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CONSISTENCY OF THE KP APPROXIMATION

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Abstract. We consider here the consistency of the KP approximation for a Boussinesq system. We consider the general case of two counterpropagating waves, which do not have to satisfy strong zero mass assumptions. We show that without such strong assumption, the KP approximation is not consistent with the Boussinesq system, but that it is close to a consistent approximation. We give precise consistency results, and also consider the case were no zero mass assumption at all is made.

1. Setting the problem up.

1.1. Introduction. The Kadomtsev-Petviashvili equation (KP) arises in the water wave problem as a bidimensional generalization of the Korteweg-de Vries equation (KdV), in the study of transversal stability of unidimensional solitons. This equation gives a model for the description of the motion of small, nearly one-dimensional long waves. More precisely, the height $\varepsilon\zeta$ of the free surface can be approximated as

$$\varepsilon\zeta(t,x,y) = \varepsilon A(\varepsilon t, \sqrt{\varepsilon} y, x-t) + O(\varepsilon^{3/2}),$$

where $0 < \varepsilon << 1$. In the absence of surface tension, the amplitude $A(T,Y,X)$ must satisfy,

$$2\partial_X\partial_T A + \tfrac{3}{2}\partial_X^2(A^2) + \tfrac{1}{3}\partial_X^4 A + \partial_Y^2 A = 0. \quad (KP)$$

In the $2D$ case, the validity of the KdV approximation has been proved in [7] and [9] as far as unidirectional waves are concerned, and in [15] for the general case. These latter authors proved that the solutions of the water wave problem split up into two wave packets, one moving to the right and the other to the left, whose amplitude A_+ and A_- evolve independently as a solution of a KdV equation

$$2\partial_T A_+ + \frac{3}{2}\partial_X(A_+^2) + \frac{1}{3}\partial_X^3 A_+ = 0 \quad \text{and} \quad 2\partial_T A_- - \frac{3}{2}\partial_X(A_-^2) - \frac{1}{3}\partial_X^3 A_- = 0.$$

For the $3D$ case, validity of the KP approximation for the water wave equations is still an open problem. In [8] and [13], the authors prove the validity of this approximation for intermediate models: a Boussinesq equation and the Benney-Luke equations respectively. However, these works only deal with uni-directional waves, and make a restrictive 'zero mass' assumption on the initial data, namely, that the initial condition $A_+(0,\cdot,\cdot)$ must be twice or even three times the derivative of a function in some Sobolev space $H^s(\mathbb{R}^2)$. In [5] the general case of two counter-propagating waves is considered for a general class of quasilinear hyperbolic

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problems (which does not cover the water wave equations), and the restrictive assumption on the initial data is not made. Generalizing the $2D$ result, it is proved that a set of two uncoupled KP equations provides a good approximation of the exact solution. Unfortunately, the generality gained in this work is at the cost of having a worse convergence rate of the approximate solution; more precisely, it is $o(1)$ in [5] while it is $O(\sqrt{\varepsilon})$ in [8] and [13][1]. In this paper, we want to concentrate on the phenomena which explain these differences in the convergence rates. These phenomena appear when proving consistency results for the KP approximation, i.e. when proving that the residual associated to it is small. We chose here to start with a Boussinesq System (BS) of equations, rather than from the full Euler Equations (EE) because technicalities would have hidden the phenomena we want to enhance here. Rigorous derivation of (BS) from (EE) is addressed in [3].
The interests of the results of this paper compared to those of [8] [13] [5] are the following: we address the general bidirectional case and only consider the classical zero mass assumption of the initial data; moreover, a careful study of the secular growth phenomena allows us to give precise estimates for the residual, better than those of [5]. We also give a result of consistency *without assuming any zero mass constraint at all* on the initial data.
We would like to enhance the link between the zero mass constraint(s) and the smallness of the residual. We can describe our main results as follows: *without restrictive zero mass assumption, the KP approximation is not consistent, but very close to a consistent approximation of (BS).* Only making strong zero mass assumptions, does the KP approximation become consistent itself.

In Sections 1.2 and 1.3 we give the Boussinesq system we work with, and the ansatz used. In Section 2, we derive the uncoupled system of KP equations and consistency is addressed in Section 3. Because we do not want to make restrictive zero mass assumptions of the initial data, we cannot prove directly consistency for the KP approximation, and we have to use a new ansatz, introduced in Section 3.1. The fact that this new ansatz is close to the first one, and that it is consistent with (BS) is proved in Section 3.2; these results are improved in two distinct directions in Section 3.3.

