

# THE LONG WAVE LIMIT FOR A GENERAL CLASS OF $2D$ QUASILINEAR HYPERBOLIC PROBLEMS

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## **Abstract**

This paper is dedicated to the study of long wave approximation for quasilinear symmetric hyperbolic systems. We use ansatz with multiple scales in order to capture both dispersive and nonlinear effects, occurring at the same time scale as a relatively small transverse correction of the unidirectional propagation.

Using the techniques developed by Joly, Métivier and Rauch for nonlinear geometric optics, we prove that the exact solutions split up

into two wave packets, one moving to the left and one to the right, and whose amplitude satisfy a system of Kadomtsev-Petviashvili equations. Whether the limit system is coupled or not is critical to the error estimate obtained.

In the last part, we formally apply these techniques to derive both uncoupled and coupled KP systems from the Euler equations with free surface in the context of water waves.

## 1 Introduction

The Kadomtsev-Petviashvili equation (KP) arises in the so called water wave problem as a bidimensional generalization of the Korteweg-de Vries equation (KdV), in the study of transversal stability of unidimensional solitons. This equation gives a model for the description of the motion of small, nearly one-dimensional long waves. More precisely, the height  $\epsilon^2\zeta$  of the free surface can be approximated as

$$\epsilon^2\zeta(t, x, y) = \epsilon^2 A(\epsilon^2 t, x - ct, \epsilon y) + O(\epsilon^3),$$

where  $c$  denotes the group velocity and  $0 < \epsilon \ll 1$ . In the absence of (or with low) surface tension, the amplitude  $A(T, X, Y)$  must satisfy, up to a rescaling,

$$2\partial_X\partial_T A + \partial_X^2(A^2) + \partial_X^4 A + \partial_Y^2 A = 0. \quad (KP)$$

Whether the KP equation provides a good approximation to the 3D water wave problem or not is still an open problem. However, many results hint that this conjecture is true. The aim of this paper is to take a new step in this direction and to obtain a system of coupled KP equations, likely to furnish a better approximation to the exact solution of the water wave problem.

**Validity of the KP approximation.** As said above, many results hint at the fact that the KP equation gives a good approximation to the solution of

the  $3D$  water wave problem. First of all, the  $2D$  case, i.e. the validity of the KdV approximation has been proved in [Cr] and [KN] as far as unidirectional waves are concerned, and in [SW] for the general case. These latter authors proved that the solutions of the water wave problem split up into two wave packets, one moving to the right and the other to the left, whose amplitude  $A_1$  and  $A_2$  evolve independently as a solution of a KdV equation

$$2\partial_T A_1 + \partial_X(A_1^2) + \partial_X^3 A_1 = 0 \quad \text{and} \quad 2\partial_T A_2 - \partial_X(A_2^2) - \partial_X^3 A_2 = 0.$$

The same kind of result is proved in [BC] for a general class of hyperbolic systems (but not for the water wave problem).

Concerning the KP equation, T. Kano [Kan] proved that analytic solutions of the  $3D$  water wave problem are approximated by the KP equation, but for times  $O(1)$  instead of  $O(1/\epsilon^2)$ , which is far too short to observe the KP dynamics.

Another step in the proof of the validity of the KP approximation has been taken in [GS] where the authors proved the result for a Boussinesq equation, and for wave packets travelling in only one direction.

We propose here to prove the result for a general class of quasi-linear hyperbolic systems. Though the water wave problem does not fall into its range, we believe that the dynamics are the same. Moreover, such a class of systems also arises in other fields of mathematical physics (optics, Euler-Poisson...).

Another interest of this paper is that it is not limited to unidirectional waves. In particular, we prove the splitting result that one expects from the  $2D$  case (KdV), i.e. that the exact solution splits into two wave trains propagating to the left and the right, and whose amplitude  $A_1$  and  $A_2$  evolve independently as a solution of a KP equation

$$2\partial_T A_j + \partial_X(A_j^2) + \partial_X^3 A_j + \partial_X^{-1} \partial_Y^2 A_j = 0 \quad j = 1, 2.$$

Moreover, using a method based on Alterman's infrared cutoffs, we manage to handle solutions  $A_j$  which are only in  $\partial_x H^s$  and not in  $\partial_x^2 H^s$  as this was the case in the previous study [GS].

**Obtention of a coupled system of KP equations.** Even though we can prove that the exact solution can be well approximated by two independent KP equations, the error estimate we obtain is only  $o(1)$  with respect to the small parameter  $\epsilon$ . One would like a better estimate,  $O(\epsilon)$  for instance, but secular growth phenomena are difficult to handle. If we consider one way propagating packet waves only, the error estimate can then be improved to  $O(\epsilon)$  (as done in [GS]), but the splitting of the exact solution into two counter propagating wave packets, which is the general case, shows that the one-way assumption is very restrictive. In [SW] the authors used a nice argument, based on a decay assumption of the solutions of the KdV equations, in order to obtain an  $O(\epsilon)$  error estimate. An alternative method has been used in [BC]; this latter method consists in obtaining a *coupled* system of KdV equations whose solution approximate the exact solution up to an  $O(\epsilon)$  error term. No decay nor one-way assumption is done in [BC], but the coupled system obtained is far more difficult to handle than two independent KdV equations.

Since the interesting solutions of the KdV equation are known to have a strong decay, the method of [SW] seems to be the most attractive. However, things may be different for the KP equation. Indeed, if the methods of [SW] can probably be generalized to the  $3D$  water wave problem, the decay assumption is far more restrictive in this case because most of the interesting solutions of the KP equation (the lump solutions for instance) have only an algebraic decay. That is why we choose to use the method of [BC], i.e. to derive a *coupled* system of KP equations, in order to obtain an  $O(\epsilon)$  error estimate.

**The water waves problem.** As said above, the water wave problem does not fall into the range of our general class of hyperbolic systems. However, the

tools introduced here can be used to derive rigorously the KP equations (both uncoupled and coupled) from the Euler equations.

## 1.1 Getting started

We present a systematic study of long wave approximation, starting from a nonlinear hyperbolic system that reads

$$\partial_t \mathbf{u}^\epsilon + A \partial_x \mathbf{u}^\epsilon + B \partial_y \mathbf{u}^\epsilon + \frac{E \mathbf{u}^\epsilon}{\epsilon} = C(\mathbf{u}^\epsilon) \partial_x \mathbf{u}^\epsilon + D(\mathbf{u}^\epsilon) \partial_y \mathbf{u}^\epsilon \quad (1.1)$$

where  $\mathbf{u}^\epsilon(t, x, y)$  is a  $\mathbb{R}^N$ -valued function. We assume that the matrices  $A$  and  $B$  are symmetric and real, whereas  $E$  is a non invertible skew-symmetric matrix. The nonlinear terms are taken as simple as possible such that for any  $u \in \mathbb{R}^N$ ,  $C(u)$  and  $D(u)$  are symmetric matrices and both  $u \mapsto C(u)$  and  $u \mapsto D(u)$  are linear.

Following the works initiated by Joly, Métivier and Rauch [JMR1], in the context of geometrical optics, we seek for approximate solutions in order to derive KP systems describing the evolution of almost one dimensional long waves. We look for an approximate solution well fit to our problem, that is under the form

$$u^\epsilon(t, x, y) = \epsilon^2 \mathcal{U}^\epsilon(\epsilon^2 t, t, x, \epsilon y) \quad \text{with} \quad \mathcal{U}^\epsilon(T, t, x, Y) = \mathcal{U}_0 + \epsilon \mathcal{U}_1 + \epsilon^2 \mathcal{U}_2 + \epsilon^3 \mathcal{U}_3. \quad (1.2)$$

We have two different time scales: one is capturing the short time evolution of the wave and the other describes the long time evolution of the wave. The critical size  $O(\epsilon^2)$  of our approximate solution is chosen such that dispersion effects and nonlinear effects occur for large time scales.

Since the above expansion is used for times of order  $O(\frac{1}{\epsilon^2})$ , one must actually control the growth of the profile with respect to the variable  $t$ . Hence, in order for the correctors to be smaller than the leading order term present in

(1.2), one must introduce the following growth condition. This condition was introduced in [BC] in the context of water waves to generalize the classical sub-linear growth condition first introduced by Joly, Métivier and Rauch [JMR2] for expansions valid for times of order  $O(\frac{1}{\epsilon})$ .

**Sub-squareroot growth condition.**

The profile  $\mathcal{V}(T, t, x, Y)$  satisfies a sub-squareroot growth condition if only if

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \sup_{(T, x, Y) \in [0, T_0] \times \mathbb{R}^2} |\mathcal{V}(T, t, x, Y)| = 0. \quad (1.3)$$

Classically, to obtain equations describing the evolution of our profiles, we plug in (1.1) the approximate solution (1.2),

$$\partial_t u^\epsilon + A \partial_x u^\epsilon + B \partial_y u^\epsilon + \frac{1}{\epsilon} E u^\epsilon - C(u^\epsilon) \partial_x u^\epsilon - D(u^\epsilon) \partial_y u^\epsilon = \sum_{j=1}^{11} \epsilon^j r_j^\epsilon \quad (1.4)$$

where for all  $j = 1 \dots 11$ , one has  $r_j^\epsilon(t, x, y) = \mathcal{R}_j(\epsilon^2 t, t, x, \epsilon y)$  with

$$\left\{ \begin{array}{l}
\mathcal{R}_1(T, t, x, Y) = E\mathcal{U}_0 \\
\mathcal{R}_2(T, t, x, Y) = \partial_t \mathcal{U}_0 + A\partial_x \mathcal{U}_0 + E\mathcal{U}_1 \\
\mathcal{R}_3(T, t, x, Y) = \partial_t \mathcal{U}_1 + A\partial_x \mathcal{U}_1 + B\partial_Y \mathcal{U}_0 + E\mathcal{U}_2 \\
\mathcal{R}_4(T, t, x, Y) = \partial_T \mathcal{U}_0 + \partial_t \mathcal{U}_2 + A\partial_x \mathcal{U}_2 + B\partial_Y \mathcal{U}_1 + E\mathcal{U}_3 - C(\mathcal{U}_0)\partial_x \mathcal{U}_0 \\
\mathcal{R}_5(T, t, x, Y) = \partial_T \mathcal{U}_1 + \partial_t \mathcal{U}_3 + A\partial_x \mathcal{U}_3 + B\partial_Y \mathcal{U}_2 \\
\quad - C(\mathcal{U}_1)\partial_x \mathcal{U}_0 - C(\mathcal{U}_0)\partial_x \mathcal{U}_1 - D(\mathcal{U}_0)\partial_Y \mathcal{U}_0 \\
\mathcal{R}_6(T, t, x, Y) = \partial_T \mathcal{U}_2 + B\partial_Y \mathcal{U}_3 - C(\mathcal{U}_0)\partial_x \mathcal{U}_2 - C(\mathcal{U}_2)\partial_x \mathcal{U}_0 - C(\mathcal{U}_1)\partial_x \mathcal{U}_1 \\
\quad - D(\mathcal{U}_0)\partial_Y \mathcal{U}_1 - D(\mathcal{U}_1)\partial_Y \mathcal{U}_0 \\
\mathcal{R}_7(T, t, x, Y) = \partial_T \mathcal{U}_3 - C(\mathcal{U}_3)\partial_x \mathcal{U}_0 - C(\mathcal{U}_0)\partial_x \mathcal{U}_3 - C(\mathcal{U}_1)\partial_x \mathcal{U}_2 \\
\quad - C(\mathcal{U}_2)\partial_x \mathcal{U}_1 - D(\mathcal{U}_1)\partial_Y \mathcal{U}_1 - D(\mathcal{U}_2)\partial_Y \mathcal{U}_0 - D(\mathcal{U}_0)\partial_Y \mathcal{U}_2 \\
\mathcal{R}_8(T, t, x, Y) = -C(\mathcal{U}_3)\partial_x \mathcal{U}_1 - C(\mathcal{U}_1)\partial_x \mathcal{U}_3 - C(\mathcal{U}_2)\partial_x \mathcal{U}_2 \\
\quad - D(\mathcal{U}_3)\partial_Y \mathcal{U}_0 - D(\mathcal{U}_0)\partial_Y \mathcal{U}_3 - D(\mathcal{U}_1)\partial_Y \mathcal{U}_2 - D(\mathcal{U}_2)\partial_Y \mathcal{U}_1 \\
\mathcal{R}_9(T, t, x, Y) = -C(\mathcal{U}_3)\partial_x \mathcal{U}_2 - C(\mathcal{U}_2)\partial_x \mathcal{U}_3 \\
\quad - D(\mathcal{U}_3)\partial_Y \mathcal{U}_1 - D(\mathcal{U}_1)\partial_Y \mathcal{U}_3 - D(\mathcal{U}_2)\partial_Y \mathcal{U}_2 \\
\mathcal{R}_{10}(T, t, x, Y) = -C(\mathcal{U}_3)\partial_x \mathcal{U}_3 - D(\mathcal{U}_3)\partial_Y \mathcal{U}_2 - D(\mathcal{U}_2)\partial_Y \mathcal{U}_3 \\
\mathcal{R}_{11}(T, t, x, Y) = -D(\mathcal{U}_3)\partial_Y \mathcal{U}_3
\end{array} \right.$$

Our strategy, following the BKW method, is to annihilate the first four  $\mathcal{R}_j$ , thus obtaining four equations on the profiles, as is

$$\mathcal{R}_1 = 0 \quad \Longrightarrow \quad E\mathcal{U}_0 = 0 \quad (1.5)$$

$$\mathcal{R}_2 = 0 \quad \Longrightarrow \quad \partial_t \mathcal{U}_0 + A\partial_x \mathcal{U}_0 + E\mathcal{U}_1 = 0 \quad (1.6)$$

$$\mathcal{R}_3 = 0 \quad \Longrightarrow \quad \partial_t \mathcal{U}_1 + A\partial_x \mathcal{U}_1 + B\partial_Y \mathcal{U}_0 + E\mathcal{U}_2 = 0 \quad (1.7)$$

$$\mathcal{R}_4 = 0 \quad \Longrightarrow \quad \partial_T \mathcal{U}_0 + \partial_t \mathcal{U}_2 + A\partial_x \mathcal{U}_2 + B\partial_Y \mathcal{U}_1 + E\mathcal{U}_3 = C(\mathcal{U}_0)\partial_x \mathcal{U}_0 \quad (1.8)$$

The paper is organized as follows: in the second section, we derive necessary conditions satisfied by the profiles  $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$  starting from the set of equa-

tions (1.5)-(1.8). We show that the main profile  $\mathcal{U}_0$  decomposes into a pair of components  $\mathcal{U}_{01}$  and  $\mathcal{U}_{02}$  solving a pair of independent KP equations.

In the third section, we prove that the set of equations obtained for the profiles is well posed and state a stability result showing that our approximate solution remains close to the exact solution  $\mathbf{u}^\epsilon$  of (1.1), in the following sense:

$$\frac{\mathbf{u}^\epsilon(t, x, y)}{\epsilon^2} - (\mathcal{U}_{01}(\epsilon^2 t, x - \lambda'(0)t, \epsilon y) + \mathcal{U}_{02}(\epsilon^2 t, x + \lambda'(0)t, \epsilon y)) = o(1),$$

in  $L^\infty([0, \frac{T_0}{\epsilon^2}]; \mathbb{R}_{x,y}^2)$ , where  $\lambda'(0)$  is the long wave limit of the phase velocity.

