(U^0),$$

and the estimate given in the theorem follows from the continuous embedding $H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$. □

Appendix A: Nondimensionalization of the Equations

It is quite straightforward to nondimensionalize the space variable X and the interface deformation ζ using the quantities $a, \lambda, H^\pm$ and H introduced in Section 1.3. We thus define

$$\widetilde{X} = \frac{X}{\lambda}, \qquad \widetilde{\zeta} = \frac{\zeta}{a},$$

where the tildes are used to denote dimensionless variables or unknowns. The non-dimensionalization of the time variable t and the velocity potential ψ requires the knowledge of a reference velocity c. Such a value can be obtained by direct observation, but it is preferable to have a formula for c in terms of the above quantities. The best way to have access to such information is through the *linear* analysis of the equations. Neglecting all the nonlinear terms in (1.16), one easily gets the system (see Remark 15 for the expression of $\mathcal{G}[0]$)

$$\begin{cases} \partial_t \zeta - \mathcal{G}[0]\psi = 0, \\ \partial_t \psi + g'\zeta = 0, \end{cases} \text{ with } \mathcal{G}[0] = |D| \frac{\tanh(H^+|D|)\tanh(H^-|D|)}{\underline{\rho}^+ \tanh(H^-|D|) + \underline{\rho}^- \tanh(H^+|D|)}.$$

In the *shallow water limit*, that is, when $H^\pm$ is small compared to the typical wavelength λ, one has $\mathcal{G}[0] \sim -\underline{H}\Delta$ (it is possible, but not necessary at this point, to give a precise meaning to the symbol $\sim$). The above linearized equation then reduces to a wave equation of speed c, with

$$c^2 = g' \frac{H^+ H^-}{\underline{\rho}^+ H^- + \underline{\rho}^- H^+} = g'H.$$

We then use this velocity to nondimensionalize t and ψ,

$$\widetilde{t} = \frac{t}{\lambda/c}, \qquad \widetilde{\psi}(\widetilde{X}) = \frac{\psi(X)}{\psi_0} \quad \text{with} \quad \psi_0 = \frac{g'a\lambda}{c}.$$

(the nondimensionalization of ψ is obtained via the equation $\partial_t \psi + g'\zeta = 0$).

We now remark that

$$\mathcal{G}^\pm[\zeta]\psi = \frac{\psi_0}{H} \mathcal{G}^\pm_\mu[\varepsilon\widetilde{\zeta}, \underline{H}^\pm]\widetilde{\psi} \quad \text{and thus} \quad \mathcal{G}[\zeta]\psi = \frac{\psi_0}{H} \mathcal{G}_\mu[\varepsilon\widetilde{\zeta}]\widetilde{\psi},$$

where $\mathcal{G}^\pm_\mu$ and $\mathcal{G}_\mu$ are as defined in (2.15) and (1.18). Since moreover

$$\nabla\psi^\pm = \frac{\psi_0}{\lambda}\widetilde{\nabla}\widetilde{\psi}^\pm,$$

it is straightforward to deduce (1.19) from (1.16).

Appendix B: Proof of (2.19), (2.23), (2.33), (2.34), (2.35) and (2.36)

Before proving these identities, we give some remarks of general interest:

- The following classical product estimate holds

$$\forall s \geqq 0 \quad \text{and} \quad t_0 > d/2, \qquad |fg|_{H^s} \lesssim |f|_{H^{s \vee t_0}}|g|_{H^s}. \qquad \text{(B.1)}$$

- For functions defined on the strip $\mathcal{S}^\pm$, the above formula can be used in the horizontal directions to obtain

$$\forall s \geqq 0 \quad \text{and} \quad t_0 > d/2, \qquad \|\Lambda^s(fg)\|_2 \lesssim \|f\|_{L^\infty_z H^{s \vee t_0}}\|\Lambda^s g\|_2. \qquad \text{(B.2)}$$

- For all $k \in \mathbb{N}$, we denote by $H^{s,k}(\mathcal{S})$ the space

$$H^{s,k}(\mathcal{S}^\pm) = \{f \in L^2(\mathcal{S}^\pm), \|f\|_{H^{s,k}} < \infty\}, \qquad \|f\|^2_{H^{s,k}} = \sum_{j=0}^{k} \|\Lambda^{s-k}\partial_z^k u\|_2^2.$$

$$\text{(B.3)}$$

- The space $H^{s+1/2,1}(\mathcal{S}^\pm)$ is continuously embedded in $L^\infty_z H^s(\mathcal{S}^\pm)$ (this is a variant of the trace lemma),

$$\forall s \geqq 0, \qquad \|f\|_{L^\infty_z H^s} \lesssim \|f\|_{H^{s+1/2,1}}. \qquad \text{(B.4)}$$

Finally, let us state the following lemma, which will be used several times.