Remark 1. It should be clear that we do *not* deal here with the convergence of the KP approximation towards the solution of the Euler equations, but only with the *consistency* of this approximation, which is a necessary (technical) step in the proof of the convergence. The non-standard part of this paper consists in the precise low-frequencies cut-off procedure used here. This is why we have detailed the computations. However, our main results can be stated very simply (see above) and the difference between our modified ansatz and the original one are described qualitatively in Remarks 4 and 5.

1.2. **The equations.** We work in this paper with a Boussinesq System (BS) which reads as follows

$$\text{(BS)}\begin{cases} V_t+\nabla\zeta+\dfrac{\varepsilon}{2}\nabla(|V|^2)=0,\\ \zeta_t+\nabla\cdot V+\varepsilon\left(\nabla\cdot(\zeta V)+\dfrac{1}{3}\Delta\nabla\cdot V\right)=0,\end{cases}$$

[1]The equations and scaling considered in these references differ one from each other. We have therefore chosen to give the estimates the methods of these papers would yield for the present case

where V denotes the velocity field and ζ the height of the free surface. The link between this system and the full Euler equations for the motion of an inviscid and incompressible liquid layer, with flat bottom, is studied in [3]. In the next sections, we construct a consistant solution for (BS).

1.3. **The ansatz.** The Boussinesq system (BS) furnishes a model for the study of $2D$ long surface waves. Here, we are concerned with small transverse perturbations of $1D$ long waves. This is modeled by looking for consistant solution to (BS) which satisfy the Kadomtsev-Petviashvili scaling,

$$(V,\zeta)(t,x,y) = (V^\varepsilon,\zeta^\varepsilon)(\varepsilon t, \sqrt{\varepsilon}y, t, x). \tag{1}$$

Moreover, we look for $V^\varepsilon := (u^\varepsilon, v^\varepsilon)^T$ and ζ^ε using the ansatz

$$\begin{array}{rcl} u^\varepsilon(T,Y,t,x) &=& u_0(T,Y,t,x) + \varepsilon u_2(T,Y,t,x), \\ v^\varepsilon(T,Y,t,x) &=& \sqrt{\varepsilon} v_1(T,Y,t,x), \\ \zeta^\varepsilon(T,Y,t,x) &=& \zeta_0(T,Y,t,x) + \varepsilon\zeta_2(T,Y,t,x), \end{array} \tag{2}$$

where T holds for εt and Y for $\sqrt{\varepsilon}y$.

Remark 2. The dissymmetry in the expansion for u^ε and v^ε is due to the KP scaling. Indeed, if the velocity potential satisfies the KP scaling, its gradient (i.e. the velocity) will be dissymmetric in the sense that its second coordinate will be $\sqrt{\varepsilon}$ times smaller than its first one.

2. **Formal derivation of the KP equations.**

2.1. **The method.** Formal derivation of the KP equations is classical; we recall it for completeness, and because all the terms are important when evaluating the size of the residual. The method consists in plugging the ansatz (1)-(2) into the Boussinesq system (BS). One then finds a residual $\mathcal{R}(V^\varepsilon,\zeta^\varepsilon)$ which can be expanded into powers of ϵ as

$$\mathcal{R}^\varepsilon := \mathcal{R}(V^\varepsilon,\zeta^\varepsilon) = \sum_{j=0}^{7} \varepsilon^{j/2}\mathcal{R}^j, \tag{3}$$

where for all $j = 0\dots 8$, $\mathcal{R}^j(T,Y,t,x)$ does not depend on ε. More precisely, we have