Section 4 is dedicated to the study of a coupled system of KP equations.

Indeed, we show that if one slightly modifies the ansatz, we can obtain a pair of functions  $(\mathbf{U}_{01}^\epsilon, \mathbf{U}_{02}^\epsilon)$  solving a coupled KP system and approximating  $\mathbf{u}^\epsilon$  better than the pair  $(\mathcal{U}_{01}, \mathcal{U}_{02})$ . The result obtained gives:

$$\left\| \frac{\mathbf{u}^\epsilon(t, x, y)}{\epsilon^2} - (\mathbf{U}_{01}^\epsilon(t, x, \epsilon y) + \mathbf{U}_{02}^\epsilon(t, x, \epsilon y)) \right\|_{L^\infty([0, \frac{T_0}{\epsilon^2}]; \mathbb{R}_{x,y}^2)} = O(\epsilon)$$

Finally, the last section is dedicated to the study of an example. Starting from the Euler equations with free surface boundary, and following our general scheme, we derive a coupled and uncoupled KP system describing a long wave approximation of this classical water-wave problem.

## 2 Algebraic analysis of Eqs (1.5)-(1.8)

### 2.1 The characteristic variety

We ought to introduce at this point some formal operators in order to simplify the set of equations (1.5)-(1.8).

**Definition 2.1** For  $(\tau, \xi, \eta) \in \mathbb{R}^3$ , let us denote by  $L(\tau, \xi, \eta)$  and  $L_1(\tau, \xi, \eta)$  the hermitian  $N \times N$  matrices,

$$L(\tau, \xi, \eta) = \tau I + A\xi + B\eta + \frac{E}{i} \quad \text{and} \quad L_1(\tau, \xi) = \tau I + A\xi.$$

The orthogonal projector onto  $\ker L(\tau, \xi, \eta)$  is denoted by  $\pi(\tau, \xi, \eta)$ , and the partial inverse  $L(0)^{-1}$  of  $L(0) = \frac{E}{i}$  is defined by the relations

$$L(0)^{-1}\pi(0) = \pi(0)L(0)^{-1} = 0 \quad \text{and} \quad L(0)L(0)^{-1} = L(0)^{-1}L(0) = I - \pi(0).$$

Let  $CharL$  be the characteristic variety of the operator  $L$ , such as

$$CharL = \left\{ (\tau, \xi, \eta) \in \mathbb{R}^3 / \det(\tau I + A\xi + B\eta + \frac{E}{i}) = 0 \right\} \quad (2.1)$$

We shall make the following assumption on the characteristic variety,

**Assumption 2.1** *The intersection of  $CharL$  with the section plane  $(\tau, \xi)$  reduces to a pair of branches which satisfies Assumption 2.1 of [BC], i.e. there exists a regular function  $\lambda$  defined on a neighborhood of 0 such that the graph of these branches is parametrized by  $(\pm\lambda(\xi), \xi)$ . We assume moreover that  $\lambda(0) = \lambda''(0) = 0$ , while  $\lambda'(0) \neq 0$ . Besides, we assume that  $CharL$  is axysymmetrical around  $(O\tau)$ .*

**Remark 2.1** Note that our assumption is true in the water-waves context and, as far as we know, in all of the physical contexts where the KP model arises.

Let  $\pi_1(\xi)$  and  $\pi_2(\xi)$  be the two orthogonal projectors defined as

$$\begin{cases} \pi_1(\xi) = \pi(\lambda(\xi), \xi, 0) \\ \pi_2(\xi) = \pi(-\lambda(\xi), \xi, 0) \end{cases} \quad \text{for } \xi \neq 0$$

Note that  $\pi(\lambda(\xi), \xi, 0)$  and  $\pi(-\lambda(\xi), \xi, 0)$  are not continuous at  $\xi = 0$ . However,  $\pi_1$  and  $\pi_2$  can be smoothly extended to 0 (see [Kat1], chap. II). One then has  $\pi(0) = \pi_1(0) + \pi_2(0)$  where  $\pi(0)$  is the projector onto the kernel of  $L(0)$ .

## 2.2 The profile equations

We recall the following straightforward lemma in order to express the usual solvability conditions of a linear equation in terms of the tools introduced above.

**Lemma 2.1** *For any  $a, b \in \mathbb{R}^N$*

$$\left( L(0)a = b \right) \Leftrightarrow \left( \pi(0)b = 0 \quad \text{and} \quad a = \Pi(0)a + L(0)^{-1}b \right)$$

Thanks to the above lemma, one turns now to the resolution of the set of equations (1.5)-(1.8).

- The first equation (1.5):  $E\mathcal{U}_0 = 0$ , from Lemma 2.1 reads as

$$\pi(0)\mathcal{U}_0 = \mathcal{U}_0. \tag{2.2}$$

This equation is non trivial since we assumed that  $L(0) = \frac{E}{i}$  is singular.

- The second equation (1.6):  $\partial_t \mathcal{U}_0 + A\partial_x \mathcal{U}_0 + E\mathcal{U}_1 = 0$  reads as,

$$L(0)\mathcal{U}_1 = iL_1(\partial)\mathcal{U}_0$$

where  $L_1(\partial) = \partial_t + A\partial_x$ . From Lemma 2.1, this is equivalent to the following solvability conditions,

$$\begin{cases} 0 = i\pi(0)L_1(\partial)\mathcal{U}_0 = i\pi(0)L_1(\partial)\pi(0)\mathcal{U}_0 & \text{thanks to (2.2)} \\ (I - \pi(0))\mathcal{U}_1 = iL(0)^{-1}L_1(\partial)\mathcal{U}_0 = iL(0)^{-1}A\partial_x\pi(0)\mathcal{U}_0 & \text{thanks to (2.2).} \end{cases} \tag{2.3}$$

- The third equation (1.7):  $\partial_t \mathcal{U}_1 + A\partial_x \mathcal{U}_1 + B\partial_Y \mathcal{U}_0 + E\mathcal{U}_2 = 0$  reads as,

$$L(0)\mathcal{U}_2 = iL_1(\partial)\mathcal{U}_1 + iB\partial_Y \mathcal{U}_0$$

which is equivalent, using Lemma 2.1, (2.3) and decomposing  $\mathcal{U}_1 = \pi(0)\mathcal{U}_1 + (I - \pi(0))\mathcal{U}_1$  to the following solvability conditions on the profiles,

$$\begin{cases} \pi(0)L_1(\partial)\pi(0)\mathcal{U}_1 = -i\pi(0)AL(0)^{-1}A\partial_x^2\pi(0)\mathcal{U}_0 - \pi(0)B\pi(0)\partial_Y\mathcal{U}_0 \\ (I - \pi(0))\mathcal{U}_2 = iL(0)^{-1}L_1(\partial)\mathcal{U}_1 + iL(0)^{-1}B\pi(0)\partial_Y\mathcal{U}_0. \end{cases} \quad (2.4)$$

- Let us turn now to the fourth profile equation (1.8):  $\partial_T\mathcal{U}_0 + \partial_t\mathcal{U}_2 + A\partial_x\mathcal{U}_2 + B\partial_Y\mathcal{U}_1 + E\mathcal{U}_3 = C(\mathcal{U}_0)\partial_x\mathcal{U}_0$ , which reads,

$$L(0)\mathcal{U}_3 = i\partial_T\mathcal{U}_0 + iL_1(\partial)\mathcal{U}_2 + iB\partial_Y\mathcal{U}_1 - iC(\mathcal{U}_0)\partial_x\mathcal{U}_0,$$

which is again equivalent to, thanks to Lemma 2.1,

$$\begin{cases} \partial_T\pi(0)\mathcal{U}_0 + \pi(0)L_1(\partial)\mathcal{U}_2 + \pi(0)B\partial_Y\mathcal{U}_1 = \pi(0)C(\mathcal{U}_0)\partial_x\mathcal{U}_0 \\ (I - \pi(0))\mathcal{U}_3 = iL(0)^{-1}L_1(\partial)\mathcal{U}_2 + iL(0)^{-1}B\partial_Y\mathcal{U}_1 - iL(0)^{-1}C(\mathcal{U}_0)\partial_x\mathcal{U}_0. \end{cases} \quad (2.5)$$

Decomposing  $\mathcal{U}_2 = \pi(0)\mathcal{U}_2 + (I - \pi(0))\mathcal{U}_2$  and using (2.4), the first equation in the above system becomes,

$$\begin{aligned} & \partial_T\pi(0)\mathcal{U}_0 + \pi(0)L_1(\partial)\pi(0)\mathcal{U}_2 + i\pi(0)AL(0)^{-1}L_1(\partial)\partial_x\mathcal{U}_1 \\ & + i\pi(0)AL(0)^{-1}B\pi(0)\partial_{x,Y}^2\mathcal{U}_0 + \pi(0)B\partial_Y\mathcal{U}_1 = \pi(0)C(\mathcal{U}_0)\partial_x\mathcal{U}_0 \end{aligned}$$

which again gives using (2.3) and writing  $\mathcal{U}_1 = \pi(0)\mathcal{U}_1 + (I - \pi(0))\mathcal{U}_1$ , the following equivalent system to (1.8),

$$\begin{cases} \partial_T\pi(0)\mathcal{U}_0 + \pi(0)L_1(\partial)\pi(0)\mathcal{U}_2 + i\pi(0)AL(0)^{-1}A\partial_x^2\pi(0)\mathcal{U}_1 \\ \quad - \pi(0)AL(0)^{-1}L_1(\partial)L(0)^{-1}A\partial_x^2\pi(0)\mathcal{U}_0 + i\pi(0)AL(0)^{-1}B\pi(0)\partial_{x,Y}^2\mathcal{U}_0 \\ \quad + \pi(0)B\pi(0)\partial_Y\mathcal{U}_1 + i\pi(0)BL(0)^{-1}A\pi(0)\partial_{x,Y}^2\mathcal{U}_0 = \pi(0)C(\mathcal{U}_0)\partial_x\mathcal{U}_0 \\ (I - \pi(0))\mathcal{U}_3 = iL(0)^{-1}L_1(\partial)\mathcal{U}_2 + iL(0)^{-1}B\partial_Y\mathcal{U}_1 - iL(0)^{-1}C(\mathcal{U}_0)\partial_x\mathcal{U}_0 \end{cases} \quad (2.6)$$

The equations obtained (2.2)-(2.6) constitute our set of solvability conditions on the profiles  $\mathcal{U}_0$ ,  $\mathcal{U}_1$ ,  $\mathcal{U}_2$  and  $\mathcal{U}_3$ .

**The operator  $\pi(0)L_1(\partial)\pi(0)$ .**

**Lemma 2.2** *If Assumption 2.1 is satisfied, one has*

$$\pi(0)L_1(\partial)\pi(0) = (\partial_t - \lambda'(0)\partial_x)\pi_1(0) + (\partial_t + \lambda'(0)\partial_x)\pi_2(0). \quad (2.7)$$

**Proof.**

The proof of this lemma can be found in [BC].

■

Applying Lemma 2.2 to (2.3) gives the two transport equations solved by each component of  $\mathcal{U}_0$ .

$$\begin{cases} (\partial_t - \lambda'(0)\partial_x)\pi_1(0)\mathcal{U}_0 = 0 \\ (\partial_t + \lambda'(0)\partial_x)\pi_2(0)\mathcal{U}_0 = 0 \end{cases} \quad (2.8)$$

**The operator**  $\pi(0)\mathbf{A}L(0)^{-1}\mathbf{A}\pi(0)\partial_{\mathbf{x}}^2$ .

**Lemma 2.3** *Under Assumption 2.1, the matrix  $\pi(0)AL(0)^{-1}A\pi(0)$  is given by,*

$$\pi(0)AL(0)^{-1}A\pi(0) = 2\lambda'(0)(\pi_2(0)\pi_1'(0) + \pi_1'(0)\pi_2(0)) \quad (2.9)$$

and thus,

$$\pi_1(0)AL(0)^{-1}A\pi_1(0) = \pi_2(0)AL(0)^{-1}A\pi_2(0) = 0.$$

**Proof.**

One can find the proof of this Lemma in [BC].

■

**The operator**  $\pi(0)\mathbf{B}\pi(0)\partial_{\mathbf{y}}$ .

**Lemma 2.4** *Under Assumption 2.1, we have that*

$$\pi_1(0)B\pi_1(0) = 0 \quad \text{and} \quad \pi_2(0)B\pi_2(0) = 0$$

**Proof.**

We prove the result for  $\pi_1(0)B\pi_1(0)$  and the proof is similar for  $\pi_2(0)B\pi_2(0)$ . Denoting by  $\tau(\xi, \eta)$  a local parametrization of  $CharL$  in a neighborhood of any smooth point, we have, by definition of  $\pi(\tau, \xi, \eta)$ ,

$$(\tau(\xi, \eta)I + A\xi + B\eta + \frac{E}{i})\pi(\tau(\xi, \eta), \xi, \eta) = 0$$

We differentiate this equation with respect to  $\eta$  and apply  $\pi(\tau, \xi, \eta)$  to find

$$\pi(\tau, \xi, \eta)B\pi(\tau, \xi, \eta) = -\partial_2\tau(\xi, \eta)\pi(\tau, \xi, \eta)$$

Besides one has that

$$\begin{aligned} \pi_1(0)B\pi_1(0) &= \lim_{\xi \rightarrow 0} \pi(\lambda(\xi), \xi, 0)B\pi(\lambda(\xi), \xi, 0) \\ &= -\lim_{\xi \rightarrow 0} \partial_2\tau(\xi, 0)\pi(\lambda(\xi), \xi, 0) \end{aligned}$$

From Assumption 2.1,  $CharL$  is axisymmetrical and thus there exists a function  $\tilde{\tau}$  such that,

$$\tau(\xi, \eta) = \tilde{\tau}(|(\xi, \eta)|) \quad \text{where} \quad |(\xi, \eta)| := (\xi^2 + \eta^2)^{1/2},$$

which implies that  $\partial_2\tau(\xi, \eta) = \frac{\eta}{|(\xi, \eta)|}\tilde{\tau}'(|(\xi, \eta)|)$  and therefore  $\partial_2\tau(\xi, 0) = 0$ . The proof is thus complete. ■

Using Lemmas 2.2-2.4 and applying  $\pi_1(0)$  to the first equation of (2.6) gives,

$$\begin{aligned} &\partial_T\pi_1(0)\mathcal{U}_0 + (\partial_t - \lambda'(0)\partial_x)\pi_1(0)\mathcal{U}_2 + i\pi_1(0)AL(0)^{-1}A\partial_x^2\pi_2(0)\mathcal{U}_1 \\ &-\pi_1(0)AL(0)^{-1}L_1(\partial)L(0)^{-1}A\partial_x^2[\pi_1(0) + \pi_2(0)]\mathcal{U}_0 + \pi_1(0)B\pi_2(0)\partial_Y\mathcal{U}_1 \\ &\quad + i\pi_1(0)AL(0)^{-1}B[\pi_1(0) + \pi_2(0)]\partial_{x,Y}^2\mathcal{U}_0 \\ &+ i\pi_1(0)BL(0)^{-1}A[\pi_1(0) + \pi_2(0)]\partial_{x,Y}^2\mathcal{U}_0 = \pi_1(0)C(\mathcal{U}_0)\partial_x\mathcal{U}_0. \end{aligned} \tag{2.10}$$

The components  $\pi_1(0)\mathcal{U}_1$  and  $\pi_2(0)\mathcal{U}_1$  of  $\mathcal{U}_1$  can be computed explicitly in terms of  $\mathcal{U}_0$  from the first equation in (2.4). Indeed, applying  $\pi_1(0)$  and  $\pi_2(0)$  to (2.4) gives, thanks to Lemmas 2.2-2.4,

$$(\partial_t - \lambda'(0)\partial_x)\pi_1(0)\mathcal{U}_1 = -i\pi_1(0)AL(0)^{-1}A\partial_x^2\pi_2(0)\mathcal{U}_0 - \pi_1(0)B\pi_2(0)\partial_Y\mathcal{U}_0$$

and likewise,

$$(\partial_t + \lambda'(0)\partial_x)\pi_2(0)\mathcal{U}_1 = -i\pi_2(0)AL(0)^{-1}A\partial_x^2\pi_1(0)\mathcal{U}_0 - \pi_2(0)B\pi_1(0)\partial_Y\mathcal{U}_0.$$

Thanks to Eqs. (2.8), we can solve both these equations by taking

$$\pi_1(0)\mathcal{U}_1 = \frac{i}{2\lambda'(0)}\pi_1(0)AL(0)^{-1}A\pi_2(0)\partial_x\mathcal{U}_0 + \frac{1}{2\lambda'(0)}\pi_1(0)B\pi_2(0)\partial_x^{-1}\partial_Y\mathcal{U}_0 \quad (2.11)$$

and

$$\pi_2(0)\mathcal{U}_1 = \frac{-i}{2\lambda'(0)}\pi_2(0)AL(0)^{-1}A\pi_1(0)\partial_x\mathcal{U}_0 - \frac{1}{2\lambda'(0)}\pi_2(0)B\pi_1(0)\partial_x^{-1}\partial_Y\mathcal{U}_0 \quad (2.12)$$

**Remark 2.2** We assume for the moment that  $\mathcal{U}_0$  is such that the expression  $\partial_x^{-1}\partial_Y^2\mathcal{U}_0$  makes sense. We will see later that we can solve the profile equations on  $\mathcal{U}_0$  in such a class of functions.