Lemma 13. *Let $t_0 > d/2$ and $u \in \dot{H}^1(\mathcal{S}^\pm)$, $\mathbf{g} \in H^{s,1}(\mathcal{S}^\pm)^{d+1}$ be such that*

$$\begin{cases} \nabla^{\mu^\pm} \cdot P(\Sigma^\pm)\nabla^{\mu^\pm} u = \nabla^{\mu^\pm} \cdot \mathbf{g}, \\ u_{|z=0} = 0, \qquad \partial_n u_{|z=\mp 1} = -\mathbf{e}_z \cdot \mathbf{g}_{|z=\mp 1}. \end{cases}$$

Then, for all $0 \leqq s \leqq t_0 + 1$,

$$\|\Lambda^s \nabla^{\mu^\pm} u\|_2 \leqq M\|\Lambda^s \mathbf{g}\|_2 \quad \text{and} \quad \|\nabla^{\mu^\pm} u\|_{H^{s,1}} \leqq M\big(\|\Lambda^s \mathbf{g}\|_2 + \|\Lambda^{s-1}\partial_z \mathbf{g} \cdot \mathbf{e}_z\|_2\big).$$

Proof. The first estimate is a consequence of Proposition 2.4 of [4]. For the second estimate, and since $\mu^{\pm}$ is assumed to be bounded, it is enough to prove that $\|\Lambda^{s-1}\partial_z^2 u\|_2 \leqq M\|\mathbf{g}\|_{H^{s,1}}$. This follows immediately from isolating the component $\partial_z^2 u$ in the equation $\nabla^{\mu^{\pm}} \cdot P\nabla^{\mu^{\pm}} u = \nabla^{\mu^{\pm}} \cdot \mathbf{g}$ and using the first estimate to control the terms involving u. □

B.1. Proof of (2.19), (2.23) and (2.24)

B.1.1. Proof of (2.19) The case $s = 0$ being proved in Prop. 3.4 and Eq. (3.3) of [4], we focus on the case $s > 0$. Remarking first that $|\mathfrak{P}\psi^{\pm}|_{H^s} = |\mathfrak{P}(\Lambda^s\psi^{\pm})|_2$, we can deduce from (2.19) with $s = 0$ that

$$\sqrt{\mu}|\mathfrak{P}\psi^{\pm}|_{H^s} \leqq M\|\nabla^{\mu^{\pm}}(\Lambda^s\psi^{\pm})^{\flat}\|$$
$$\leqq M\big(\|\nabla^{\mu^{\pm}}((\Lambda^s\psi^{\pm})^{\flat} - \Lambda^s\phi^{\pm})\|_2 + \|\Lambda^s\nabla^{\mu^{\pm}}\phi^{\pm}\|_2,$$

where $(\Lambda^s\psi^{\pm})^{\flat}$ stands for the solution to (2.17) with Dirichlet data $\Lambda^s\psi^{\pm}$. Remarking that $u = (\Lambda^s\psi^{\pm})^{\flat} - \Lambda^s\phi^{\pm}$ is as in Lemma 13 with $\mathbf{g} = [\Lambda^s, P^{\pm}]\nabla^{\mu^{\pm}}\phi^{\pm}$, we have

$$\|\nabla^{\mu^{\pm}}((\Lambda^s\psi^{\pm})^{\flat} - \Lambda^s\phi^{\pm})\|_2 \leqq \|\mathbf{g}\|_2$$
$$\leqq M\|\Lambda^{s-1}\nabla^{\mu^{\pm}}\phi^{\pm}\|_2, \qquad (B.5)$$

where we used the commutator estimate (2.29) to derive the second inequality. The end of the proof is then straightforward.

B.1.2. Proof of (2.23) Integrating by parts in (2.17), one easily gets

$$\int_{\mathbb{R}^d} \Lambda^s \mathcal{G}^{\pm}\psi_1 \Lambda^s\psi_2 = \int_{\mathcal{S}^{\pm}} \Lambda^s(P(\Sigma^{\pm})\nabla^{\mu^{\pm}}\phi_1^{\pm}) \cdot \Lambda^s\nabla^{\mu^{\pm}}\psi_2^{\dagger}, \qquad (B.6)$$

where $\phi_1^{\pm}$ denotes the solution to (2.17) with Dirichlet data ψ_1, while

$$\psi_2^{\dagger}(\cdot, z) = \chi(\sqrt{\mu^{\pm}}z|D|)\psi_2,$$

with χ a smooth compactly supported function equal to 1 in a neighborhood of the origin. One then gets

$$\left|(\Lambda^s \mathcal{G}^{\pm}\psi_1, \Lambda^s\psi_2)\right| \leqq \|\Lambda^s(P(\Sigma^{\pm})\nabla^{\mu^{\pm}}\phi_1^{\pm})\|_2\|\Lambda^s\nabla^{\mu^{\pm}}\psi_2^{\dagger}\|_2.$$

From (B.2) one gets, for all $0 \leqq s \leqq t_0 + 1$,

$$\|\Lambda^s(P(\Sigma^{\pm})\nabla^{\mu^{\pm}}\phi_1^{\pm})\|_2 \lesssim (1 + \|P(\Sigma^{\pm}) - \mathrm{Id}\,\|_{L_z^{\infty}H^{t_0+1}})\|\Lambda^s\nabla^{\mu^{\pm}}\phi_1^{\pm}\|_2$$
$$\leqq M\|\Lambda^s\nabla^{\mu^{\pm}}\phi_1^{\pm}\|_2,$$

where the second identity follows from (2.7); (2.23) then follows from (2.18) and the observation that $\|\Lambda^s\nabla^{\mu^{\pm}}\psi_2^{\dagger}\|_2 \lesssim \sqrt{\mu}|\mathfrak{P}\psi_2|_{H^s}$ (Proposition 2.2 of [4]).