$$\begin{aligned}
\mathcal{R}^0 &= (\partial_t u_0 + \partial_x\zeta_0, 0, \partial_t\zeta_0 + \partial_x u_0)^T, && (4)\\
\mathcal{R}^1 &= (0, \partial_t v_1 + \partial_Y\zeta_0, 0)^T, && (5)\\
\mathcal{R}^2 &= \Big(\partial_T u_0 + \partial_t u_2 + \partial_x\zeta_2 + u_0\partial_x u_0, 0,\\
&\qquad \partial_T\zeta_0 + \partial_t\zeta_2 + \partial_x u_2 + \partial_Y v_1 + \frac{1}{3}\partial_x^3 u_0 + \partial_x(\zeta_0 u_0)\Big)^T && (6)\\
\mathcal{R}^3 &= (0, \partial_T v_1 + \partial_Y\zeta_2 + u_0\partial_Y u_0, 0)^T, && (7)\\
\mathcal{R}^4 &= \Big(\partial_T u_2 + \frac{1}{2}\partial_x(2u_0u_2 + v_1^2), 0,\\
&\qquad \partial_T\zeta_2 + \frac{1}{3}\partial_x^3 u_2 + \frac{1}{3}\partial_Y^2\partial_x u_0 + \partial_x(\zeta_0 u_2 + \zeta_2 u_0) + \partial_Y(\zeta_0 v_1)\Big)^T && (8)\\
\mathcal{R}^5 &= \Big(0, \frac{1}{2}\partial_Y(2u_0u_2 + v_1^2), 0\Big)^T, && (9)\\
\mathcal{R}^6 &= \Big(u_2\partial_x u_2, 0, \partial_x(\zeta_2 u_2) + \partial_Y(\zeta_2 v_1) + \frac{1}{3}\partial_Y^2(\partial_x u_2 + \partial_Y v_1)\Big) && (10)\\
\mathcal{R}^7 &= (0, u_2\partial_Y u_2, 0)^T. && (11)
\end{aligned}$$

The next task is then to chose the profiles u_0, u_2, v_1, ζ_0 and ζ_2 in order to cancel the first terms of expansion (3). This yields the profile equations which determine these terms, as the next section shows.

2.2. **Derivation of the profile equations.** We now proceed to cancel the first terms of the previous expansion.

Order ε^0. Thanks to (4), it is obvious that $\mathcal{R}^0 = 0$ provided that

$$\begin{cases} u_0(T,Y,t,x) = A_+(T,Y,x-t) - A_-(T,Y,x+t)\\ \zeta_0(T,Y,t,x) = A_+(T,Y,x-t) + A_-(T,Y,x+t). \end{cases} \tag{12}$$

Remark 3. This means that the leading terms of u^ε and ζ^ε are a superposition of a right going wave A_+ and a left going one $\pm A_-$. If one chooses the initial condition $(u^{0,\varepsilon}, v^{0,\varepsilon}, \zeta^{0,\varepsilon}) := (u_0^0(0,Y,0,x), \varepsilon v_1^0(0,Y,0,x), \zeta_0^0(0,Y,0,x))$ such that $u_0^0(0,Y,0,x) = \zeta_0^0(0,Y,0,x)$, then the left going wave vanishes from the leading term of the ansatz.

Order $\varepsilon^{\frac{1}{2}}$. Next, thanks to (5), the condition $\mathcal{R}^1 = 0$ gives v_1 in terms of ζ_0,

$$\partial_t v_1 + \partial_Y\zeta_0 = 0. \tag{13}$$

If $A_+(T,Y,X)$ and $A_-(T,Y,X)$ are derivatives with respect to X, then the notation $\partial_X^{-1}A_\pm$ makes sense, and a solution to (13) is given by

$$v_1(T,Y,t,x) = \partial_X^{-1}\partial_Y A_+(T,Y,x-t) - \partial_X^{-1}\partial_Y A_-(T,Y,x+t). \tag{14}$$

Order ε. The condition $\mathcal{R}^2 = 0$ reads, thanks to (6),

$$\begin{cases} \partial_T u_0 + \partial_t u_2 + \partial_x\zeta_2 + u_0\partial_x u_0 = 0,\\ \partial_T\zeta_0 + \partial_t\zeta_2 + \partial_x u_2 + \partial_Y v_1 + \dfrac{1}{3}\partial_x^3 u_0 + \partial_x(\zeta_0 u_0) = 0, \end{cases}$$

or equivalently

$$\begin{cases} (\partial_t+\partial_x)(\zeta_2+u_2)+\partial_T(u_0+\zeta_0)+\frac{1}{3}\partial_x^3 u_0+u_0\partial_x u_0+\partial_x(\zeta_0 u_0)+\partial_Y v_1=0,\\ (\partial_t-\partial_x)(\zeta_2-u_2)+\partial_T(\zeta_0-u_0)+\frac{1}{3}\partial_x^3 u_0-u_0\partial_x u_0+\partial_x(\zeta_0 u_0)+\partial_Y v_1=0. \end{cases}$$