We now plug back  $\pi_1(0)\mathcal{U}_1$  and  $\pi_2(0)\mathcal{U}_1$  into (2.10), thus yielding

$$\begin{aligned} & \partial_T\pi_1(0)\mathcal{U}_0 + \mathbf{M}_1\partial_x^3\pi_1(0)\mathcal{U}_0 - \frac{1}{2\lambda'(0)}\pi_1(0)B\pi_2(0)B\pi_1(0)\partial_x^{-1}\partial_Y^2\pi_1(0)\mathcal{U}_0 \\ & + \mathbf{N}_1\partial_{x,Y}^2\pi_1(0)\mathcal{U}_0 + \mathbf{O}_1(\partial)\mathcal{U}_0 + (\partial_t - \lambda'(0)\partial_x)\pi_1(0)\mathcal{U}_2 \\ & = \pi_1(0)C(\pi_1(0)\mathcal{U}_0)\partial_x\pi_1(0)\mathcal{U}_0 \end{aligned} \quad (2.13)$$

where the matrices  $\mathbf{M}_1$  and  $\mathbf{N}_1$  are given by

$$\begin{aligned} \mathbf{M}_1 &= \frac{1}{2\lambda'(0)}\pi_1(0)AL(0)^{-1}A\pi_2(0)AL(0)^{-1}A\pi_1(0) \\ &- \pi_1(0)AL(0)^{-1}AL(0)^{-1}A\pi_1(0) - \lambda'(0)\pi_1(0)AL(0)^{-2}A\pi_1(0) \end{aligned}$$

and

$$\begin{aligned}
\mathbf{N}_1 &= i\pi_1(0)AL(0)^{-1}B\pi_1(0) + i\pi_1(0)BL(0)^{-1}A\pi_1(0) \\
&\quad - \frac{i}{2\lambda'(0)}\pi_1(0)B\pi_2(0)AL(0)^{-1}A\pi_1(0) \\
&\quad - \frac{i}{2\lambda'(0)}\pi_1(0)AL(0)^{-1}A\pi_2(0)B\pi_1(0),
\end{aligned}$$

which simplifies using Lemma 2.3,

$$\begin{aligned}
\mathbf{N}_1 &= i\pi_1(0)AL(0)^{-1}B\pi_1(0) + i\pi_1(0)BL(0)^{-1}A\pi_1(0) \\
&\quad - i\pi_1(0)B\pi_2(0)\pi_1'(0)\pi_1(0) - i\pi_1(0)\pi_1'(0)\pi_2(0)B\pi_1(0).
\end{aligned}$$

Finally the operator  $\mathbf{O}_1(\partial)$  is given by,

$$\begin{aligned}
\mathbf{O}_1(\partial)\mathcal{U}_0 &= i[\pi_1(0)AL(0)^{-1}B\pi_2(0) + \pi_1(0)BL(0)^{-1}A\pi_2(0)]\partial_{x,Y}^2\pi_2(0)\mathcal{U}_0 \\
&\quad + \pi_1(0)AL(0)^{-1}(\lambda'(0) - A)L(0)^{-1}A\pi_2(0)\partial_x^3\pi_2(0)\mathcal{U}_0 \\
&\quad - \pi_1(0)C(\mathcal{U}_0)\partial_x\mathcal{U}_0 + \pi_1(0)C(\pi_1(0)\mathcal{U}_0)\partial_x\pi_1(0)\mathcal{U}_0
\end{aligned}$$

**Remark 2.3** In the right hand side of Eq. (2.13), we have kept only the products of the component  $\pi_1(0)\mathcal{U}_0$ . All the other terms of the nonlinearity, i.e. all the products involving  $\pi_2(0)\mathcal{U}_0$  are put in  $\mathbf{O}_1$ . The reason of this is that these latter nonlinear terms are asymptotically non relevant for the evolution of  $\pi_1(0)\mathcal{U}_0$ , as we will see in the next section.

Similarly, we obtain an equation governing the long time evolution of  $\pi_2(0)\mathcal{U}_0$ , as is

$$\begin{aligned}
&\partial_T\pi_2(0)\mathcal{U}_0 + \mathbf{M}_2\partial_x^3\pi_2(0)\mathcal{U}_0 + \frac{1}{2\lambda'(0)}\pi_2(0)B\pi_1(0)B\pi_2(0)\partial_x^{-1}\partial_Y^2\pi_2(0)\mathcal{U}_0 \\
&\quad + \mathbf{N}_2\partial_{x,Y}^2\pi_2(0)\mathcal{U}_0 + \mathbf{O}_2(\partial)\mathcal{U}_0 + (\partial_t + \lambda'(0)\partial_x)\pi_2(0)\mathcal{U}_2 \\
&= \pi_2(0)C(\pi_2(0)\mathcal{U}_0)\partial_x\pi_2(0)\mathcal{U}_0
\end{aligned} \tag{2.14}$$

where  $\mathbf{M}_2$ ,  $\mathbf{N}_2$  and  $\mathbf{O}_2(\partial)$  are deduced from  $\mathbf{M}_1$ ,  $\mathbf{N}_1$  and  $\mathbf{O}_1(\partial)$  by swapping  $\pi_1(0)$  and  $\pi_2(0)$ , as well as  $\lambda'(0)$  and  $-\lambda'(0)$ .

In order to simplify Eqs. (2.13)-(2.14), we state at this point the following algebraic proposition.

**Proposition 2.1** *i) One has the following relation*

$$\mathbf{M}_1 = \frac{\lambda'''(0)}{6}\pi_1(0) \quad \text{and} \quad \mathbf{M}_2 = -\frac{\lambda'''(0)}{6}\pi_2(0).$$

*ii) One has*

$$\pi_1(0)B\pi_2(0)B\pi_1(0) = \lambda'(0)^2\pi_1(0) \quad \text{and} \quad \pi_2(0)B\pi_1(0)B\pi_2(0) = \lambda'(0)^2\pi_2(0)$$

*iii) The following equalities hold*

$$\mathbf{N}_1 = 0 \quad \text{and likewise} \quad \mathbf{N}_2 = 0.$$

**Proof.**

**i)** The proof of this result is given in [BC], Lemma 2.3.

**ii)** Since both of the equalities stated by the proposition are proved in the same way, we just demonstrate the first one. We recall that the symbol  $L$  is defined for all  $\tau$  and  $\Theta = (\xi, \eta)$  as

$$L(\tau, \Theta) = \tau Id + A\xi + B\eta + \frac{E}{i},$$

and define the associated tangent symbol  $L^0$  as

$$L^0(\tau, \Theta) = \pi(0)L(\tau, \Theta)\pi(0),$$

so called since its characteristic variety  $\mathcal{C}^0$  is the tangent cone at the origins to the characteristic variety  $\mathcal{C}_L$  associated to  $L$  ([L], Prop. 2). Thanks to Assumption 2.1, we thus know that  $\mathcal{C}^0$  is given by

$$\mathcal{C}^0 = \{(\tau, \Theta) \in \mathbb{R}^3, \tau = \pm\lambda'(0)|\Theta|\}, \quad (2.15)$$

so that one has

$$L^0(\tau, \Theta) = \tau\pi(0) - \lambda'(0)|\Theta|(\pi_1^0(\Theta) - \pi_2^0(\Theta)),$$

where  $\pi_1^0(\Theta)$  (resp.  $\pi_2^0(\Theta)$ ) is the orthogonal projector onto  $\ker L^0(\lambda'(0)|\Theta|, \Theta)$  (resp. onto  $\ker L^0(-\lambda'(0)|\Theta|, \Theta)$ ). One of the key points of this proof is that we have therefore

$$L^0(\lambda'(0)|\Theta|, \Theta)^{-1} = \frac{1}{2\lambda'(0)|\Theta|} \pi_2^0(\Theta), \quad (2.16)$$

where the exponent minus one of the left hand side obviously denotes the partial inverse.

The next step of the proof is to show that one has, for all  $\xi > 0$ ,

$$\pi_1^0((\xi, 0)) = \pi_1(0) \quad \text{and} \quad \pi_2^0((\xi, 0)) = \pi_2(0). \quad (2.17)$$

By definition of  $\pi_1$ , one has, for all  $h > 0$ ,

$$(\lambda(h\xi)Id + Ah\xi + \frac{E}{i})\pi_1(h\xi) = 0,$$

and multiplying this equality on the left by  $\pi(0)$  yields

$$\lambda(h\xi)\pi(0)\pi_1(h\xi) + \pi(0)Ah\xi\pi_1(h\xi) = 0.$$

Dividing by  $h$  and letting  $h \rightarrow 0$ , one obtains

$$\xi\lambda'(0)\pi_1(0) + \pi(0)A\xi\pi_1(0) = 0 \quad \text{that is,} \quad L^0(\lambda'(0)\xi, \xi, 0)\pi_1(0) = 0,$$

which proves that the range of  $\pi_1(0)$  is included in the range of  $\pi_1^0((\xi, 0))$ .

Similarly, we can prove that the range of  $\pi_2(0)$  is included in the range of  $\pi_2^0((\xi, 0))$ , and since we have  $\pi(0) = \pi_1(0) + \pi_2(0) = \pi_1^0((\xi, 0)) + \pi_2^0((\xi, 0))$ , Eq. (2.17) is proved.

Now, using Eqs. (2.16)-(2.17), we can write

$$L^0(\lambda'(0)\xi, (\xi, 0))^{-1} = \frac{1}{2\lambda'(0)\xi} \pi_2(0),$$

so that

$$\pi_1(0)B\pi_2(0)B\pi_1(0) = 2\lambda'(0)\xi\pi_1^0((\xi, 0))BL^0(\lambda'(0)\xi, (\xi, 0))^{-1}B\pi_1^0((\xi, 0)).$$

Since  $(\lambda'(0)\xi, \xi, 0)$  is a smooth point of  $\mathcal{C}^0$ , we can deduce (from Prop. 4.2 of [JMR2] for instance) that

$$\pi_1(0)B\pi_2(0)B\pi_1(0) = \lambda'(0)\xi\partial_2^2\tau^0(\xi, 0)\pi_1(0),$$

where  $\tau^0$  is a local parameterization of  $\mathcal{C}^0$  in a neighborhood of  $(\lambda'(0)\xi, \xi, 0)$ . Thanks to (2.15), we know that we have necessarily  $\tau^0(\Theta) = \lambda'(0)|\Theta|$ , and thus  $\partial_2^2\tau^0(\xi, 0) = \lambda'(0)/\xi$ , which finishes the proof of the second point of the proposition.

**iii)** Here again, we only prove the first of the two identities stated, since the second one can be obtained in the same way. We introduce here the regular mapping  $\Pi(\xi) = \pi_1(\xi) + \pi_2(\xi)$ . Note that  $\Pi(0) = \pi(0)$ . We use here the following identity ([BC], Eq. (2.16))

$$L(0)^{-1}A\pi(0) + (Id - \pi(0))\Pi'(0) = 0,$$

and multiply it on the left by  $\pi(0)B$  to obtain an expression of  $\pi(0)B\Pi'(0)$ ,

$$\pi(0)B\Pi'(0) = \pi(0)B\pi(0)\Pi'(0) - \pi(0)BL(0)^{-1}A\pi(0).$$

Since  $\pi(0) = \pi_1(0) + \pi_2(0)$  and  $\Pi'(0) = \pi_1'(0) + \pi_2'(0)$ , this equation reads

$$\begin{aligned} & \pi(0)B\pi_1'(0) + \pi(0)B\pi_2'(0) \\ &= \pi(0)B\pi(0)\pi_1'(0) + \pi(0)B\pi(0)\pi_2'(0) - \pi(0)BL(0)^{-1}A\pi(0) \\ &= \pi_1(0)B\pi_2(0)\pi_1'(0) + \pi_2(0)B\pi_1(0)\pi_1'(0) \\ & \quad + \pi_1(0)B\pi_2(0)\pi_2'(0) + \pi_2(0)B\pi_1(0)\pi_2'(0) - \pi_1(0)BL(0)^{-1}A\pi(0), \end{aligned}$$

where we have used in the last equality the fact that  $\pi_1(0)B\pi_1(0) = 0$  and  $\pi_2(0)B\pi_2(0) = 0$ . Multiplying on both sides by  $\pi_1(0)$  yields

$$\begin{aligned} & \pi_1(0)B\pi_1'(0)\pi_1(0) + \pi_1(0)B\pi_2'(0)\pi_1(0) = \\ & \pi_1(0)B\pi_2(0)\pi_1'(0)\pi_1(0) + \pi_1(0)B\pi_2(0)\pi_2'(0)\pi_1(0) - \pi_1(0)BL(0)^{-1}A\pi_1(0) \end{aligned}$$

Since  $\pi_1'(0)\pi_2(0) + \pi_1(0)\pi_2'(0) = 0$ , this reads

$$\begin{aligned} \pi_1(0)B\pi_1'(0)\pi_1(0) - \pi_1(0)B\pi_2(0)\pi_1'(0) = \\ \pi_1(0)B\pi_2(0)\pi_1'(0)\pi_1(0) - \pi_1(0)B\pi_2(0)\pi_1'(0) - \pi_1(0)BL(0)^{-1}A\pi_1(0), \end{aligned}$$

and therefore

$$\pi_1(0)B\pi_1'(0)\pi_1(0) = \pi_1(0)B\pi_2(0)\pi_1'(0)\pi_1(0) - \pi_1(0)BL(0)^{-1}A\pi_1(0).$$