B.1.3. Proof of (2.24)   Let us first prove (2.24) in the case $1 \leqq s \leqq t_0 + 1$. With the same notations as in Section 6.3, one gets

$$
\begin{aligned}
([\Lambda^s, \mathcal{G}^{\pm}]\psi_1, \Lambda^s \psi_2) &= \int_{\mathcal{S}^{\pm}} [\Lambda^s, P(\Sigma^{\pm})] \nabla^{\mu^{\pm}} \psi_1^{\flat} \cdot \Lambda^s \nabla^{\mu^{\pm}} \psi_2^{\dagger} \\
&\quad + \int_{\mathcal{S}^{\pm}} P(\Sigma^{\pm}) \nabla^{\mu^{\pm}} [\Lambda^s \psi_1^{\flat} - (\Lambda^s \psi_1)^{\flat}] \cdot \Lambda^s \nabla^{\mu^{\pm}} \psi_2^{\dagger}.
\end{aligned}
$$

Using the Cauchy–Schwarz inequality, (2.29) (for the first component of the right-hand side) and (B.5) (for the second component) we get

$$
\left| ([\Lambda^s, \mathcal{G}^{\pm}]\psi_1, \Lambda^s \psi_2) \right| \leqq M \| \Lambda^{s-1} \nabla^{\mu^{\pm}} \psi_1^{\flat} \|_2 \| \Lambda^s \nabla^{\mu^{\pm}} \psi_2^{\dagger} \|_2.
$$

Since $s - 1 \geqq 0$, (2.18) and the observation that $\| \Lambda^s \nabla^{\mu^{\pm}} \psi_2^{\dagger} \|_2 \lesssim \sqrt{\mu} |\mathfrak{P} \psi_2|_{H^s}$ yield the result.

Let us now prove (2.24) for $0 \leqq s \leqq 1$. Let us write first that

$$
\begin{aligned}
&([\Lambda^s, \mathcal{G}^{\pm}]\psi_1, \Lambda^s \psi_2) \\
&= -(\Lambda^s [\Lambda, \mathcal{G}^{\pm}]\Lambda^{-1} \psi_1, \Lambda^s \psi_2) + ([\Lambda^{s+1}, \mathcal{G}^{\pm}]\Lambda^{-1} \psi_1, \Lambda^s \psi_2).
\end{aligned}
$$

Since $s + 1 \geqq 1$, we can use the computations above to show that the second term of the right-hand side satisfies the desired estimate. We thus focus on the first term. Proceeding as above, we write

$$
\begin{aligned}
(\Lambda^s [\Lambda, \mathcal{G}^{\pm}]\Lambda^{-1} \psi_1, \Lambda^s \psi_2) &= \int_{\mathcal{S}^{\pm}} \Lambda^s [\Lambda, P(\Sigma^{\pm})] \nabla^{\mu^{\pm}} (\Lambda^{-1} \psi_1)^{\flat} \cdot \Lambda^s \nabla^{\mu^{\pm}} \psi_2^{\dagger} \\
&\quad + \int_{\mathcal{S}^{\pm}} \Lambda^s \big(P(\Sigma^{\pm}) \nabla^{\mu^{\pm}} [\Lambda (\Lambda^{-1} \psi_1)^{\flat} - \psi_1^{\flat}] \big) \cdot \Lambda^s \nabla^{\mu^{\pm}} \psi_2^{\dagger}.
\end{aligned}
$$

Since $\| \Lambda^s [\Lambda, P(\Sigma^{\pm})] \|_{H^{s,0} \to L^2(\mathcal{S}^{\pm})} + \| \Lambda^s \big(P(\Sigma^{\pm}) \cdot \big) \|_{H^{s,0} \to L^2(\mathcal{S}^{\pm})} \leqq M$ (recall that $0 \leqq s \leqq 1$), we deduce (proceeding as for (B.5) for the second component) that

$$
\left| (\Lambda^s [\Lambda, \mathcal{G}^{\pm}]\Lambda^{-1} \psi_1, \Lambda^s \psi_2) \right| \leqq M \| \Lambda^s \nabla^{\mu^{\pm}} (\Lambda^{-1} \psi_1)^{\flat} \|_2 \| \Lambda^s \nabla^{\mu^{\pm}} \psi_2^{\dagger} \|_2,
$$

and the result follows as for the case $s \geqq 1$.