Thanks to (12), these equations also read

$$\begin{cases} (\partial_t+\partial_x)(\zeta_2+u_2)+2\partial_T A_+ +\frac{1}{3}\partial_X^3(A_+-A_-)+\partial_Y v_1\\ \qquad\qquad +(3A_+-A_-)\partial_X A_+ -(A_++A_-)\partial_X A_-=0,\\ (\partial_t-\partial_x)(\zeta_2-u_2)+2\partial_T A_- +\frac{1}{3}\partial_X^3(A_+-A_-)+\partial_Y v_1\\ \qquad\qquad +(A_++A_-)\partial_X A_+ +(A_+-3A_-)\partial_X A_-=0. \end{cases}$$

If we want to avoid secular growth of the corrector terms ζ_2 and u_2, a *necessary condition* for the solvability of these equations is that (see e.g. [10] [5]) $A_+(T,Y,X)$ and $A_-(T,Y,X)$ satisfy two *uncoupled* KP equations,

$$2\partial_T A_\pm \pm \frac{1}{3}\partial_X^3 A_\pm \pm \partial_X^{-1}\partial_Y^2 A_\pm \pm 3A_\pm\partial_X A_\pm=0 \tag{15}$$

where we have also used (14). These equations can be solved using the following (far from optimal) theorem

Theorem 1 ([17] [14])**.** *Let $s\geq 0$. For any $A_\pm^0:=\partial_X\widetilde{A_\pm^0}$ with $\widetilde{A_\pm^0}\in H^{s+4}(\mathbb{R}^2)$, there exists $T_0>0$ such that the KP equations (15) have a unique solution*

$$A_\pm\in C^0([-T_0,T_0],H^{s+3})\cap C^1([-T_0,T_0],H^s),$$
$$\widetilde{A_\pm}:=\partial_X^{-1}A_\pm\in C^0([-T_0,T_0],H^{s+3}),$$

with initial data $A_\pm(0,\cdot,\cdot)=A_\pm^0$.

The equations (15) must be complemented by the following equations for the correctors,

$$\begin{cases} (\partial_t+\partial_x)(\zeta_2+u_2)=\frac{1}{3}\partial_X^3 A_- +\partial_X^{-1}\partial_Y^2 A_- +\partial_X(A_+A_-)+\frac{1}{2}\partial_X A_-^2\\ (\partial_t-\partial_x)(\zeta_2-u_2)=-\frac{1}{3}\partial_X^3 A_+ -\partial_X^{-1}\partial_Y^2 A_+ -\partial_X(A_+A_-)-\frac{1}{2}\partial_X A_+^2, \end{cases} \tag{16}$$

where $A_\pm$ is always evaluated at $(T,Y,x\mp t)$.

3. **Consistency estimates.** Since the ansatz (1)-(2) has been chosen in order for $\mathcal{R}^0$, $\mathcal{R}^1$ and $\mathcal{R}^2$ to vanish, one could think that because the residual $\mathcal{R}^\varepsilon$ of (3) satisfies

$$\sup_{T\in[0,T_0]}|\mathcal{R}^\varepsilon(T,\cdot,\frac{T}{\varepsilon},\cdot)|_{H^s(\mathbb{R}^2_{Y,x})}\leq\sum_{j=3}^{7}\varepsilon^{j/2}\sup_{T\in[0,T_0]}|\mathcal{R}^j(T,\cdot,\frac{T}{\varepsilon},\cdot)|_{H^s(\mathbb{R}^2_{Y,x})},$$

it is of order $O(\varepsilon^{3/2})$.
However, there are two mistakes in reasoning like this: first, secular growth can cause the quantities $\sup_{T\in[0,T_0]}|\mathcal{R}^j(T,\cdot,T/\varepsilon,\cdot)|_{H^s(\mathbb{R}^2_{Y,x})}$ to increase as ε tends to 0; second, under the assumptions of Th. 1, some terms in the expansion (5)-(11) do not make sense. This latter point is the most tricky and is the reason why we cannot prove directly a consistency result for the KP approximation, and why we have to change the ansatz.