Taking the adjoint of this equality and adding both of them thus yields

$$\begin{aligned} \pi_1(0)B\pi_1'(0)\pi_1(0) + \pi_1(0)\pi_1'(0)B\pi_1(0) = \\ \pi_1(0)B\pi_2(0)\pi_1'(0)\pi_1(0) - \pi_1(0)BL(0)^{-1}A\pi_1(0) \\ + \pi_1(0)\pi_1'(0)\pi_2(0)B\pi_1(0) - \pi_1(0)AL(0)^{-1}B\pi_1(0). \end{aligned}$$

The right hand side of this equality is exactly the term the proposition claims to be zero. The left hand side also reads  $\pi_1(0)B\pi_1'(0) + \pi_1'(0)B\pi_1(0)$  since  $\pi_1'(0) = \pi_1'(0)\pi_1(0) + \pi_1(0)\pi_1'(0)$  and  $\pi_1(0)B\pi_1(0) = 0$ . It is therefore equal to  $(\pi_1 B \pi_1)'(0)$  and hence vanishes, since for all  $\xi$ , one has  $\pi_1(\xi)B\pi_1(\xi) = 0$ . The proof is thus complete. ■

Using this lemma, Eqs. (2.13)-(2.14) read

$$\begin{aligned} \partial_T \pi_1(0)\mathcal{U}_0 + \frac{\lambda'''(0)}{6} \partial_x^3 \pi_1(0)\mathcal{U}_0 - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \pi_1(0)\mathcal{U}_0 \\ + (\partial_t - \lambda'(0)\partial_x)\pi_1(0)\mathcal{U}_2 + \mathbf{O}_1(\partial)\mathcal{U}_0 \\ = \pi_1(0)C(\pi_1(0)\mathcal{U}_0)\partial_x \pi_1(0)\mathcal{U}_0, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \partial_T \pi_2(0)\mathcal{U}_0 - \frac{\lambda'''(0)}{6} \partial_x^3 \pi_2(0)\mathcal{U}_0 + \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \pi_2(0)\mathcal{U}_0 \\ + (\partial_t + \lambda'(0)\partial_x)\pi_2(0)\mathcal{U}_2 + \mathbf{O}_2(\partial)\mathcal{U}_0 \\ = \pi_2(0)C(\pi_2(0)\mathcal{U}_0)\partial_x \pi_2(0)\mathcal{U}_0. \end{aligned} \tag{2.19}$$

### 2.3 The average operators

**Transport operators.** We introduce for convenience the two transport operators  $T_1(\partial)$  and  $T_2(\partial)$ ,

$$\begin{cases} T_1(\partial) = \partial_t - \lambda'(0)\partial_x \\ T_2(\partial) = \partial_t + \lambda'(0)\partial_x \end{cases} \quad (2.20)$$

We use average operators in order to eliminate  $\mathcal{U}_2$  from Eqs. (2.13)-(2.14) governing the slow evolution of the profile  $\mathcal{U}_0$ . We recall that we assume that  $\mathcal{U}_2$  respects some growth condition.

As in [L], an average operator is defined relatively to a transport operator. Hence for  $T_1$  and  $T_2$ , we define two average operators  $G_{T_1}$  and  $G_{T_2}$ ,

**Definition 2.2** For  $h > 0$  and  $w$  sufficiently smooth,

$$\begin{aligned} G_{T_1}^h w(T, t, x, Y) &= \frac{1}{h} \int_0^h w(T, x - \lambda'(0)s, t + s, Y) ds \\ G_{T_2}^h w(T, t, x, Y) &= \frac{1}{h} \int_0^h w(T, x + \lambda'(0)s, t + s, Y) ds \end{aligned}$$

and,

$$\begin{cases} G_{T_1} w = \lim_{h \rightarrow \infty} G_{T_1}^h w \\ G_{T_2} w = \lim_{h \rightarrow \infty} G_{T_2}^h w \end{cases} \quad (2.21)$$

when this limit exists.

These operators were described and introduced in detail in [L]. We recall their properties and refer to [L] for the corresponding proofs.

**Proposition 2.2 (Properties of the average operator)** Let  $T$  be a transport operator such that  $T(\partial) = \partial_t - c\partial_x$ , then

- i) If  $w$  satisfies  $T(\partial)w = 0$ , then  $G_T w$  exists and  $G_T w = w$ .
- ii) If  $w$  satisfies  $T'(\partial)w = 0$  where  $T'(\partial) = \partial_t - c'\partial_x$  and if  $c \neq c'$  then  $G_T w$  exists and  $G_T w = 0$ .

iii) If  $w$  respects a sub-linear (and a fortiori a sub-squareroot) growth condition, then  $G_T T(\partial)w$  is well defined and  $G_T T(\partial)w = 0$ .

iv) Let  $W := ww'$  where  $w$  and  $w'$  are such that  $T(\partial)w = 0$  and  $T'(\partial)w' = 0$ . If  $T(\partial) = T'(\partial)$ , then  $G_T W = W$ . In any other case  $G_T W = 0$ .

The first two properties mean that when we apply  $G_T$  to the linear terms of the equations, it leaves only those transported by  $T(\partial)$  and eliminates the rest. The third property allows us to get rid of the correctors in the equations as it was the motivation in the construction of these operators. And the important last property allows us to eliminate all the product terms where the factors are transported by different operators. And as we said earlier, it is thanks to this last property that we will reduce dramatically the nonlinear terms and thus uncouple the system (2.13)-(2.14) in order to derive a pair of independent KdV equations for the evolution of each component of  $\mathcal{U}_0$ .

## 2.4 Consequences for the profile equations

As we are looking for solvability conditions on the system (2.18)-(2.19), let us apply the operator  $G_{T_1}$  on (2.18) and  $G_{T_2}$  on (2.19), which gives thanks to the properties of these operators,

$$\begin{aligned} & \partial_T \pi_1(0) \mathcal{U}_0 + \frac{\lambda'''(0)}{6} \partial_x^3 \pi_1(0) \mathcal{U}_0 - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \pi_1(0) \mathcal{U}_0 \\ & + \underbrace{G_{T_1}(\partial_t - \lambda'(0) \partial_x) \pi_1(0) \mathcal{U}_2}_{=0 \text{ prop. iii)}} + \underbrace{G_{T_1} \mathbf{O}_1(\partial) \pi_2(0) \mathcal{U}_0}_{=0 \text{ prop. ii) and iv)}} = G_{T_1}(\pi_1(0) C(\mathcal{U}_0) \partial_x \mathcal{U}_0) \end{aligned} \quad (2.22)$$

Thanks to point iv) of Prop 2.2, we know that the nonlinear terms present in  $\mathbf{O}_1(\partial)$  vanish under the action of  $G_{T_1}$  since at least one of their components is transported by  $T_2$ . The right hand side of the equation, though nonlinear, is left invariant by  $G_{T_1}$  since all its components are transported by  $T_1$ . The first three terms of the above equation are left invariant by  $G_{T_1}$  as a consequence

of point  $i$ ) of Prop. 2.2

We obtain similarly an analog equation governing  $\pi_2(0)\mathcal{U}_0$ . The system (2.18)-(2.19) therefore reduces to

$$\begin{cases} \left( \partial_T + \frac{\lambda'''(0)}{6} \partial_x^3 - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \right) \pi_1(0)\mathcal{U}_0 = \pi_1(0)C(\pi_1(0)\mathcal{U}_0) \partial_x \pi_1(0)\mathcal{U}_0 \\ \left( \partial_T - \frac{\lambda'''(0)}{6} \partial_x^3 + \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \right) \pi_2(0)\mathcal{U}_0 = \pi_2(0)C(\pi_2(0)\mathcal{U}_0) \partial_x \pi_2(0)\mathcal{U}_0 \end{cases} \quad (2.23)$$

and the corrector  $\mathcal{U}_2$  verifies thereafter the necessary conditions

$$\begin{cases} T_1(\partial)\pi_1(0)\mathcal{U}_2 = -\mathbf{O}_1(\partial)\pi_2(0)\mathcal{U}_0 \\ T_2(\partial)\pi_2(0)\mathcal{U}_2 = -\mathbf{O}_2(\partial)\pi_1(0)\mathcal{U}_0 \end{cases} \quad (2.24)$$

### 3 Properties of the approximate solution $u^\epsilon$

#### 3.1 The leading term of the ansatz (1.2)

We have obtained a set of necessary conditions on the profile  $\mathcal{U}^\epsilon(T, t, x, Y)$ . We want to know in particular the leading order term  $\mathcal{U}_0$ . Therefore our aim is to simultaneously solve (2.23), the transport equations (2.8), together with the polarization condition (2.2).

The slow evolution of both components of  $\mathcal{U}_0$  satisfy a KP equation. The KP equation appearing in (2.23) may either be of KP-I or KP-II type depending on the sign of the odd derivatives of  $\lambda$  at the origin. As we only need a local existence theorem, this distinction is not important here, and the following result, which is not sharp, will be sufficient for our study.

**Theorem 3.1** *Let  $s \geq 3$  and  $\alpha, \beta$  and  $\gamma$  be real constants. Let  $f_0 \in H^s(\mathbb{R}^2)$  satisfying*

$$f_0 = \varphi_x^0 \quad \text{with some} \quad \varphi^0 \in H_{loc}^s(\mathbb{R}^2), \quad \nabla \varphi^0 \in H^s(\mathbb{R}^2).$$

Then there exists a constant  $t_0 > 0$  such that the equation

$$(f_t + \alpha f f_x + \beta f_{xxx})_x + \gamma f_{yy} = 0$$

has a unique solution  $f \in C^0([-t_0, t_0]; H^s(\mathbb{R}^2)) \cap Lip([-t_0, t_0]; H^{s-3}(\mathbb{R}^2))$  satisfying  $f(0) = f^0$ .

**Proof.**

This is a classical result from S. Ukai [U].

■

**Corollary 3.1** *Let  $s \geq 3$  and  $\mathcal{U}_0^0 \in (H^s(\mathbb{R}_{x,Y}^2))^N$  satisfying*

$$\pi(0)\mathcal{U}_0^0 = \mathcal{U}_0^0 \quad \text{and} \quad \mathcal{U}_0^0 = \varphi_x^0 \quad \text{with} \quad \varphi^0 \in (H^{s+1}(\mathbb{R}_{x,Y}^2))^N.$$

*i) Then there exists  $T_0 > 0$  and a unique profile*

$$\mathcal{U}_0 \in C([0, T_0] \times \mathbb{R}_t; (H^s(\mathbb{R}_{x,Y}^2))^N) \cap Lip([0, T_0] \times \mathbb{R}_t; (H^{s-3}(\mathbb{R}_{x,Y}^2))^N)$$

*satisfying the initial condition  $\mathcal{U}_0(0, 0, \cdot, \cdot) = \mathcal{U}_0^0(\cdot, \cdot)$  and the set of equations*

$$\pi(0)\mathcal{U}_0 = \mathcal{U}_0, \quad (\partial_t - \lambda'(0)\partial_x)\pi_1(0)\mathcal{U}_0 = 0 \quad \text{and} \quad (\partial_t + \lambda'(0)\partial_x)\pi_2(0)\mathcal{U}_0 = 0$$

*as well as the uncoupled KP equations (2.23).*

*ii) If moreover the nonlinearity  $C(u)\partial_x u$  may write under the form  $\partial_x \nabla P(u)$ , with  $P$  being a homogeneous polynomial of degree 3, then  $\mathcal{U}_0$  has the form*

$$\mathcal{U}_0(T, t, x, Y) = \pi_1(0)\varphi_x(T, x + \lambda'(0)t, Y) + \pi_2(0)\varphi_x(T, x - \lambda'(0)t, Y),$$

*with  $\varphi \in L^\infty([0, T_0]; (H^s(\mathbb{R}_{x,Y}^2))^N)$ .*

**Remark 3.1** The second point of the above corollary justifies the notation  $\partial_x^{-1}$  used throughout this paper. Moreover, this point is of crucial importance in the estimates of  $\mathcal{U}_1$ .

**Proof.**

Because of the first two equations of the system satisfied by  $\mathcal{U}_0$ , we have to look for  $\mathcal{U}_0$  under the form

$$\mathcal{U}_0(T, t, x, Y) = \pi_1(0)\widetilde{\mathcal{U}}_0(T, x + \lambda'(0)t, Y) + \pi_2(0)\widetilde{\mathcal{U}}_0(T, x - \lambda'(0)t, Y).$$

Existence and uniqueness of  $\widetilde{\mathcal{U}}_0$  is a direct consequence of the first point of Theorem 3.1.

The proof of point ii) follows the proof of Theor. 3.1ii) of [Sa]. We prove the result for the component  $\pi_1(0)\widetilde{\mathcal{U}}_0$  without loss of generality.

Denoting by  $S(T)$  the semi-group associated to the linear part of Eq. (2.23)<sub>1</sub>,  $\pi_1(0)\widetilde{\mathcal{U}}_0$  writes

$$\pi_1(0)\widetilde{\mathcal{U}}_0(T) = S(T)\pi_1(0)\mathcal{U}_0^0 + \int_0^T S(T-\tau)\pi_1(0)C(\pi_1(0)\widetilde{\mathcal{U}}_0(\tau))\partial_x\pi_1(0)\widetilde{\mathcal{U}}_0(\tau)d\tau.$$

Since  $\mathcal{U}_0^0 = \varphi_x^0$  with  $\varphi^0 \in H^s(\mathbb{R}^2)^N$ , point ii) is obviously true if we can write  $C(\pi_1(0)\widetilde{\mathcal{U}}_0)\partial_x\pi_1(0)\widetilde{\mathcal{U}}_0 = \partial_x\nabla P(\pi_1(0)\widetilde{\mathcal{U}}_0)$ , which is exactly what we have assumed. ■

From now on, we assume that we are in the situation stated by point ii) of the above corollary, i.e. we suppose that the nonlinearity derives from a gradient, i.e. is symmetric and conservative. In the present case, it reads,

**Assumption 3.1** *The nonlinearity  $C(u)\partial_x u$  also writes*

$$C(u)\partial_x u = \partial_x\nabla P(u),$$

where  $P$  is a homogeneous polynomial of degree 3.

### 3.2 Corrector terms of the ansatz

The leading profile  $\mathcal{U}_0$  being given by Cor. 3.1, we now construct the missing terms of the ansatz in accordance with the necessary conditions obtained previously.

$\pi(0)\mathcal{U}_1$  is fully explicit from the relations (2.11)-(2.12) whereas  $(I - \pi(0))\mathcal{U}_1$  is obtained from the second equation in (2.3).

The second corrector  $\mathcal{U}_2$  shall also be determined from  $\mathcal{U}_0$ . Indeed,  $(I - \pi(0))\mathcal{U}_2$  is given by (2.4) and each component of  $\pi(0)\mathcal{U}_2$  is the solution of a well posed hyperbolic equation given in (2.24).

We turn now to the third corrector  $\mathcal{U}_3$ . We have that its component  $(I - \pi(0))\mathcal{U}_3$  is given by (2.6) whereas  $\pi(0)\mathcal{U}_3$  can be taken equal to 0.