B.2. Proof of (2.33), (2.34), (2.35) and (2.36)

Throughout this proof, we assume that $\Sigma^{\pm}$ is a regularizing diffeomorphism of the form $\Sigma^{\pm}(X, z) = (X, z + \sigma^{\pm}(X, z))$ with $\sigma^{\pm}$ as in Example 2. We denote by $d^j P^{\pm}(\mathbf{h})$ ($j \in \mathbb{N}$) the jth derivative of the mapping $\zeta \mapsto P(\Sigma^{\pm})$ in the direction $\mathbf{h} = (h_1, \ldots, h_j)$, and by $d^j \phi(\mathbf{h})$ the j-th derivative in the direction $\mathbf{h}$ of $\zeta \mapsto \phi$, where ϕ solves (2.17).

B.2.1. Proof of (2.33) With the same notations as in Section 6.3, one has

$$(\Lambda^{s-1/2} \mathcal{G}^{\pm} \psi, \varphi) = \int_{\mathcal{S}^+} \Lambda^s (P(\Sigma^{\pm}) \nabla^{\mu^{\pm}} \phi^{\pm}) \cdot \Lambda^{-1/2} \nabla^{\mu^{\pm}} \varphi^{\dagger};$$

since $\|\Lambda^{-1/2} \nabla^{\mu^{\pm}} \varphi^{\dagger}\|_2 \lesssim \mu^{1/4} |\varphi|_2$, it follows after differentiating this identity with respect to ζ and by a duality argument that for all $0 \leq s \leq t_0 + 1$, one has

$$|d^j \mathcal{G}^{\pm}(\mathbf{h}) \psi|_{H^{s-1/2}} \lesssim \mu^{1/4} \sum \|\Lambda^s d^{j_1} P^{\pm}(\mathbf{h}_I) \nabla^{\mu^{\pm}} d^{j_2} \phi(\mathbf{h}_{II})\|_2,$$

where the summation is taken over all integers j_1 and j_2 such that $j_1 + j_2 = j$ and on all the j_1 and j_2-uplets $\mathbf{h}_I$ and $\mathbf{h}_{II}$ whose coordinates form a permutation of the coordinates of $\mathbf{h}$. The result follows therefore from the estimate, for all $0 \leq s \leq t_0 + 1/2$ (and using Notation 3),

$$\|\Lambda^s d^{j_1} P^{\pm}(\mathbf{h}_I) \nabla^{\mu^{\pm}} d^{j_2} \phi(\mathbf{h}_{II})\|_2 \leq M \varepsilon^j \sqrt{\mu} |h_l|_{H^{s+1/2}} |\check{\mathbf{h}}_l|_{H^{s \vee t_0 + 3/2}} |\mathfrak{P} \psi|_{H^{s \vee t_0 + 1/2}};$$

the rest of this section is thus devoted to the proof of (B.7). We first state three lemmas and use them to prove the result, and then we prove the lemmas.

Lemma 14. *For all $0 \leq s \leq t_0 + 1, k \in \mathbb{N}^*$, $\mathbf{h} = (h_1, \ldots, h_k) \in H^{s \vee t_0 + 1}(\mathbb{R}^d)^k$ and $1 \leq l \leq k$, one has*

$$\|\Lambda^s d^k P^{\pm}(\mathbf{h})\|_2 \leq M \varepsilon^k |h_l|_{H^{s+1/2}} |\check{\mathbf{h}}_l|_{H^{s \vee t_0 + 1}}.$$

Lemma 15. *For all $0 \leq s \leq t_0 + 1/2, k \in \mathbb{N}^*$, $\mathbf{h} = (h_1, \ldots, h_k) \in H^{t_0 + 2}(\mathbb{R}^d)^k$ and $\psi \in \dot{H}^{s \vee t_0 + 1}(\mathbb{R}^d)$. Then $d^k \phi(\mathbf{h})$ satisfies*

$$\|\nabla^{\mu^{\pm}} d^k \phi(\mathbf{h})\|_{H^{s \vee t_0 + 1/2, 1}} \leq M \varepsilon^k \sqrt{\mu} |\mathbf{h}|_{H^{s \vee t_0 + 3/2}} |\mathfrak{P} \psi|_{H^{s \vee t_0 + 1/2}}.$$

Lemma 16. *Under the assumptions of Lemma 15, one also has*

$$\|\Lambda^s \nabla^{\mu^{\pm}} d^k \phi(\mathbf{h})\|_2 \leq M \varepsilon^k \sqrt{\mu} |h_l|_{H^{s+1/2}} |\check{\mathbf{h}}_l|_{H^{s \vee t_0 + 3/2}} |\mathfrak{P} \psi|_{H^{s \vee t_0 + 1/2}}.$$

Let us assume first that $h_l \in \mathbf{h}_I$. Then, from (B.1) and (B.4), one deduces

$$\|\Lambda^s d^{j_1} P^{\pm}(\mathbf{h}_I) \nabla^{\mu^{\pm}} d^{j_2} \phi(\mathbf{h}_{II})\|_2 \lesssim \|\Lambda^s d^{j_1} P^{\pm}(\mathbf{h}_I)\|_2 \|\nabla^{\mu^{\pm}} d^{j_2} \phi(\mathbf{h}_{II})\|_{H^{s \vee t_0 + 1/2, 1}},$$

and (B.7) easily follows from the first two lemmas.