3.1. **A new ansatz.** The problem encountered with ansatz (1)-(2) is that expansion (5)-(11) involves derivatives of $\widetilde{A_\pm} := \partial_X^{-1} A_\pm$ with respect to the slow variable T, while under the assumptions of Th. 1, $\widetilde{A_\pm}$ is only continuous with respect to T. In [8] and [13], the authors make restrictive assumptions on the initial data $A_\pm^0$ (namely, that $\partial_X^{-2} A_\pm^0$ or even $\partial_X^{-3} A_\pm^0$ are in H^s) which ensure the needed regularity on $\widetilde{A_\pm}$. As explained in the introduction, we do not want to strengthen the assumptions of Th. 1. Because all the difficulties come from the fact that the operator ∂_X^{-1} is singular for low frequencies, we choose to work with a new ansatz which is deduced form the previous one by cutting off the low frequencies, thus following a procedure now classical in geometric optics [1] [2] [16].
With this idea in mind, let χ^δ be defined on $\mathbb{R}$ as $\chi^\delta(s) = 0$ if $|s| \leq \delta$ and $\chi^\delta(s) = 1$ otherwise. A straightforward but fundamental result is that $\partial_X^{-1}\chi^\delta(D) f$, with $D := -i\partial_x$, is well defined in $H^s(\mathbb{R}^2)$ for all $f \in H^s(\mathbb{R}^2)$, and that one has

$$|\partial_X^{-1}\chi^\delta(D) f|_{H^s} \leq \frac{1}{\delta}|f|_{H^s}. \tag{17}$$

Our new ansatz $(V^{\varepsilon,\delta}, \zeta^{\varepsilon,\delta})$ reads

$$\begin{aligned}
u^{\varepsilon,\delta}(T,Y,t,x) &= \left[A_+^\delta(T,Y,x-t) - A_-^\delta(T,Y,x+t)\right] + \varepsilon u_2^\delta(T,Y,t,x), \\
v^{\varepsilon,\delta}(T,Y,t,x) &= \sqrt{\varepsilon}\left[\partial_Y \widetilde{A_+^\delta}(T,Y,x-t) - \partial_Y \widetilde{A_-^\delta}(T,Y,x+t)\right], \\
\zeta^{\varepsilon,\delta}(T,Y,t,x) &= \left[A_+^\delta(T,Y,x-t) + A_-^\delta(T,Y,x+t)\right] + \varepsilon \zeta_2^\delta(T,Y,t,x),
\end{aligned}$$

where $A_\pm^\delta := \chi^\delta(D) A_\pm$ and $\widetilde{A_\pm^\delta} := \partial_X^{-1} A_\pm^\delta$, while $(u_2^\delta, \zeta_2^\delta)$ is found using $(16)^\delta$, which are the same equations as (16) except that in the r.h.s. all the $A_\pm$ are replaced by $A_\pm^\delta$.
A first consequence of this choice of ansatz is that we can write the term $\partial_X^{-1}\partial_Y^2 A_\pm^\delta$ which appears in the right hand side of $(16)^\delta$ as $\partial_X^{-1}\partial_Y^2 A_\pm^\delta = \partial_X(\partial_X^{-2}\partial_Y^2 A_\pm^\delta)$ where $\partial_X^{-2}\partial_Y^2 A_\pm^\delta \in H^s(\mathbb{R}^2)$, with $|\partial_X^{-2}\partial_Y^2 A_\pm^\delta|_{H^s} \leq \frac{1}{\delta}|\widetilde{A_\pm^\delta}|_{H^{s+2}}$. It follows that $(16)^\delta$ can be solved explicitely as

$$\begin{aligned}
(\zeta_2^\delta + u_2^\delta)(T,Y,t,x) &= \frac{1}{2}\left[\frac{1}{3}\partial_X^2 A_-^\delta + \partial_X^{-2}\partial_Y^2 A_-^\delta + \frac{1}{2}(A_-^\delta)^2\right](x+t) \\
&+ \frac{1}{2}A_+^\delta(x-t)\left[A_-^\delta(x+t) - A_-^\delta(x-t)\right] \\
&+ \frac{1}{2}\partial_X A_+^\delta(x-t)\left[\widetilde{A_-^\delta}(x+t) - \widetilde{A_-^\delta}(x-t)\right],
\end{aligned} \tag{18}$$

together with a similar expression for $(\zeta_2^\delta - u_2^\delta)$, and omitting the variables T, Y in the r.h.s.

Remark 4. An important consequence of (18) is that the correctors u_2^δ and ζ_2^δ do not grow secularly, while ζ_2 and u_2 grew sublinearly, i.e. as $o(t)$, in Sobolev norm.