Now that all the components of the ansatz (1.2) are known, we can state a regularity result.

**Proposition 3.1** *If  $\mathcal{U}_0$  lies in  $C([0, T_0] \times \mathbb{R}_t; H^s(\mathbb{R}_{x,Y}^2)^N)$  then  $\mathcal{U}_1, \mathcal{U}_2$  and  $\mathcal{U}_3$  lie in  $C([0, T_0] \times \mathbb{R}_t; H^{s-3}(\mathbb{R}_{x,Y}^2)^N)$*

**Proof.**

Theorem 3.1 and its corollary gives that the main order term  $\mathcal{U}_0$  lies in the correct space. From (2.11),(2.12) ,(2.24) and (2.5) we can deduce that the profiles  $\mathcal{U}_{1 \leq j \leq 3}$  defining the corrector terms lie in  $C([0, T_0] \times \mathbb{R}_t; H^{s-3}(\mathbb{R}_{x,Y}^2)^N)$ . Indeed, from Assumption 3.1, Cor. 3.1 and Eqs. (2.11)-(2.12), there exists  $\varphi \in C([0, T_0]; H^s(\mathbb{R}_{x,Y}^2)^N)$  such that

$$\begin{aligned} \pi_1(0)\mathcal{U}_1(T, x, t, Y) &= \frac{i}{2\lambda'(0)}\pi_1(0)AL(0)^{-1}A\pi_2(0)\partial_x^2\varphi(T, x - \lambda'(0)t, Y) \\ &\quad + \frac{1}{2\lambda'(0)}\pi_1(0)B\pi_2(0)\partial_Y\varphi(T, x - \lambda'(0)t, Y) \end{aligned} \tag{3.25}$$

and an analog expression holds for  $\pi_2(0)\mathcal{U}_1$ , which implies that  $\mathcal{U}_1$  lies in the correct destination space. It is then obvious to finish the proof of the proposition for the other profiles defined explicitly. ■

Note that if the profiles  $\mathcal{U}_{1 \leq j \leq 3}$  lie in  $C([0, T_0] \times \mathbb{R}_t; H^{s-3}(\mathbb{R}_{x,Y}^2)^N)$ , (with  $s - 3 > 1$  such that  $H^{s-3}$  embeds in  $L^\infty$ ), one deduces that the functions  $u_j^\epsilon$  defined

as

$$u_j^\epsilon(t, x, y) = \mathcal{U}_j(\epsilon^2 t, t, x, \epsilon y)$$

lie in  $L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x,Y}^2)$ .

It is now crucial to show that the corrector term  $(\epsilon u_1^\epsilon + \epsilon^2 u_2^\epsilon + \epsilon^3 u_3^\epsilon)(t, x, y) = (\epsilon \mathcal{U}_1 + \epsilon^2 \mathcal{U}_2 + \epsilon^3 \mathcal{U}_3)(\epsilon^2 t, t, x, \epsilon y)$  remains smaller than the leading order term  $u_0^\epsilon(t, x, y)$  for times of order  $O(\frac{1}{\epsilon^2})$ .

Thanks to the previous calculations, we know that

- $\mathcal{U}_1$  is bounded in  $C([0, T_0] \times \mathbb{R}_t; H^{s-3}(\mathbb{R}_{x,Y}^2)^N)$  since  $\varphi$  is bounded thanks to Corollary 4.1 (point ii). Hence,  $u_1^\epsilon(t, x, y) = \mathcal{U}_1(\epsilon^2 t, t, x, \epsilon y)$  is bounded independently of  $\epsilon$  in  $L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x,y}^2)$  and one has that,

$$\|\epsilon u_1^\epsilon\|_{L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x,y}^2)} = O(\epsilon)$$

- One recalls that each component of  $\mathcal{U}_2$  solves the following hyperbolic equation (2.24):

$$\begin{cases} T_1(\partial)\pi_1(0)\mathcal{U}_2 = -\mathbf{O}_1(\partial)\pi_2(0)\mathcal{U}_0 \\ T_2(\partial)\pi_2(0)\mathcal{U}_2 = -\mathbf{O}_2(\partial)\pi_1(0)\mathcal{U}_0 \end{cases}$$

where  $\mathcal{U}_0$  is bounded in  $C([0, T_0] \times \mathbb{R}_t; H^s(\mathbb{R}_{x,Y}^2)^N)$ . In a simplified way according to the variables  $(x, t)$  and taking  $(Y, T)$  as parameters, one can write the first equation for instance as,

$$(\partial_t - \lambda'(0)\partial_x)u = \partial_x \mathbf{f}(x - \lambda'(0)t) + \mathbf{g}(x - \lambda'(0)t)\mathbf{h}(x + \lambda'(0)t) \quad (3.26)$$

where  $\mathbf{f}, \mathbf{g}$  and  $\mathbf{h}$  are smooth functions. Again (3.26) is true thanks to assumption ii) in Corollary 4.1. Since  $C(u)\partial_x u = \partial_x \partial_u P(u)$ , one has that the terms in the r.h.s transported in the sole direction  $+\lambda'(0)$  can be written in the form of a derivative  $\partial_x \mathbf{f}(x - \lambda'(0)t)$ .

Now thanks to propositions 3.2 and 3.3 of [BC], one has that the first term in the r.h.s. gives a bounded contribution in  $u$  whereas the second

term induces a sub-square root growth in time for  $u$ . This result applies for both components of  $\mathcal{U}_2$ , so that the profile  $\mathcal{U}_2$  respects a sub-square root growth condition (1.3) that reads,

$$\lim_{t \rightarrow \infty} \frac{1}{\sqrt{t}} \|\mathcal{U}_2(T, t, x, Y)\|_{L^\infty([0, T_0]; H^{s-1}(\mathbb{R}_{x, Y}^2))} = 0$$

which implies for the corrector  $u_2^\epsilon$  that,

$$\sup_{t \in [0, \frac{T_0}{\epsilon^2}]} \|\epsilon^2 \mathcal{U}_2(\epsilon^2 t, t, \cdot, \cdot)\|_{H^{s-1}} = o(\epsilon) \quad \text{and} \quad \|\epsilon^2 u_2^\epsilon\|_{L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x, y}^2)} = o(\epsilon)$$

- For the profile  $\mathcal{U}_3$ , which is defined in terms of the previous profiles, we have for the corrector  $u_3^\epsilon$  an analog result, that reads

$$\sup_{t \in [0, \frac{T_0}{\epsilon^2}]} \|\epsilon^3 \mathcal{U}_3(\epsilon^2 t, t, \cdot, \cdot)\|_{H^{s-3}(\mathbb{R}_{x, Y}^2)^N} = o(\epsilon^2)$$

and hence  $\|\epsilon^3 u_3^\epsilon\|_{L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x, y}^2)} = o(\epsilon^2)$ .

We have thus proved the following smallness result for the corrector

**Proposition 3.2** *Suppose  $s > 4$ . Then the corrector terms are small in the sense that*

$$\sup_{t \in [0, \frac{T_0}{\epsilon^2}]} \|\epsilon^2(\epsilon \mathcal{U}_1 + \epsilon^2 \mathcal{U}_2 + \epsilon^3 \mathcal{U}_3)(\epsilon^2 t, t, \cdot, \cdot)\|_{H^{s-3}(\mathbb{R}_{x, Y}^2)^N} = O(\epsilon^3)$$

and

$$\|\epsilon^3 u_1^\epsilon + \epsilon^4 u_2^\epsilon + \epsilon^5 u_3^\epsilon\|_{L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x, y}^2)} = O(\epsilon^3).$$

### 3.3 Estimate for the residual

The ansatz  $u^\epsilon$  is only an approximate solution of (1.1). We have already computed the residual  $r^\epsilon(t, x, y)$  in the first part of this work. We recall that it is given by

$$r^\epsilon(t, x, y) = \mathcal{R}(\epsilon^2 t, t, x, \epsilon y) \quad \text{with} \quad \mathcal{R}(T, t, x, Y) = \sum_{j=1}^{11} \epsilon^j \mathcal{R}_j(T, t, x, Y).$$

The usual procedure would consist in proving a smallness result for  $\mathcal{R}^\epsilon$  and  $r^\epsilon$ . Unfortunately, this is not possible because we lack a functional space to estimate  $\mathcal{R}^\epsilon$ . Indeed, the term  $\mathcal{R}_5$ , for instance, involves  $\partial_\tau \mathcal{U}_1$  and hence  $\partial_\tau \varphi$ , while we only know that  $\varphi \in C([0, T_0], H^s(\mathbb{R}_{x,Y}^2)^N)$ . This problem is equivalent to computing  $\partial_x^{-2} \mathcal{U}_0$ , which may be not possible.

Using the method of infrared cutoffs introduced by Alterman [A], [AR] and also used in [BL], [T], we change a little our ansatz by removing the infrared frequencies which prevent the computation of  $\partial_x^{-2} \mathcal{U}_0$ . First introduce Alterman's infrared cutoff.

**Definition 3.1** *Let  $\chi$  be a smooth compactly supported function in  $\mathbb{R}$  such that  $0 \leq \chi \leq 1$ ,  $\chi(r) = 0$  if  $|r| \leq 1$  and  $\chi(r) = 1$  for  $r \geq 2$ . For all  $\delta > 0$ , define the Fourier multiplier  $\chi^\delta(|D|)$  as*

$$\chi^\delta(|D|) : f \mapsto \mathcal{F}^{-1}(\chi(\frac{|\cdot|}{\delta})\widehat{f}(\cdot)).$$

We then define  $U_0^\delta := \chi^\delta(|D|)\mathcal{U}_0$ , and as we had constructed  $U_1, U_2$  and  $U_3$  in terms of  $U_0$ , we construct  $U_1^\delta, U_2^\delta$  and  $U_3^\delta$  in terms of  $U_0^\delta$ , exactly in the same way. One can check that Prop. 3.2 remains true for these new profiles.

The new residual  $\mathcal{R}^{\epsilon, \delta}$  associated to these profiles is written

$$\mathcal{R}^{\epsilon, \delta}(\tau, t, x, Y) = \sum_{j=1}^{11} \epsilon^j \mathcal{R}_j^\delta(\tau, t, x, Y),$$

where for all  $j \neq 4$ ,  $\mathcal{R}_j^\delta$  is deduced from  $\mathcal{R}_j$  by replacing  $\mathcal{U}_1, \mathcal{U}_2$  and  $\mathcal{U}_3$  by  $\mathcal{U}_1^\delta, \mathcal{U}_2^\delta$  and  $\mathcal{U}_3^\delta$ . In particular,  $\mathcal{R}_1^\delta = \mathcal{R}_2^\delta = \mathcal{R}_3^\delta = 0$ . However, we do not have  $\mathcal{R}_4^\delta = 0$  since  $\mathcal{U}_0^\delta$  does not solve exactly Eq. 2.23 but

$$\begin{aligned} \left( \partial_T + \frac{\lambda'''(0)}{6} \partial_x^3 - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \right) \pi_1(0) \mathcal{U}_0^\delta &= \chi^\delta(|D|) \pi_1(0) C(\pi_1(0) \mathcal{U}_0) \partial_x \pi_1(0) \mathcal{U}_0 \\ \left( \partial_T - \frac{\lambda'''(0)}{6} \partial_x^3 + \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \right) \pi_2(0) \mathcal{U}_0^\delta &= \chi^\delta(|D|) \pi_2(0) C(\pi_2(0) \mathcal{U}_0) \partial_x \pi_2(0) \mathcal{U}_0 \end{aligned}$$

Therefore,  $\mathcal{R}_4^\delta$  is given by

$$\mathcal{R}_4^\delta = \chi^\delta(|D|) \pi_1(0) C(\pi_1(0) \mathcal{U}_0) \partial_x \pi_1(0) \mathcal{U}_0 - \pi_1(0) C(\pi_1(0) \mathcal{U}_0^\delta) \partial_x \pi_1(0) \mathcal{U}_0^\delta$$

$$+ \chi^\delta(|D|)\pi_2(0)C(\pi_2(0)\mathcal{U}_0)\partial_x\pi_2(0)\mathcal{U}_0 - \pi_2(0)C(\pi_2(0)\mathcal{U}_0^\delta)\partial_x\pi_2(0)\mathcal{U}_0^\delta.$$

We can now state the following result on  $\mathcal{R}^{\epsilon,\delta}$ , as well as on  $r^{\epsilon,\delta}(t, x, y) := \mathcal{R}^{\epsilon,\delta}(\epsilon^2 t, t, x, \epsilon y)$ .

**Proposition 3.3** *Suppose  $s > 7$ . Then the modified residual satisfies the following estimates*

$$\sup_{t \in [0, \frac{T_0}{\epsilon^2}]} \|\mathcal{R}^{\epsilon,\delta}(\epsilon^2 t, t, \cdot, \cdot)\|_{H^{s-6}(\mathbb{R}_{x,Y}^2)^N} = (f(\delta) + \frac{\epsilon}{\delta}) O(\epsilon^4) + o(\epsilon^4)$$

and

$$\|r^{\epsilon,\delta}\|_{L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x,y}^2)} = (f(\delta) + \frac{\epsilon}{\delta}) O(\epsilon^4) + o(\epsilon^4),$$

where  $f$  is a positive nonincreasing function such that  $f(\delta) \rightarrow 0$  when  $\delta \rightarrow 0$ .

**Proof.**

As said above,  $\mathcal{R}_1^\delta = \mathcal{R}_2^\delta = \mathcal{R}_3^\delta = 0$ . Since for any function  $v \in H^s(\mathbb{R}^2)$ , one has  $\chi^\delta(|D|)v \rightarrow v$  when  $\delta \rightarrow 0$  by Lebesgue's Dominated Convergence Theorem, a direct computation yields that  $\sup_{t \in [0, \frac{T_0}{\epsilon^2}]} \|\mathcal{R}_4^\delta(\epsilon^2 t, t, \cdot, \cdot)\|_{H^{s-6}(\mathbb{R}_{x,Y}^2)^N} = f(\delta)$ , where  $f$  is as stated in the proposition.

Following the estimations laid out in [BC] and [L] for the residual, and thanks to the sub-square root growth condition satisfied by  $\mathcal{U}_2^\delta$  and  $\mathcal{U}_3^\delta$  and the boundedness of  $\mathcal{U}_1^\delta$ , all the components of  $\mathcal{R}_5^\delta$  except  $\partial_\tau\pi(0)\mathcal{U}_1^\delta$  have at most a sub-square root growth in  $H^{s-4}$ . In order to bound  $\partial_\tau\pi_1(0)\mathcal{U}_1^\delta$  (the component  $\partial_\tau\pi_2(0)\mathcal{U}_1^\delta$  must be treated in the same way), we write

$$\begin{aligned} \partial_\tau\pi_1(0)\mathcal{U}_1^\delta &= \frac{i}{2\lambda'(0)}\pi_1(0)AL(0)^{-1}A\pi_2(0)\partial_x\partial_\tau\mathcal{U}_0^\delta \\ &+ \frac{1}{2\lambda'(0)}\pi_1(0)B\pi_2(0)\partial_x^{-1}\partial_Y\partial_\tau\mathcal{U}_0^\delta \\ &= \text{terms bounded in } H^{s-4} + \mathbf{P}\partial_Y^3\partial_x^{-2}\mathcal{U}_0^\delta, \end{aligned}$$

where  $\mathbf{P}$  is a matricial operator.