Now, if $h_l \in \mathbf{h}_{II}$ then we write, rather,

$$\|\Lambda^s d^{j_1} P^{\pm}(\mathbf{h}_I) \nabla^{\mu^{\pm}} d^{j_2} \phi(\mathbf{h}_{II})\|_2 \lesssim \|d^{j_1} P^{\pm}(\mathbf{h})\|_{L^{\infty} H^{s \vee t_0}} \|\Lambda^s \nabla^{\mu^{\pm}} d^{j_2} \phi(\mathbf{h}_{II})\|_2,$$

and Lemma 16 gives (B.7) (the bound on $\|d^k P^{\pm}(\mathbf{h})\|_{L^{\infty} H^{s \vee t_0}}$ being a straightforward consequence of (B.7) below).

Proof of Lemma 14. Computing explicitly the matrix $P(\Sigma^{\pm})$ given by (2.5), one obtains

$$P(\Sigma^{\pm}) = \begin{pmatrix} \mathrm{Id}_{d \times d} + \partial_z \sigma^{\pm} & -\sqrt{\mu} \nabla \sigma^{\pm} \\ -\sqrt{\mu} (\nabla \sigma^{\pm})^T & \dfrac{1 + \mu |\nabla \sigma^{\pm}|^2}{1 + \partial_z \sigma^{\pm}}; \end{pmatrix}. \tag{B.7}$$

Since $\zeta \mapsto \sigma^{\pm}$ is linear, it is easy to deduce the result from (B.1) and the regularizing properties of $\sigma^{\pm}$ as given in Example 2. $\square$

Proof of Lemma 15. By differentiating (2.17) k times with respect to ζ in the direction $\mathbf{h}$, one gets

$$\begin{cases} \nabla^{\mu\pm} \cdot P(\Sigma^\pm)\nabla^{\mu\pm}d^k\phi^\pm(\mathbf{h}) = \nabla^{\mu\pm} \cdot \mathbf{g}, \\ d^k\phi^\pm(\mathbf{h})_{|z=0} = 0, \qquad \partial_n d^k\phi^\pm(\mathbf{h})_{|z=\mp 1} = -\mathbf{e}_z \cdot \mathbf{g}_{|z=\mp 1}, \end{cases} \tag{B.8}$$

where $\mathbf{g}$ is a sum of terms of the form

$$d^{k_1}P^\pm(\mathbf{h}_I)\nabla^{\mu\pm}d^{k_2}\phi^\pm(\mathbf{h}_{II}) =: A(\mathbf{h}_I, \mathbf{h}_{II}), \tag{B.9}$$

where $k_1, k_2 \in \mathbb{N}$, $k_2 < k$, $k_1 + k_2 = k$, and $\mathbf{h}_I$ and $\mathbf{h}_{II}$ are respectively k_1 and k_2-uplets whose coordinates form a permutation of the coordinates of $\mathbf{h}$. Since by Lemma 13 we have

$$\|\nabla^{\mu\pm}d^k\phi^\pm(\mathbf{h})\|_{H^{s\vee t_0+1/2,1}} \leqq M\|\mathbf{g}\|_{H^{s\vee t_0+1/2,1}},$$

we can deduce the result from the fact that

$$\|A(\mathbf{h}_I, \mathbf{h}_{II})\|_{H^{s\vee t_0+1/2,1}} \leqq M\varepsilon^k\sqrt{\mu}|\mathbf{h}|_{H^{s\vee t_0+3/2}}|\mathfrak{P}\psi|_{H^{t_0+1}}, \tag{B.10}$$

which we now turn to prove. From (B.9) and (B.2), one gets

$$\|A(\mathbf{h}_I, \mathbf{h}_{II})\|_{H^{s\vee t_0+1/2,1}} \lesssim \|\nabla^{\mu\pm}d^{k_2}\phi(\mathbf{h}_{II})\|_{H^{s\vee t_0+1/2,1}}$$
$$\times (\|d^{k_1}P^\pm(\mathbf{h}_I)\|_{L_z^\infty H^{s\vee t_0+1/2}} + \|\partial_z d^{k_1}P^\pm(\mathbf{h}_I)\|_{L_z^\infty H^{t_0}})$$

and therefore, using (B.7) and the fact that $\zeta \mapsto \sigma^\pm[\zeta]$ s linear,

$$\|A(\mathbf{h}_I, \mathbf{h}_{II})\|_{H^{s\vee t_0+1/2,1}} \leqq M\varepsilon^{k_1}|\mathbf{h}_I|_{H^{s\vee t_0+3/2}}\|\nabla^{\mu\pm}d^{k_2}\phi(\mathbf{h}_{II})\|_{H^{s\vee t_0+1/2,1}}.$$