3.2. **Estimates of the residual.** To the new ansatz $(V^{\varepsilon,\delta}, \zeta^{\varepsilon,\delta})$ corresponds a residual term $\mathcal{R}^{\varepsilon,\delta}$ which can be written as in (3),

$$\mathcal{R}^{\varepsilon,\delta} := \mathcal{R}(V^{\varepsilon,\delta}, \zeta^{\varepsilon,\delta}) = \sum_{j=0}^{7} \varepsilon^{j/2}\mathcal{R}^{j,\delta}. \tag{19}$$

As in Section 2, it is clear that $\mathcal{R}^{0,\delta} = \mathcal{R}^{1,\delta} = 0$, but we do not have $\mathcal{R}^{2,\delta} = 0$ while we had $\mathcal{R}^2 = 0$. Indeed, the splitting (15)-(16) shows that in order for $\mathcal{R}^{2,\delta}$

to vanish, $A_\pm^\delta$ should satisfy the KP equations (15), while the equations actually satisfied by $A_\pm^\delta$ are

$$2\partial_T A_\pm^\delta \pm \frac{1}{3}\partial_X^3 A_\pm^\delta \pm \partial_X^{-1}\partial_Y^2 A_\pm^\delta \pm 3\chi^\delta(D)(A_\pm \partial_X A_\pm) = 0.$$

It follows that

$$\begin{aligned}\mathcal{R}^{2,\delta} &= \frac{3}{4}\Big(\partial_X({A_+^\delta}^2 + {A_-^\delta}^2) - \chi^\delta(D)\partial_X(A_+^2 + A_-^2), 0,\\ &\quad \partial_X({A_+^\delta}^2 + {A_-^\delta}^2) - \chi^\delta(D)\partial_X(A_+^2 - A_-^2)\Big).\end{aligned}$$

In order to estimate $\mathcal{R}^{2,\delta}$, we need the following lemma.

Lemma 1. *Let $s > 1$ and $f \in H^s(\mathbb{R}^2)$, and denote $f^\delta := \chi^\delta(D)f$;*
i. *One has $|{f^\delta}^2 - \chi^\delta(D)f^2|_{H^s} = o(1)$ as δ goes to 0;*
ii. *If $f \in H^{s+1}(\mathbb{R}^2)$ and $f = \partial_X \widetilde{f}$, $\widetilde{f} \in H^{s+1}(\mathbb{R}^2)$, then*

$$|\partial_X({f^\delta}^2 - \chi^\delta(D)f^2)|_{H^s} \leq \mathrm{Cst}\ \delta\left(|f|_{H^{s+1}}|\widetilde{f}|_{H^{s+1}} + |f|_{H^s}^2\right);$$

iii. *If, moreover, the partial Fourier transforms $\mathcal{F}_x f$ and $\mathcal{F}_x \widetilde{f}$ belong both to $L^\infty(\mathbb{R}_\xi; H^s(\mathbb{R}_Y))$, then*

$$|\partial_X({f^\delta}^2 - \chi^\delta(D)f^2)|_{H^s} \leq \mathrm{Cst}\ \delta^{3/2}\left(|f|_{H^{s+1}}|\mathcal{F}_x\widetilde{f}|_{L^\infty H^s} + |f|_{H^s}|\mathcal{F}_x f|_{L^\infty H^s}\right).$$

Proof. From the triangular inequality, one has $|{f^\delta}^2 - \chi^\delta(D)f^2|_{H^s} \leq |{f^\delta}^2 - f^2|_{H^s} + |(1-\chi^\delta(D))f^2|_{H^s}$ and hence

$$|{f^\delta}^2 - \chi^\delta(D)f^2|_{H^s} \leq 2|(1-\chi^\delta(D))f|_{H^s}|f|_{H^s} + |(1-\chi^\delta(D))f^2|_{H^s}. \tag{20}$$

Point i. of the lemma follows from (20) by a simple dominated convergence theorem. As for (20), one also obtains

$$|\partial_X({f^\delta}^2 - \chi^\delta(D)f^2)|_{H^s} \leq 2|(1-\chi^\delta(D))f|_{H^{s+1}}|f|_{H^{s+1}} + |(1-\chi^\delta(D))\partial_X f^2|_{H^s}.$$

For any function $g \in H^s$, one also has $|(1-\chi^\delta(D))\partial_X g|_{H^s} \leq \mathrm{Cst}\ \delta|g|_{H^s}$ since $|(1-\chi^\delta(\xi))^2\xi^2| \leq \delta^2$, which yields point ii.
The last point of the lemma follows from the fact that for any function g,

$$\begin{aligned}|(1-\chi^\delta(D))\partial_X g|_{H^s} &\leq \mathrm{Cst}\ \left(\int_{|\xi|\leq\delta}|\xi|^2 d\xi\right)^{1/2}|\mathcal{F}_x g|_{L^\infty H^s}\\ &\leq \mathrm{Cst}\ \delta^{3/2}|\mathcal{F}_x g|_{L^\infty H^s}.\end{aligned}$$