Since  $\mathcal{U}_0^\delta = \chi^\delta\mathcal{U}_0$ , the last term of the above equation reads  $\mathbf{P}\partial_Y^3\partial_x^{-2} =$

$\mathbf{P}\partial_Y^3\partial_x^{-1}\chi^\delta(|D|)\varphi$ . Since for any  $v \in H^s(\mathbb{R}^2)$ ,  $\|\partial_x^{-1}\chi^\delta(|D|)v\|_{H^s} \leq \text{Cte}/\delta\|v\|_{H^s}$  this term is in  $H^{s-3}(\mathbb{R}^d)$  with norm bounded by  $\frac{\text{Cte}}{\delta}\|\varphi\|_{H^{s-3}}$ , the constant being independant of  $\delta$ . Hence we can conclude that in  $H^{s-4} \subset H^{s-6}$ , one has  $\|\epsilon^5\mathcal{R}_5^\delta\| \leq o(\epsilon^4) + \text{Cte} \epsilon^5/\delta$ .

The following terms of the residual can be evaluated with the same techniques, the biggest loss of derivatives occuring for  $\mathcal{R}_7^\delta$ .

We take  $s > 7$  in the proposition in order to deduce the  $L^\infty$  estimates from the  $H^{s-6}$  estimates. ■

**Remark 3.2** Making as in [GS] the quite restrictive assumptions that  $\varphi$  is  $C^1$  in time and that the wave packets travel in one direction only, one would not need to consider the troncated profile  $\mathcal{U}_0^\delta$ , and would obtain a better  $O(\epsilon^5)$  residual.

### 3.4 Stability of the KP approximation

We have shown so far that there exists a (modified) approximate solution  $u^{\epsilon,\delta}(t, x, y)$  to (1.1) for times of order  $O(\frac{1}{\epsilon^2})$  and whose residual is small. We now prove that the (untruncated) leading term of the approximate solution remains close to the exact solution  $\mathbf{u}^\epsilon$  for times of order  $O(\frac{1}{\epsilon^2})$ . More precisely, as in [S] and [GS], we show that for any  $T_0 > 0$  such that the KP approximation exists, the exact solution of Eq. (1.1) exists on  $[0, T_0/\epsilon^2]$ , and that the approximation remains good for such times.

**Theorem 3.2** *Let  $u_{in}^\epsilon(x, y) = \mathcal{U}^0(x, \epsilon y) + \epsilon\mathcal{V}^\epsilon(x, \epsilon y)$ , where  $\mathcal{U}^0 = \pi(0)\mathcal{U}^0 = \pi(0)\partial_x\varphi^0$ , with  $\varphi^0 \in H^{s+1}(\mathbb{R}_{x,Y}^2)^N$  and  $s > 8$ , and  $(\mathcal{V}^\epsilon)_\epsilon$  is a bounded family of profiles in  $H^s(\mathbb{R}_{x,Y}^2)^N$ . For any  $T_0 > 0$ , such that the unique solution  $(\mathcal{U}_{01}, \mathcal{U}_{02})$*

of the uncoupled KP equations

$$\begin{cases} \partial_T \mathcal{U}_{01} + \frac{\lambda'''(0)}{6} \partial_x^3 \mathcal{U}_{01} - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \mathcal{U}_{01} = \pi_1(0) C(\mathcal{U}_{01}) \partial_x \mathcal{U}_{01}, \\ \partial_T \mathcal{U}_{02} - \frac{\lambda'''(0)}{6} \partial_x^3 \mathcal{U}_{02} + \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \mathcal{U}_{02} = \pi_2(0) C(\mathcal{U}_{02}) \partial_x \mathcal{U}_{02}, \\ \mathcal{U}_{0j}(0, x, Y) = \pi_j(0) \mathcal{U}^0(x, Y), \quad j = 1, 2, \end{cases} \quad (3.27)$$

lies in  $C([0, T_0]; H^s(\mathbb{R}_{x,Y}^2))^{2N}$ , there exists  $\epsilon_0 > 0$  such that for all  $0 < \epsilon < \epsilon_0$  there is a unique solution  $\mathbf{u}^\epsilon \in C([0, \frac{T_0}{\epsilon^2}]; H^s(\mathbb{R}_{x,y}^2)^N) \cap C^1([0, \frac{T_0}{\epsilon^2}]; H^{s-1}(\mathbb{R}_{x,y}^2)^N)$  of Eq. (1.1),

$$\partial_t \mathbf{u}^\epsilon + A \partial_x \mathbf{u}^\epsilon + B \partial_y \mathbf{u}^\epsilon + \frac{E \mathbf{u}^\epsilon}{\epsilon} = C(\mathbf{u}^\epsilon) \partial_x \mathbf{u}^\epsilon + D(\mathbf{u}^\epsilon) \partial_y \mathbf{u}^\epsilon$$

with  $\mathbf{u}^\epsilon(0, x, y) = \epsilon^2 u_{in}^\epsilon(x, y)$ .

Moreover, under Assumptions 2.1 and 3.1, one has

$$\left| \frac{\mathbf{u}^\epsilon}{\epsilon^2} - u_{app}^\epsilon \right|_{L^\infty([0, \frac{T_0}{\epsilon^2}]; \mathbb{R}_{x,y}^2)} = o(1) \quad \text{as } \epsilon \rightarrow 0$$

where  $u_{app}^\epsilon(t, x, y)$  is defined as

$$u_{app}^\epsilon(t, x, y) = \mathcal{U}_{01}(\epsilon^2 t, x - \lambda'(0)t, \epsilon y) + \mathcal{U}_{02}(\epsilon^2 t, x + \lambda'(0)t, \epsilon y).$$

**Proof.**

We will look for the exact solution  $\mathbf{u}^\epsilon(t, x, y)$  as a perturbation of the approximate solution. In order to do so, first introduce

$$\mathbf{U}_{app}^{\epsilon, \delta}(t, x, Y) := \mathcal{U}_{01}^\delta(\epsilon^2 t, x - \lambda'(0)t, Y) + \mathcal{U}_{02}^\delta(\epsilon^2 t, x + \lambda'(0)t, Y),$$

and  $u_{app}^{\epsilon, \delta}(t, x, y) = \mathbf{U}_{app}^{\epsilon, \delta}(t, x, \epsilon y)$ . Similarly, we write

$$\mathbf{U}_j^{\epsilon, \delta}(t, x, Y) := \mathcal{U}_j^\delta(\epsilon^2 t, t, x, Y), \quad j = 1, 2, 3,$$

and

$$\mathbf{U}^{\epsilon, \delta} := (\mathbf{U}_{app}^{\epsilon, \delta} + \epsilon \mathbf{U}_1^{\epsilon, \delta} + \epsilon^2 \mathbf{U}_2^{\epsilon, \delta} + \epsilon^2 \mathbf{U}_3^{\epsilon, \delta}).$$

We look for  $\mathbf{u}^\epsilon$  under the form  $\mathbf{u}^\epsilon(t, x, y) = \epsilon^2 \mathbf{U}^\epsilon(t, x, \epsilon y)$ , where  $\mathbf{U}^\epsilon = \mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta$ , and the error term  $\tilde{\mathbf{U}}^\delta(t, x, Y)$  satisfies

$$\begin{aligned} \partial_t \tilde{\mathbf{U}}^\delta + (A - \epsilon^2 C(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)) \partial_x \tilde{\mathbf{U}}^\delta + \epsilon(B - \epsilon^2(D(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta))) \partial_Y \tilde{\mathbf{U}}^\delta + \frac{E}{\epsilon} \tilde{\mathbf{U}}^\delta \\ = \epsilon^2 C(\tilde{\mathbf{U}}^\delta) \partial_x \mathbf{U}^{\epsilon, \delta} + \epsilon^3 D(\tilde{\mathbf{U}}^\delta) \partial_Y \mathbf{U}^{\epsilon, \delta} - \frac{1}{\epsilon^2} \mathbf{R}^{\epsilon, \delta}, \end{aligned} \quad (3.28)$$

where  $\mathbf{R}^{\epsilon, \delta}(t, x, Y) := \mathcal{R}^{\epsilon, \delta}(\epsilon^2 t, t, x, Y)$  satisfies the estimate of Prop. 3.3; the initial condition is  $\tilde{\mathbf{U}}^\delta(t = 0, x, Y) = \mathcal{U}^0(x, Y) - \mathcal{U}^{0, \delta}(x, Y) + \epsilon \mathcal{V}^\epsilon(x, Y) - \epsilon \mathcal{U}_1^\delta(x, Y) - \epsilon^2 \mathcal{U}_2^\delta(x, Y) - \epsilon^3 \mathcal{U}_3^\delta(x, Y)$ , where the family  $(\mathcal{V}^\epsilon)$  is as stated in the theorem.

Local existence of  $\tilde{\mathbf{U}}^\delta$  in  $H^{s-6}(\mathbb{R}_{x,Y}^2)$  can be proved using classical techniques (see [AG], chap. III.B for instance). We still have to prove that for any  $T_0 > 0$  as in the theorem,  $\tilde{\mathbf{U}}^\delta$  exists on  $[0, T_0/\epsilon^2]$ .

In order to obtain the  $H^{s-6}$  estimate associated to Eq. (3.28), we write the equation satisfied by  $\Lambda^{s-6} \tilde{\mathbf{U}}^\delta$ , where  $\Lambda^s := (1 + |D|^s)^{1/2}$ . One has

$$\begin{aligned} \partial_t \Lambda^{s-6} \tilde{\mathbf{U}}^\delta + (A - \epsilon^2 C(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)) \partial_x \Lambda^{s-6} \tilde{\mathbf{U}}^\delta + \epsilon(B - \epsilon^2(D(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta))) \partial_Y \Lambda^{s-6} \tilde{\mathbf{U}}^\delta \\ + \frac{E}{\epsilon} \Lambda^{s-6} \tilde{\mathbf{U}}^\delta \end{aligned} \quad (3.29)$$

$$\begin{aligned} = -\epsilon^2 [\Lambda^{s-6}, C(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)] \partial_x \tilde{\mathbf{U}}^\delta - \epsilon^3 [\Lambda^{s-6}, D(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)] \partial_Y \tilde{\mathbf{U}}^\delta \\ + \epsilon^2 \Lambda^{s-6} C(\tilde{\mathbf{U}}^\delta) \partial_x \mathbf{U}^{\epsilon, \delta} + \epsilon^3 \Lambda^{s-6} D(\tilde{\mathbf{U}}^\delta) \partial_Y \mathbf{U}^{\epsilon, \delta} - \frac{1}{\epsilon^2} \Lambda^{s-6} \mathbf{R}^{\epsilon, \delta}. \end{aligned} \quad (3.30)$$

Since, for any profile  $A$  and  $B$ , and all  $s' > 2$ , one has

$$|[\Lambda^{s'}, C(A)] \partial_x B|_{L^2} \leq C |\nabla C(A)|_{s'-1} |\partial_x B|_{s'-1},$$

we obtain easily (since  $s - 6 > 2$ ) from the  $L^2$  estimate of (3.30) that

$$\begin{aligned} |\tilde{\mathbf{U}}^\delta(t)|_{s-6}^2 &\leq |\tilde{\mathbf{U}}^\delta(t=0)|_{s-6}^2 + \frac{1}{\epsilon^2} \int_0^t |\mathbf{R}^{\epsilon, \delta}(s)|_{s-6} |\tilde{\mathbf{U}}^\delta(s)|_{s-6} ds \\ &+ \epsilon^2 \|\partial_x(C(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)) + \epsilon \partial_Y(D(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta))\|_\infty \int_0^t |\tilde{\mathbf{U}}^\delta(s)|_{s-6}^2 ds \\ &+ \epsilon^2 C \int_0^t |C(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta) + \epsilon D(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)|_{s-6} |\tilde{\mathbf{U}}^\delta(s)|_{s-6}^2 ds \\ &+ \epsilon^2 \int_0^t |C(\tilde{\mathbf{U}}^\delta) \partial_x \mathbf{U}^{\epsilon, \delta} + \epsilon D(\tilde{\mathbf{U}}^\delta) \partial_Y \mathbf{U}^{\epsilon, \delta}|_{s-6} |\tilde{\mathbf{U}}^\delta(s)|_{s-6} ds. \end{aligned}$$

Inspired by [GS], introduce  $t_0(\delta, \epsilon) > 0$  defined as

$$t_0(\delta, \epsilon) := \sup\{t \in [0, \frac{T_0}{\epsilon^2}], |\tilde{\mathbf{U}}^\delta(t)|_{s-6} \leq 1\}.$$

By definition of  $T_0$  and  $t_0(\delta, \epsilon)$ , and since  $H^{s-7}$  embeds in  $L^\infty$ , we have, for all  $0 \leq t \leq t_0(\delta, \epsilon)$ ,

$$\begin{aligned} \|\partial_x(A - \epsilon^2 C(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)) + \epsilon \partial_Y(B - \epsilon^2(D(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta))\|_\infty &\leq \epsilon^2 C \\ |C(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta) + \epsilon D(\mathbf{U}^{\epsilon, \delta} + \tilde{\mathbf{U}}^\delta)|_{s-6} &\leq C \\ |C(\tilde{\mathbf{U}}^\delta) \partial_x \mathbf{U}^{\epsilon, \delta} + \epsilon D(\tilde{\mathbf{U}}^\delta) \partial_Y \mathbf{U}^{\epsilon, \delta}|_{s-6} &\leq C |\tilde{\mathbf{U}}^\delta|_{s-6}, \end{aligned}$$

and hence by Gronwall's lemma,

$$|\tilde{\mathbf{U}}^\delta(t)|_{s-6} \leq \left( |\tilde{\mathbf{U}}^\delta(0)|_{s-6} + \frac{T_0}{\epsilon^4} \sup_{t \in [0, \frac{T_0}{\epsilon^2}]} |\mathbf{R}^{\epsilon, \delta}|_{s-6} \right) \exp(C\epsilon^2 t) := \mathcal{E}(\delta, \epsilon) \exp(C\epsilon^2 t). \quad (3.31)$$

Thanks to Props. 3.2-3.3, and since  $|\mathcal{U}^0 - \mathcal{U}^{0, \delta}|_{s-6} \rightarrow 0$  when  $\delta \rightarrow 0$ , we can choose  $\delta_0$  small enough, and then  $\epsilon_0 > 0$  small enough such that for all  $0 < \epsilon < \epsilon_0$ , one has  $\mathcal{E}(\delta_0, \epsilon) \exp(CT_0) \leq 1$ . Therefore, for such values of  $\epsilon$ , we can deduce from Eq. (3.31) and the definition of  $t_0(\delta, \epsilon)$  that one has  $t_0(\delta_0, \epsilon) = T_0/\epsilon^2$ .