Since $k_2 < k$, one can deduce (B.10) by a simple induction on k (the case $k = 0$ being a consequence of (2.18)). □

Proof of Lemma 16. Proceeding as in the proof of Lemma 15, we can check that the result follows from

$$\|\Lambda^s A(\mathbf{h}_I, \mathbf{h}_{II})\|_2 \leqq M\varepsilon^k\sqrt{\mu}|h_l|_{H^{s+1/2}}|\check{\mathbf{h}}_l|_{H^{s\vee t_0+3/2}}|\mathfrak{P}\psi|_{H^{s\vee t_0+1/2}}, \tag{B.11}$$

that we now turn to prove. We have to distinguish two cases:

– If h_l belongs to $\mathbf{h}_I$ we use (B.2) and (B.4) to write

$$\|\Lambda^s A(\mathbf{h}_I, \mathbf{h}_{II})\|_2 \leqq \|\Lambda^s d^{k_1}P^\pm(\mathbf{h}_I)\|_2\|\nabla^{\mu\pm}d^{k_2}\phi(\mathbf{h}_{II})\|_{L_z^\infty H^{s\vee t_0}}$$
$$\leqq \|\Lambda^s d^{k_1}P^\pm(\mathbf{h}_I)\|_2\|\nabla^{\mu\pm}d^{k_2}\phi(\mathbf{h}_{II})\|_{H^{s\vee t_0+1/2,1}},$$

and (B.11) follows directly from Lemmas 14 and 15.
– If h_l belongs to $\mathbf{h}_{II}$, we rather write

$$\|\Lambda^s A(\mathbf{h}_I, \mathbf{h}_{II})\|_2 \lesssim \|d^{k_1}P^\pm(\mathbf{h}_I)\|_{L_z^\infty H^{s\vee t_0}}\|\Lambda^s\nabla^{\mu\pm}d^{k_2}\phi(\mathbf{h}_{II})\|_2,$$

and (B.11) follows easily by induction as in the proof of Lemma 15. □

B.2.2. Proof of (2.34) With the same notations as in Section 6.3, one has

$$(\Lambda^{s-1/2}\mathcal{G}^{\pm}\psi, \varphi) = \int_{\mathcal{S}^+} \Lambda^{s+1/2}(P(\Sigma^{\pm})\nabla^{\mu}\phi^{\pm}) \cdot \Lambda^{-1}\nabla^{\mu}\varphi^{\dagger};$$

since $\|\Lambda^{-1}\nabla^{\mu}\varphi^{\dagger}\|_2 \lesssim \sqrt{\mu}|\varphi|_2$, it follows after differentiating this identity with respect to ζ and by a duality argument that for all $0 \leqq s \leqq t_0 + 1$, one has

$$|d^j\mathcal{G}^{\pm}(\mathbf{h})\psi|_{H^{s-1/2}} \lesssim \sqrt{\mu}\sum \|\Lambda^{s+1/2}d^{j_1}P^{\pm}(\mathbf{h}_I)\nabla^{\mu^{\pm}}d^{j_2}\phi(\mathbf{h}_{II})\|_2,$$

where the summation is as in Section 6.3. The result is therefore a direct consequence of (B.7) (with a shift of $1/2$ derivative).

B.2.3. Proof of (2.35) We proceed as in Section 6.3 after differentiating (B.6) with respect to ζ. One thus gets

$$\left|(\Lambda^s d^j\mathcal{G}^{\pm}(\mathbf{h})\psi_1, \Lambda^s\psi_2)\right| \leqq \sqrt{\mu}\sum \|\Lambda^s d^{j_1}P^{\pm}(\mathbf{h}_I)\nabla^{\mu^{\pm}}d^{j_2}\phi^{\pm}(\mathbf{h}_{II})\|_2|\mathfrak{P}\psi_2|_{H^s},$$

where the summation is the same as in the previous section. Elliptic estimates on (B.8) and an induction on j show that

$$\|\Lambda^s d^{j_1}P^{\pm}(\mathbf{h}_I)\nabla^{\mu^{\pm}}d^{j_2}\phi^{\pm}(\mathbf{h}_{II})\|_2 \leqq M\varepsilon^j\sqrt{\mu}|\mathbf{h}|_{H^{s\vee t_0+1}}|\mathfrak{P}\psi_1|_{H^s},$$

and the result follows.

B.2.4. Proof of (2.36) We proceed as for (2.35) but use (B.7) to control $\|\Lambda^s d^{j_1}P^{\pm}(\mathbf{h}_I)\nabla^{\mu^{\pm}}d^{j_2}\phi^{\pm}(\mathbf{h}_{II})\|_2$.

Appendix C: Proof of (5.15) and (5.16)

C.1. Proof of (5.15)

We prove (5.15) in the case $\iota = 0$ (that is, $J^\iota = 1$); the generalization to the case $\iota > 0$ requires only simple technical modifications and is thus omitted.