□

It follows from Lemma 1.ii that under the assumptions of Th. 1, one has

$$|\mathcal{R}^{2,\delta}|_{H^s} \leq C\left(|\widetilde{A_\pm}|_{H^{s+2}}\right)\delta. \tag{21}$$

We now turn to estimate the residual terms $\mathcal{R}^{j,\delta}$ of (19). These terms are deduced from (5)-(11) by adding the δ index everywhere. Using the explicit expression (18) of the correctors u_2^δ and ζ_2^δ, together with the fundamental estimate (17), these residual terms can be bounded, under the assumptions of Th. 1, as

$$\begin{aligned}|\mathcal{R}^{j,\delta}|_{H^s} &\leq C\left(|\widetilde{A_\pm}|_{H^{s+4}}\right)(1+\frac{1}{\delta}), \quad \text{for} \quad j = 3,5,6,\\ |\mathcal{R}^{j,\delta}|_{H^s} &\leq C\left(|\widetilde{A_\pm}|_{H^{s+6}}\right)(1+\frac{1}{\delta^2}), \quad \text{for} \quad j = 4,7.\end{aligned}$$

It follows that

$$\sup_{T\in[0,T_0]} |\mathcal{R}^{\varepsilon,\delta}(T,\cdot,\varepsilon T,\cdot)|_{H^s} \leq C\left(|\widetilde{A_\pm}|_{H^{s+6}}\right)(\varepsilon\delta + \frac{\varepsilon^{3/2}}{\delta} + \frac{\varepsilon^2}{\delta^2}).$$

A rapid analysis shows that the best δ possible is given by $\delta = \varepsilon^{1/4}$ for which the r.h.s. is of order $O(\varepsilon^{5/4})$. Most of the work for the proof of the following theorem has thus been done.

Theorem 2. *Let $s > 2$ and $A^0_\pm := \partial_X \widetilde{A^0_\pm}$ with $\widetilde{A^0_\pm} \in H^{s+6}(\mathbb{R}^2)$. With $A_\pm$ as given by Th. 1, the KP approximation*

$$(u^\varepsilon_{KP}, \zeta^\varepsilon_{KP})^T(t,x,y) := \begin{pmatrix} A_+(\varepsilon t, \sqrt{\varepsilon}y, t-x) - A_-(\varepsilon t, \sqrt{\varepsilon}y, t+x) \\ A_+(\varepsilon t, \sqrt{\varepsilon}y, t-x) + A_-(\varepsilon t, \sqrt{\varepsilon}y, t+x) \end{pmatrix}$$

converges towards a couple of functions $(u^\varepsilon_{cons}, \zeta^\varepsilon_{cons})$ consistent with (BS) in the sense that

$$|(u^\varepsilon_{KP}, \zeta^\varepsilon_{KP}) - (u^\varepsilon_{cons}, \zeta^\varepsilon_{cons})|_{L^\infty([0,\frac{T_0}{\varepsilon}]\times\mathbb{R}^2_{x,y})} + \frac{1}{\varepsilon}|\mathcal{R}^\varepsilon_{cons}|_{L^\infty([0,\frac{T_0}{\varepsilon}]\times\mathbb{R}^2_{x,y})} = O(\varepsilon^{1/4}),$$

where $\mathcal{R}^\varepsilon_{cons}$ is the error made when plugging $(V^\varepsilon_{cons}, \zeta^\varepsilon_{cons})$ in (BS), with $V^\varepsilon_{cons} := (u^\varepsilon_{cons}, \sqrt{\varepsilon} v^\varepsilon_{cons})$, $|v^\varepsilon_{cons}|_{H^s} = O(1)$.