Thanks to Eq. (3.31), we also deduce that

$$\lim_{\epsilon \rightarrow 0} \sup_{t \in [0, \frac{T_0}{\epsilon^2}]} |\tilde{\mathbf{U}}^{\delta_0}(t)|_{s-6} \leq \left( |\mathcal{U}^0 - \mathcal{U}^{0, \delta_0}|_{s-6} + f(\delta_0) \right) \exp(CT_0). \quad (3.32)$$

We now prove that for any  $\mu > 0$ , one can choose  $0 < \epsilon < \epsilon_0$  small enough such that  $\sup_{t \in [0, T_0/\epsilon^2]} |\mathbf{U}_{app}^\epsilon(t, \cdot, \cdot) - \mathbf{U}^\epsilon(t, \cdot, \cdot)|_{s-6} \leq \mu$ , where  $\mathbf{U}_{app}^\epsilon$  is defined as  $\mathbf{U}_{app}^{\epsilon, \delta}$  with obvious notations. This will prove the convergence result in the space of profiles. The  $L^\infty$  estimate of the theorem is then deduced from the embedding  $H^{s-6}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ . One has

$$\begin{aligned} \sup_{t \in [0, \frac{T_0}{\epsilon^2}]} |\mathbf{U}_{app}^\epsilon(t, \cdot, \cdot) - \mathbf{U}^\epsilon(t, \cdot, \cdot)|_{s-6} &\leq \sup_{t \in [0, \frac{T_0}{\epsilon^2}]} |\mathbf{U}_{app}^\epsilon - \mathbf{U}_{app}^{\epsilon, \delta_0}|_{s-6} \\ &+ \sup_{t \in [0, \frac{T_0}{\epsilon^2}]} |\mathbf{U}_{app}^{\epsilon, \delta_0} - \mathbf{U}^{\epsilon, \delta_0}|_{s-6} + \sup_{t \in [0, \frac{T_0}{\epsilon^2}]} |\tilde{\mathbf{U}}^{\delta_0}|_{s-6}. \end{aligned}$$

If  $\delta_0$  has been chosen small enough, and using Eq. (3.32), one can see that the first and the third term of the r.h.s. of the inequality are smaller than  $\mu/3$  for  $\epsilon$  small enough. This is also true for the second term, thanks to Prop. 3.2, which proves the desired result. ■

## 4 The coupled system

### 4.1 Statement of the result

The motivation of this part relies on a simple idea which has also been used in [BC] to derive a system of coupled KdV equations.

We have obtained above a stability result for the approximate solution defined thanks to the uncoupled system of KP equations (3.27). However, this result only gives an error estimate  $o(1)$  with respect to  $\epsilon$ . One may think that coupling effects on both modes of the solution make impossible a better estimate, say  $O(\epsilon)$  for instance. Schneider and Wayne proved in [SW] that these coupling effects are negligible for the KdV case, under a decay assumption on the solutions of the KdV equations. As said previously, we believe that for the KP case, coupling effects must be taken into account.

What follows gives a partial answer to this conjecture, since we can define a new approximate solution to Eq. (1.1), found by solving a *coupled* system of KP equations, and which gives an error estimate  $O(\epsilon)$ . We say that the answer to the above conjecture is only partial since we do not have any lower bound of the error estimates, neither in the uncoupled nor in the coupled case.

We have seen in the above sections that the uncoupled system is obtained, *via* the average operators, as a solvability condition for the slow-evolution equations (2.18)-(2.19) and the transport equations (2.8). Therefore, if one wants to obtain a coupled system, then one of these two sets of equations cannot hold.

The trick consists in solving Eqs. (2.18)-(2.19), while the transport equation (2.8) will only be solved *up to the order*  $O(\epsilon^2)$ .

We recall that Eq. (2.18) reads

$$\begin{aligned} & \partial_T \pi_1(0) \mathcal{U}_0 + \frac{\lambda'''(0)}{6} \partial_x^3 \pi_1(0) \mathcal{U}_0 - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \pi_1(0) \mathcal{U}_0 \\ & + (\partial_t - \lambda'(0) \partial_x) \pi_1(0) \mathcal{U}_2 + \mathbf{O}_1(\partial) \pi_2(0) \mathcal{U}_0 \\ & = \pi_1(0) C(\pi_1(0) \mathcal{U}_0) \partial_x \pi_1(0) \mathcal{U}_0. \end{aligned}$$

If  $T_1(\partial) \pi_1(0) \mathcal{U}_0 = 0$  then, applying the average projector  $G_{T_1}$  to this equation, one obtains the first equation of the uncoupled system (2.23), together with the necessary condition (2.24) on the corrector  $\pi_1(0) \mathcal{U}_2$ ,

$$T_1(\partial) \pi_1(0) \mathcal{U}_2 = -\mathbf{O}_1(\partial) \pi_2(0) \mathcal{U}_0.$$

In this section, we choose another way of solving Eq. (2.18), keeping all the nonlinear terms of the r.h.s., and we split it as follows

$$\begin{cases} \partial_T \pi_1(0) \mathcal{U}_0 + \frac{\lambda'''(0)}{6} \partial_x^3 \pi_1(0) \mathcal{U}_0 - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \pi_1(0) \mathcal{U}_0 \\ \quad = \pi_1(0) C(\pi_1(0) \mathcal{U}_0 + \pi_2(0) \mathcal{U}_0) \partial_x (\pi_1(0) \mathcal{U}_0 + \pi_2(0) \mathcal{U}_0) \\ T_1(\partial) \pi_1(0) \mathcal{U}_2 = -\mathbf{O}'_1(\partial) \mathcal{U}_0, \end{cases} \quad (4.33)$$

where  $\mathbf{O}'_1(\partial) = \mathbf{O}_1(\partial) - \text{nonlinear terms}$  that is,

$$\begin{aligned} \mathbf{O}'_1(\partial) \mathcal{U}_0 & = i[\pi_1(0) AL(0)^{-1} B \pi_2(0) + \pi_1(0) BL(0)^{-1} A \pi_2(0)] \partial_{x,Y}^2 \pi_2(0) \mathcal{U}_0 \\ & - \pi_1(0) AL(0)^{-1} (\lambda'(0) - A) L(0)^{-1} A \pi_2(0) \partial_x^3 \pi_2(0) \mathcal{U}_0. \end{aligned}$$

We insist on the fact that this set of equations *is not compatible with the transport equations (2.8)*. Therefore, we do not keep the long time scale variable in time in our profiles, and consider only, from now on, profiles of the form  $V^\epsilon(t, x, Y)$ .

**Remark 4.1** We keep a dependence on  $Y$  instead of  $y$  for the same reasons as in the proof of Theorem 3.2, i.e. in order to have bounded  $H^s$  norms, uniformly in  $\epsilon$ .

The ansatz we use in this section to approximate the exact solution  $\mathbf{u}^\epsilon$  of Eq. (1.1) writes  $\mathbf{u}^\epsilon(t, x, y) = \epsilon^2 \mathbf{U}^\epsilon(t, x, \epsilon y)$  where  $\mathbf{U}^\epsilon$  expands as

$$\mathbf{U}^\epsilon(t, x, Y) = \mathbf{U}_0^\epsilon(t, x, Y) + \epsilon \mathbf{U}_1^\epsilon(t, x, Y) + \epsilon^2 \mathbf{U}_2^\epsilon(t, x, Y). \quad (4.34)$$

Using the notations  $\mathbf{U}_{01}^\epsilon = \pi_1(0) \mathbf{U}_0^\epsilon$ ,  $\mathbf{U}_{02}^\epsilon = \pi_2(0) \mathbf{U}_0^\epsilon$  and  $\mathbf{U}_0^\epsilon = \mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon$ , the first of the equations of System (4.33) reads

$$\partial_t \mathbf{U}_{01}^\epsilon - \lambda'(0) \partial_x \mathbf{U}_{01}^\epsilon + \epsilon^2 \left[ \frac{\lambda'''(0)}{6} \partial_x^3 \mathbf{U}_{01}^\epsilon - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \mathbf{U}_{01}^\epsilon - \pi_1(0) C(\mathbf{U}_0^\epsilon) \partial_x \mathbf{U}_{01}^\epsilon \right] = 0.$$

**Remark 4.2** As we can see on this equation, the transport equation  $T_1(\partial) \mathbf{U}_{01}$  is satisfied up to a  $O(\epsilon^2)$  term.

A similar equation is obtained in the same way for  $\mathbf{U}_{02}$ . The second equation of System (4.33) allows us to find  $\pi_1(0) \mathbf{U}_2$ , and  $\pi_2(0) \mathbf{U}_2$  is found with the same method. All the other corrector terms are obtained as in the above sections. For instance, one has

$$\pi_1(0) \mathbf{U}_1 = \frac{i}{2\lambda'(0)} \pi_1(0) A L(0)^{-1} A \pi_2(0) \partial_x \mathbf{U}_{02} + \frac{1}{2\lambda'(0)} \pi_1(0) B \pi_2(0) \partial_x^{-1} \partial_Y \mathbf{U}_{02} \quad (4.35)$$

and

$$\pi_2(0) \mathbf{U}_1 = -\frac{i}{2\lambda'(0)} \pi_2(0) A L(0)^{-1} A \pi_1(0) \partial_x \mathbf{U}_{01} - \frac{1}{2\lambda'(0)} \pi_2(0) B \pi_1(0) \partial_x^{-1} \partial_Y \mathbf{U}_{01}. \quad (4.36)$$

There are two obstructions to obtain a  $O(\epsilon)$  error term. The first one is the secular growth of the interaction terms, and is taken into account by the coupled system. The second one is the method of infrared cutoffs we have used for the uncoupled system in order to avoid the computation of  $\partial_x^{-2} \mathcal{U}_0$ . This problem would remain here, and that is why we have to make a regularity assumption as in [GS]. The main result of this section states as follows.

**Theorem 4.1** *Let  $u_{in}^\epsilon(x, y) = \mathcal{U}^0(x, \epsilon y) + \epsilon \mathcal{V}^\epsilon(x, \epsilon y)$ , where  $\mathcal{U}^0 = \pi(0)\mathcal{U}^0 = \pi(0)\partial_x \varphi^0$ , with  $\varphi^0 \in H^{s+1}(\mathbb{R}_{x,Y}^2)^N$  and  $s > 8$ , and  $(\mathcal{V}^\epsilon)_\epsilon$  is a bounded family of profiles in  $H^s(\mathbb{R}_{x,Y}^2)^N$ . Let  $T_0 > 0$  be such that the unique solution  $(\mathbf{U}_{01}^\epsilon, \mathbf{U}_{02}^\epsilon)$  of the coupled KP equations*

$$\begin{cases} \partial_t \mathbf{U}_{01}^\epsilon - \lambda'(0) \partial_x \mathbf{U}_{01}^\epsilon \\ \quad + \epsilon^2 \left[ \frac{\lambda'''(0)}{6} \partial_x^3 \mathbf{U}_{01}^\epsilon - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \mathbf{U}_{01}^\epsilon - \pi_1(0) C(\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon) \partial_x (\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon) \right] = 0, \\ \partial_t \mathbf{U}_{02}^\epsilon + \lambda'(0) \partial_x \mathbf{U}_{02}^\epsilon \\ \quad - \epsilon^2 \left[ \frac{\lambda'''(0)}{6} \partial_x^3 \mathbf{U}_{02}^\epsilon - \frac{\lambda'(0)}{2} \partial_x^{-1} \partial_Y^2 \mathbf{U}_{02}^\epsilon + \pi_2(0) C(\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon) \partial_x (\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon) \right] = 0, \\ \mathbf{U}_{0j}^\epsilon(0, x, Y) = \pi_j(0) \mathcal{U}^0(x, Y), \quad j = 1, 2, \end{cases} \quad (4.37)$$

lies in  $C([0, T_0/\epsilon^2]; H^s(\mathbb{R}_{x,Y}^2))^{2N}$ . Suppose moreover that  $\partial_x^{-1} \mathbf{U}_{01}^\epsilon$  and  $\partial_x^{-1} \mathbf{U}_{02}^\epsilon$  are in  $C^1([0, T_0]; H^{s-3}(\mathbb{R}^2)^N)$ . Then there exists  $\epsilon_0 > 0$  such that for all  $0 < \epsilon < \epsilon_0$  there is a unique solution  $\mathbf{u}^\epsilon \in C([0, \frac{T_0}{\epsilon^2}]; H^s(\mathbb{R}_{x,y}^2)^N) \cap C^1([0, \frac{T_0}{\epsilon^2}]; H^{s-1}(\mathbb{R}_{x,y}^2)^N)$  of Eq. (1.1),

$$\partial_t \mathbf{u}^\epsilon + A \partial_x \mathbf{u}^\epsilon + B \partial_y \mathbf{u}^\epsilon + \frac{E \mathbf{u}^\epsilon}{\epsilon} = C(\mathbf{u}^\epsilon) \partial_x \mathbf{u}^\epsilon + D(\mathbf{u}^\epsilon) \partial_y \mathbf{u}^\epsilon$$

with  $\mathbf{u}^\epsilon(0, x, y) = \epsilon^2 u_{in}^\epsilon(x, y)$ .

Moreover, under Assumptions 2.1 and 3.1, one has

$$\left| \frac{\mathbf{u}^\epsilon}{\epsilon^2} - u_{APP}^\epsilon \right|_{L^\infty([0, \frac{T_0}{\epsilon^2}]; \mathbb{R}_{x,y}^2)} = O(\epsilon) \quad \text{as } \epsilon \rightarrow 0$$

where  $u_{APP}^\epsilon(t, x, y)$  is defined as

$$u_{APP}^\epsilon(t, x, y) = \mathbf{U}_{01}^\epsilon(t, x, \epsilon y) + \mathbf{U}_{02}^\epsilon(t, x, \epsilon y),$$

## 4.2 An existence-uniqueness result for the approximate solution

In this section we prove an existence-uniqueness result for the coupled system of KP equations (4.37). This result says that there is a unique solution  $(\mathbf{U}_{01}^\epsilon, \mathbf{U}_{02}^\epsilon)$

for (4.37), which exists for times  $O(1/\epsilon^2)$ . The approximate solution  $\mathbf{U}_{APP}^\epsilon = \mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon$  therefore exists for such times.

**Proposition 4.1** *Let  $s \geq 3$  and  $\mathcal{U}_0^0 \in (H^s(\mathbb{R}_{x,Y}^2))^N$  satisfying*

$$\pi(0)\mathcal{U}_0^0 = \mathcal{U}_0^0 \quad \text{and} \quad \mathcal{U}_0^0 = \varphi_x^0 \quad \text{with} \quad \varphi^0 \in (H^{s+1}(\mathbb{R}_{x,Y}^2))^n.$$

*i) Then there exists  $T_0$  and a unique family of profiles*

$$(\mathbf{U}_{01}^\epsilon, \mathbf{U}_{02}^\epsilon) \in C([0, \frac{T_0}{\epsilon^2}]; (H^s(\mathbb{R}_{x,Y}^2))^N)^2 \cap Lip([0, \frac{T_0}{\epsilon^2}]; (H^{s-3}(\mathbb{R}_{x,Y}^2))^N)^2,$$

*uniformly bounded in  $\epsilon$ , satisfying the initial condition  $\mathbf{U}_{0j}^\epsilon(0, \cdot, \cdot) = \pi_j(0)\mathcal{U}^0(\cdot, \cdot)$  for  $j = 1, 2$ , and the coupled system of KP equations (4.37);*

*ii) Under Assumption 3.1, i.e. if the nonlinearity  $C(u)\partial_x u$  derives from a gradient, then  $\mathbf{U}_{0j}^\epsilon$ ,  $j = 1, 2$ , have the form  $\mathbf{U}_{0j}^\epsilon = \pi_j(0)\varphi_x^\epsilon$ ,  $j = 1, 2$ , with  $\varphi^\epsilon \in L^\infty([0, \frac{T_0}{\epsilon^2}]; (H^s(\mathbb{R}_{x,Y}^2))^N)$ , and  $(\varphi^\epsilon)_{0 < \epsilon < 1}$  is uniformly bounded in  $\epsilon$ .*

**Proof.**

We cannot obtain the result stated in the Proposition as a direct consequence of Theor. 3.1, as it has been done for the uncoupled case in Cor. 3.1. However, the proof of S. Ukai [U] of Theor. 3.1 generalizes easily to the present case, and gives an existence time  $O(1/\epsilon^2)$ . We sketch here the main steps of this proof, and particularly focus on the fact that it gives an existence time  $O(1/\epsilon^2)$ .