By definition of $Ins[U]$, one has

$$\left(Ins[U]\mathcal{T}_{(\alpha)}, \zeta_{(\alpha)}\right) = -\varepsilon^2\mu\underline{\rho}^+\underline{\rho}^-\left(\llbracket\underline{V}^{\pm}\rrbracket \cdot \mathcal{E}(\llbracket\underline{V}^{\pm}\rrbracket\mathcal{T}_{(\alpha)}), \zeta_{(\alpha)}\right)$$
$$+\left(\mathfrak{a}\mathcal{T}_{(\alpha)}, \zeta_{(\alpha)}\right) - \frac{1}{Bo}\left(\nabla \cdot \mathcal{K}\nabla\mathcal{T}_{(\alpha)}, \zeta_{(\alpha)}\right).$$

The fact that the second and third terms of this identity are controlled by $\mathfrak{m}^N(U)$ is a direct consequence of the last two points of Proposition 4. We are thus led to control the first term. Let us remark first that, by definition of $\mathcal{T}[U]$,

$$\left(\llbracket\underline{V}^{\pm}\rrbracket \cdot \mathcal{E}(\llbracket\underline{V}^{\pm}\rrbracket\mathcal{T}_{(\alpha)}), \zeta_{(\alpha)}\right) = \left(\llbracket\underline{V}^{\pm}\rrbracket \cdot \mathcal{E}(\llbracket\underline{V}^{\pm}\rrbracket\nabla \cdot (\zeta_{(\alpha)}\underline{V}^+)), \zeta_{(\alpha)}\right)$$
$$+\underline{\rho}^-\underline{H}^-\left(\llbracket\underline{V}^{\pm}\rrbracket \cdot \mathcal{E}(\llbracket\underline{V}^{\pm}\rrbracket\mathcal{G}(\mathcal{G}^-)^{-1}\nabla \cdot (\zeta_{(\alpha)}\llbracket\underline{V}^{\pm}\rrbracket)), \zeta_{(\alpha)}\right)$$
$$= A_1 + A_2.$$

Owing to (5.7), the inequality (5.15) follows from the estimate

$$\varepsilon^2 \mu \underline{\rho}^+ \underline{\rho}^- \left(|A_1| + |A_2| \right) \leqq \mathfrak{m}^N(U)$$
$$\times \left[1 + \varepsilon^2 \underline{\rho}^+ \underline{\rho}^- \left(\left| (1 + \mu^{1/4} |D|^{1/2}) F \right|_2^2 + \left| (1 + \mu^{1/4} |D|^{1/2}) G \right|_2^2 \right) \right], \quad \text{(C.1)}$$

with $F = \zeta_{(\alpha)} [\![\underline{V}^\pm]\!]$ and $G = \zeta_{(\alpha)} \partial_j \nabla [\![\underline{V}^\pm]\!]$, which we prove now.

– *Control of A_1.* We decompose further A_1 into $A_1 = A_{11} + A_{12} + A_{13} + A_{14}$ with (summing over the repeated index j, denoting $\widetilde{F} = (1 + \mu^{1/4} |D|^{1/2}) F$ and with $\widetilde{G}$ as in (3.16)),

$$A_{11} = \left((1 + \sqrt{\mu} |D|)^{1/2} (\zeta_{(\alpha)} \underline{V}_j^+ \partial_j [\![\underline{V}^\pm]\!]), \frac{\nabla}{|D|} \mathfrak{P} \widetilde{G}^{-1} \nabla \cdot F \right),$$

$$A_{12} = \mu^{1/4} \left(\partial_j [|D|^{1/2}, \underline{V}_j^+] F, \mathfrak{P}' \widetilde{G}^{-1} \nabla \cdot F \right),$$

$$A_{13} = - \left(\underline{V}_j^+ \widetilde{F}, \frac{(1 + \sqrt{\mu} |D|)}{(1 + \mu^{1/4} |D|^{1/2})} \frac{\nabla \partial_j}{|D|^2} \mathrm{Op}(\tau) \nabla \cdot \widetilde{F} \right),$$

$$A_{14} = - \left(\underline{V}_j^+ \widetilde{F}, \frac{(1 + \sqrt{\mu} |D|)}{(1 + \mu^{1/4} |D|^{1/2})} \frac{\nabla \partial_j}{|D|^2} [\mathfrak{P}^2 \widetilde{G}^{-1} \frac{1}{(1 + \mu^{1/4} |D|^{1/2})} - \mathrm{Op}(\tau)] \nabla \cdot \widetilde{F} \right),$$

where $\mathfrak{P}' = \frac{|D|}{1 + \mu^{1/4} |D|}$ and the symbol τ is given by (with $S^\pm$ as in (3.3))

$$\tau(X, \xi) = \frac{|\xi|^2}{(1 + \sqrt{\mu} |\xi|)} \frac{\langle \underline{H}^\pm \rangle}{(\underline{\rho}^+ \underline{H}^+ S^- - \underline{\rho}^- \underline{H}^- S^+)} \frac{1}{(1 + \mu^{1/4} |\xi|^{1/2})}.$$