Proof. If we take $(V^\varepsilon_{cons}, \zeta^\varepsilon_{cons}) = (V^{\varepsilon,\varepsilon^{1/4}}, \zeta^{\varepsilon,\varepsilon^{1/4}})$, the analysis made above shows that the residual is of order $O(\varepsilon^{5/4})$ in $H^s(\mathbb{R}^2_{X,Y})$. The difference

$$\mathcal{E}^\varepsilon := (u^{\varepsilon,\varepsilon^{1/4}}, \zeta^{\varepsilon,\varepsilon^{1/4}}) - (u^\varepsilon_{KP}, \zeta^\varepsilon_{KP})$$

satisfies the estimate, in (X,Y) variables,

$$\begin{aligned} |\mathcal{E}^\varepsilon|_{H^s} &\leq |(1-\chi^{\varepsilon^{1/4}}(D))(A_+,A_-)|_{H^s} + \sqrt{\varepsilon}|v_1^{\varepsilon,\varepsilon^{1/4}}|_{H^s} + \varepsilon|(u_2^{\varepsilon,\varepsilon^{1/4}}, \zeta_2^{\varepsilon,\varepsilon^{1/4}})|_{H^s} \\ &\leq \text{Cst } \varepsilon^{1/4}|\widetilde{A_\pm}|_{H^s} + \sqrt{\varepsilon}|\widetilde{A_\pm}|_{H^{s+1}} + \varepsilon C(|\widetilde{A_\pm}|_{H^{s+3}})(1 + \frac{1}{\varepsilon^{1/4}}), \end{aligned}$$

where we have used Lemma 1.ii and the estimates already used on v_1^δ and $(\zeta_2^\delta, u_2^\delta)$. From these Sobolev estimates, one deduces L^∞ estimates in profile variables because $s > 1$, and one just has to substitute εt to T and $\sqrt{\varepsilon}y$ to Y to obtain the result of the theorem. □

Remark 5. i. If $(V^\varepsilon_{ex}, \zeta^\varepsilon_{ex})$ is an exact solution of (BS), it follows from the above theorem that formally, one expects $|(u^\varepsilon_{ex}, \zeta^\varepsilon_{ex}) - (u^\varepsilon_{KP}, \zeta^\varepsilon_{KP})|_{L^\infty([0,\frac{T_0}{\varepsilon}]\times\mathbb{R}^2)}$ to be of order $O(\varepsilon^{1/4})$.

ii. The reader can observe that the results would not be better if we kept only one direction of propagation (which is possible according to Remark 3). This means that the coupling effects between the two counterpropagating waves are not the reason why we cannot find the same estimates as for the KdV approximation. This is quite a striking result since the solutions of the KP equation do not decay fastly.

3.3. **Improved results.** As previously said, assuming that $\partial_X^{-2}\partial_Y^2 A^0_\pm \in H^s$ is too restrictive, but we can make more reasonable assumptions on the initial data in order to improve Th. 2. For instance, inspired by the study of secular growth [11], we can assume that the Fourier transform of the initial conditions are bounded.

Theorem 3. *Under the same assumptions as in Th. 2, if moreover $\mathcal{F}_x A^0_\pm$ and $\mathcal{F}_x \widetilde{A^0_\pm}$ are in $L^\infty(\mathbb{R}_\xi; H^s(\mathbb{R}_Y))$ then the $O(\varepsilon^{1/4})$ of the estimate of Th. 2 can be improved up to $O(\varepsilon^{3/10})$.*

Proof. It is easy to check that $\mathcal{F}A_\pm$ and $\widetilde{\mathcal{F}A_\pm}$ are in $L^\infty(\mathbb{R}^2)$ for all times, provided that it is initially true. Using Lemma 1.iii, the estimate (21) on $\mathcal{R}^{2,\delta}$ can be improved up to $O(\delta^{3/2})$. Choosing $\delta = \varepsilon^{1/5}$ one obtains as above that the residual satisfies $\mathcal{R}(V^{\varepsilon,\varepsilon^{1/5}}, \zeta^{\varepsilon,\varepsilon^{1/5}}) = O(\varepsilon^{13/10})$. Since moreover, still using Lemma 1.iii, one has $|(1-\chi^{\varepsilon^{1/5}}(D))A_\pm|_{H^s} = O(\varepsilon^{3/10})$, the result wanted is obtained as in the proof of Th. 2. □

Another intersting improvement of Th. 2 consists in weakening the assumptions, and particularly the zero mass constraint. Solving the Cauchy problem without such a constraint is possible (see e.g. [6] [12]) and using Lemma 1.i and the same kind of proof as for Th. 2-3, one obtains,

Theorem 4. *Let $s > 6$ and $A_\pm \in C([0,T_0], H^s(\mathbb{R}^2_{X,Y}))$ be a solution of the KP equations (15) with initial condition $A^0_\pm \in H^s(\mathbb{R}^2)$. Then Th. 2 is still valid, but one has to replace the $O(\varepsilon^{1/4})$ of the estimate by $o(1)$.*

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