Using the Cauchy–Schwarz inequality, standard commutator estimates and Remark 8, we get that A_{11} and A_{12} are controlled as in (C.1). For A_{13}, we remark that $\mathbf{O}(U) = \underline{V}_j^+ \frac{(1 + \sqrt{\mu} |D|)}{(1 + |\mu|^{1/4} |D|^{1/2})} \frac{\nabla \partial_j}{|D|} \mathrm{Op}(\tau) \nabla^T$ is a first order operator with a skew symmetric principal symbol and that, as a consequence of the pseudodifferential estimates of [40], one has $\mu \| \mathbf{O}(U) + \mathbf{O}(U)^* \|_{L^2 \to L^2} \leqq \mathfrak{m}^N(U)$. It follows that A_{13} is also bounded by the right-hand side of (C.1). For A_{14}, one deduces from the Cauchy–Schwarz inequality that

$$|A_{14}| \leqq |\underline{V}_j^+ \widetilde{F}|_2 \left(\left| [\mathfrak{P}^2 \widetilde{G}^{-1} \frac{1}{(1 + \mu^{1/4} |D|^{1/2})} - \mathrm{Op}(\tau)] \nabla \cdot \widetilde{F} \right|_2 \right.$$
$$\left. + \mu^{1/4} \left| [\mathfrak{P}^2 \widetilde{G}^{-1} \frac{1}{(1 + \mu^{1/4} |D|^{1/2})} - \mathrm{Op}(\tau)] \nabla \cdot \widetilde{F} \right|_{H^{1/2}} \right),$$

and Corollary 5 (with $k = 0$ for the first term and $k = 1$ for the second one) implies that $|A_{14}| \leqq \frac{1}{\mu} \mathfrak{m}^N(U) |\widetilde{F}|_2^2$, from which (C.1) follows for for A_{14} and therefore for A_1.

– *Control of A_2.* Let us first rewrite A_2 as (with $F = \zeta_{(\alpha)} [\![\underline{V}^\pm]\!]$)

$$A_2 = \underline{\rho}^- \frac{1}{\underline{H}^+} (\mathcal{G}^+ \widetilde{G}^{-1} \nabla \cdot F), [\![\underline{V}^\pm]\!] \cdot \nabla \widetilde{G}^{-1} \nabla \cdot F).$$

Using (2.25) with $u = \widetilde{G}^{-1}\nabla \cdot F$ and $\mathbf{v} = [\![V^\pm]\!]$, and using Remark 8 yields, with $\widetilde{F}$ as above,

$$|A_2| \leqq \frac{1}{\mu}\mathfrak{m}^N(U)|\widetilde{F}|_2^2,$$

which implies (C.1).

6.4. Proof of (5.16)

As for (5.15), we consider only the case $\iota = 0$ (that is, $J^\iota = \text{Id}$). We first write

$$\left(\frac{1}{\mu}GT^*\psi_{(\alpha)}, \psi_{(\alpha)}\right) = -\left(\frac{1}{\mu}\underline{V}^+ \cdot \nabla\psi_{(\alpha)}, G\psi_{(\alpha)}\right)$$

$$-\frac{1}{\mu}\left([\![V^\pm]\!] \cdot \nabla(G^-)^{-1}G\psi_{(\alpha)}, G\psi_{(\alpha)}\right)$$

$$= B_1 + B_2.$$

For B_1, we use the fact that $G = \frac{1}{\underline{H}^+}G^+ \circ J[\zeta]^{-1}$ (see Remark 14) to get

$$B_1 = -\frac{1}{\mu}\left(\underline{V}^+ \cdot \nabla J[\zeta]g, G^+g\right),$$

with $g = J[\zeta]^{-1}\psi_{(\alpha)}$. Since $J[\zeta] = \underline{\rho}^+\text{Id} - \underline{\rho}^-\frac{H^-}{H^+}(G^-)^{-1}G^+$, we deduce that

$$B_1 = -\frac{\underline{\rho}^+}{\mu}\left(\underline{V}^+ \cdot \nabla g, G^+g\right) - \frac{\underline{\rho}^-}{\mu}\frac{H^-}{H^+}\left(\underline{V}^+ \cdot \nabla\widetilde{g}, G^-\widetilde{g}\right),$$

with $\widetilde{g} = (G^-)^{-1}G^+g$. It follows therefore from (2.25), Lemma 2 and Proposition 1 that $|B_1| \leqq \mathfrak{m}^N(U)$. For B_2, we just write

$$B_2 = \frac{1}{\mu}\left([\![V^\pm]\!] \cdot \nabla g^\sharp, G^-g^\sharp\right),$$

with $g^\sharp = (G^-)^{-1}G\psi_{(\alpha)}$. Using (2.25) again, we also get $|B_2| \leqq \mathfrak{m}^N(U)$, and the proof of (5.16) is complete.

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