We write the coupled system (4.37) under the form

$$\left\{ \begin{array}{l} \partial_t \mathbf{U}_{01}^\epsilon - \lambda'(0)\partial_x \mathbf{U}_{01}^\epsilon \\ \quad + \epsilon^2 \left[ \frac{\lambda'''(0)}{6} \partial_x^3 \mathbf{U}_{01}^\epsilon - \frac{\lambda'(0)}{2} \partial_Y \mathbf{V}_{01}^\epsilon - \pi_1(0)C(\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon)\partial_x(\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon) \right] \\ \partial_t \mathbf{U}_{02}^\epsilon + \lambda'(0)\partial_x \mathbf{U}_{02}^\epsilon \\ \quad - \epsilon^2 \left[ \frac{\lambda'''(0)}{6} \partial_x^3 \mathbf{U}_{02}^\epsilon - \frac{\lambda'(0)}{2} \partial_Y \mathbf{V}_{02}^\epsilon + \pi_2(0)C(\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon)\partial_x(\mathbf{U}_{01}^\epsilon + \mathbf{U}_{02}^\epsilon) \right] \\ \partial_x \mathbf{V}_{01}^\epsilon = \partial_Y \mathbf{U}_{01}^\epsilon \\ \partial_x \mathbf{V}_{02}^\epsilon = \partial_Y \mathbf{U}_{02}^\epsilon. \end{array} \right.$$

This system is considered as the limit as  $\eta \rightarrow 0$  of the evolution system

$$\left\{ \begin{array}{l} \partial_t \mathbf{U}_{01}^{\epsilon, \eta} - \lambda'(0) \partial_x \mathbf{U}_{01}^{\epsilon, \eta} \\ \quad + \epsilon^2 \left[ \frac{\lambda'''(0)}{6} \partial_x^3 \mathbf{U}_{01}^{\epsilon, \eta} - \frac{\lambda'(0)}{2} \partial_Y \mathbf{V}_{01}^{\epsilon, \eta} - \pi_1(0) C(\mathbf{U}_{01}^{\epsilon, \eta} + \mathbf{U}_{02}^{\epsilon, \eta}) \partial_x (\mathbf{U}_{01}^{\epsilon, \eta} + \mathbf{U}_{02}^{\epsilon, \eta}) \right] \\ \partial_t \mathbf{U}_{02}^{\epsilon, \eta} + \lambda'(0) \partial_x \mathbf{U}_{02}^{\epsilon, \eta} \\ \quad - \epsilon^2 \left[ \frac{\lambda'''(0)}{6} \partial_x^3 \mathbf{U}_{02}^{\epsilon, \eta} - \frac{\lambda'(0)}{2} \partial_Y \mathbf{V}_{02}^{\epsilon, \eta} + \pi_2(0) C(\mathbf{U}_{01}^{\epsilon, \eta} + \mathbf{U}_{02}^{\epsilon, \eta}) \partial_x (\mathbf{U}_{01}^{\epsilon, \eta} + \mathbf{U}_{02}^{\epsilon, \eta}) \right] \\ \eta \partial_t \mathbf{V}_{01}^{\epsilon, \eta} - \partial_x \mathbf{V}_{01}^{\epsilon, \eta} + \partial_Y \mathbf{U}_{01}^{\epsilon, \eta} = 0 \\ \eta \partial_t \mathbf{V}_{02}^{\epsilon, \eta} + \partial_x \mathbf{V}_{02}^{\epsilon, \eta} - \partial_Y \mathbf{U}_{02}^{\epsilon, \eta} = 0. \end{array} \right.$$

Multiplying the last two equations of the above system by  $-(\lambda'(0)/2)\epsilon^2$ , and introducing  $\mathbf{W}^{\epsilon, \eta} := (\mathbf{U}_{01}^{\epsilon, \eta}, \mathbf{U}_{02}^{\epsilon, \eta}, \mathbf{V}_{01}^{\epsilon, \eta}, \mathbf{V}_{02}^{\epsilon, \eta})^T$ , this systems reads in matricial form

$$A_0 \partial_t \mathbf{W}^{\epsilon, \eta} + A_1 (\mathbf{W}^{\epsilon, \eta}) \partial_x \mathbf{W}^{\epsilon, \eta} + A_2 \partial_Y \mathbf{W}^{\epsilon, \eta} + A_3 \partial_x^3 \mathbf{W}^{\epsilon, \eta} = 0, \quad (4.38)$$

where

$$A_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{\lambda'(0)}{2} \epsilon^2 \eta & 0 \\ 0 & 0 & 0 & -\frac{\lambda'(0)}{2} \epsilon^2 \eta \end{pmatrix},$$

while  $A_1(\mathbf{W}^{\epsilon, \eta})$  reads

$$\begin{pmatrix} -\lambda'(0) - \epsilon^2 \pi_1(0) C(\mathbf{U}^{\epsilon, \eta}) \pi_1(0) & -\epsilon^2 \pi_1(0) C(\mathbf{U}^{\epsilon, \eta}) \pi_2(0) & 0 & 0 \\ -\epsilon^2 \pi_2(0) C(\mathbf{U}^{\epsilon, \eta}) \pi_1(0) & \lambda'(0) - \epsilon^2 \pi_2(0) C(\mathbf{U}^{\epsilon, \eta}) \pi_2(0) & 0 & 0 \\ 0 & 0 & \frac{\lambda'(0)}{2} \epsilon^2 & 0 \\ 0 & 0 & 0 & -\frac{\lambda'(0)}{2} \epsilon^2 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & -\epsilon^2 \frac{\lambda'(0)}{2} & 0 \\ 0 & 0 & 0 & \epsilon^2 \frac{\lambda'(0)}{2} \\ -\epsilon^2 \frac{\lambda'(0)}{2} & 0 & 0 & 0 \\ 0 & \epsilon^2 \frac{\lambda'(0)}{2} & 0 & 0 \end{pmatrix},$$

and

$$A_3 = \begin{pmatrix} \epsilon^2 \frac{\lambda'''(0)}{6} & 0 & 0 & 0 \\ 0 & -\epsilon^2 \frac{\lambda'''(0)}{6} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Choosing the sign of  $\eta$  such that  $-(\lambda'(0)/2)\epsilon^2\eta > 0$ , the matrix  $A_0$  is positive definite so that using results of Kato [Kat2], one can obtain the following lemma.

**Lemma 4.1** *There is a  $T^{\epsilon,\eta} > 0$  such that the system (4.38) has a unique solution  $\mathbb{W}^{\epsilon,\eta} \in C([-T^{\epsilon,\eta}, T^{\epsilon,\eta}]; (H^s(\mathbb{R}^2))^N) \cap C^1([-T^{\epsilon,\eta}, T^{\epsilon,\eta}]; (H^{s-3}(\mathbb{R}^2))^N)$  satisfying the initial condition*

$$\mathbb{W}^{\epsilon,\eta}(0, \cdot, \cdot) = \mathbb{W}^0(\cdot, \cdot) := (\pi_1(0)\mathcal{U}^0, \pi_2(0)\mathcal{U}^0, \pi_1(0)\partial_y\varphi^0, \pi_2(0)\partial_y\varphi^0)(\cdot, \cdot).$$

Moreover, the map  $\mathbb{W}^0 \rightarrow \mathbb{W}^{\epsilon,\eta}$  is continuous in the solution space.

We now prove that the lemma remains true with  $T^{\epsilon,\eta} = T_0/\epsilon^2$  and  $T_0$  independent of  $\eta$  and  $\epsilon$ .

Following [U,Kat2,KIM,M], we introduce the norm

$$\|\mathbb{W}^{\epsilon,\eta}\|_{s,\epsilon,\eta}^2 := |(\mathbf{U}_{01}^{\epsilon,\eta}, \mathbf{U}_{02}^{\epsilon,\eta})|_s^2 - \epsilon^2 \eta \frac{\lambda'(0)}{2} |(\mathbf{V}_{01}^{\epsilon,\eta}, \mathbf{V}_{02}^{\epsilon,\eta})|_s^2,$$

where  $|\cdot|_s$  denotes the usual  $H^s$  norm.

Introducing  $M(\mathbf{U}^{\epsilon,\eta}) := \begin{pmatrix} \pi_1(0)C(\mathbf{U}^{\epsilon,\eta})\pi_1(0) & \pi_1(0)C(\mathbf{U}^{\epsilon,\eta})\pi_2(0) \\ \pi_2(0)C(\mathbf{U}^{\epsilon,\eta})\pi_1(0) & \pi_2(0)C(\mathbf{U}^{\epsilon,\eta})\pi_2(0) \end{pmatrix}$ , we obtain from (4.38) the following estimate

$$\begin{aligned} \frac{d}{dt} \|\mathbb{W}^{\epsilon,\eta}\|_{s,\epsilon,\eta}^2 &= \epsilon^2 \underbrace{\langle \partial_x(M(\mathbf{U}^{\epsilon,\eta}))\Lambda^s\mathbf{U}^{\epsilon,\eta}, \Lambda^s\mathbf{U}^{\epsilon,\eta} \rangle}_{I_1} \\ &+ 2\epsilon^2 \underbrace{\langle [\Lambda^s, M(\mathbf{U}^{\epsilon,\eta})]\partial_x\mathbf{U}^{\epsilon,\eta}, \Lambda^s\mathbf{U}^{\epsilon,\eta} \rangle}_{I_2} \end{aligned} \quad (4.39)$$

where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product of  $L^2$ , and  $\Lambda^s := (1 + |D|^s)^{1/2}$ . Both terms  $I_1$  and  $I_2$  are controlled as follows

$$|I_1| \leq |\partial_x M(\mathbf{U}^{\epsilon, \eta})|_\infty |\mathbf{U}^{\epsilon, \eta}|_s^2 \leq C |\partial_x M(\mathbf{U}^{\epsilon, \eta})|_{s-1} |\mathbf{U}^{\epsilon, \eta}|_s^2, \quad (4.40)$$

since  $H^{s-1}$  embeds in  $L^\infty$  (one has  $s-1 \geq 2 > 1$ ) and  $M(\mathbf{U}^{\epsilon, \eta}) \in H^s(\mathbb{R}^2)^{N^2}$ .

We also have

$$|I_2| \leq |[\Lambda^s, M(\mathbf{U}^{\epsilon, \eta})] \partial_x \mathbf{U}^{\epsilon, \eta}|_0 |\mathbf{U}^{\epsilon, \eta}|_s.$$

As proved in [Kat3], we have therefore

$$|I_2| \leq C |\nabla(M(\mathbf{U}^{\epsilon, \eta}))|_{s-1} |\partial_x \mathbf{U}^{\epsilon, \eta}|_{s-1} |\mathbf{U}^{\epsilon, \eta}|_s \leq C |\nabla \mathbf{U}^{\epsilon, \eta}|_{s-1} |\mathbf{U}^{\epsilon, \eta}|_s^2. \quad (4.41)$$

From inequalities (4.39)-(4.41) we obtain

$$\frac{d}{dt} \|\mathbb{W}^{\epsilon, \eta}\|_{s, \epsilon, \eta}^2 \leq \epsilon^2 C |\nabla \mathbf{U}^{\epsilon, \eta}|_{s-1} |\mathbf{U}^{\epsilon, \eta}|_s^2,$$

where  $C$  does not depend on  $\epsilon$  nor on  $\eta$ , and since  $|\mathbf{U}^{\epsilon, \eta}|_s \leq \|\mathbb{W}^{\epsilon, \eta}\|_{s, \epsilon, \eta}$ , we have

$$\frac{d}{dt} \|\mathbb{W}^{\epsilon, \eta}\|_{s, \epsilon, \eta}^2 \leq \epsilon^2 C \|\mathbb{W}^{\epsilon, \eta}\|_{s, \epsilon, \eta}^3,$$

and we can deduce from a usual continuation argument that the time of existence  $T^{\epsilon, \eta}$  given by Lemma 4.1 is bigger than  $T_0/\epsilon^2$  with  $T_0 > 0$  independent of  $\epsilon$  and  $\eta$ .

The end of the proof of the proposition relies on a compactness result for the sequence  $((\mathbf{U}_{01}^{\epsilon, \eta}, \mathbf{U}_{02}^{\epsilon, \eta}))_\eta$ . We refer to [U] for the end of the proof (see also [Sa] for point ii) ).

■

### 4.3 End of the proof

Before proving that the approximate solution  $u_{APP}^\epsilon$  remains close to an exact solution of (1.1), we prove that the corrector terms remain smaller than the

leading terms of the ansatz for times  $O(1/\epsilon^2)$ . This result has been proved in Prop. 3.2 for the uncoupled case, and the following proposition gives it for the present coupled case.

**Proposition 4.2** *Suppose  $s > 4$ . Then the corrector terms are small in the sense that*

$$\sup_{t \in [0, \frac{T_0}{\epsilon^2}]} \|\epsilon^2(\epsilon \mathbf{U}_1^\epsilon + \epsilon^2 \mathbf{U}_2^\epsilon + \epsilon^2 \mathbf{U}_3^\epsilon)(t, \cdot, \cdot)\|_{H^{s-3}(\mathbb{R}_{x,Y}^2)^N} = O(\epsilon^3)$$

and

$$\|\epsilon^3 \mathbf{u}_1^\epsilon + \epsilon^4 \mathbf{u}_2^\epsilon + \epsilon^5 \mathbf{u}_2^\epsilon\|_{L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x,y}^2)} = O(\epsilon^3),$$

where  $\mathbf{u}_j^\epsilon(t, x, y) := \mathbf{U}_j^\epsilon(t, x, \epsilon y)$ , for all  $(t, x, y)$  and  $j = 1, 2, 3$ .

**Proof.**

For most of the corrector terms, the proof is as in Prop. 3.2, since these terms are obtained from the leading term  $\mathbf{U}_0^\epsilon$  as in the coupled case, and since Prop. 4.1 asserts that this leading term  $\mathbf{U}_0^\epsilon$  has the required boundedness properties. The main difficulty relies in the proof of the fact that  $\epsilon^2 \mathbf{U}_2^\epsilon$  remains small enough. The proof of this result in the uncoupled case relies heavily on the structure of  $\mathbf{O}_1(\partial)$  since  $\pi(0)\mathcal{U}_2$  is found solving Eq. (2.24). Here, we must find  $\pi(0)\mathbf{U}_2^\epsilon$  solving the second equation of (4.33) where  $\mathbf{O}_1(\partial)$  have been replaced by  $\mathbf{O}'_1(\partial)$ .

In this case, one can prove that  $\pi(0)\mathbf{U}_2^\epsilon$  is bounded independently of  $\epsilon$  (see [BC], Prop. 4.1). It is then routine to conclude the proof of the proposition. ■

The next step in the proof of Theor. 4.1 consists in estimating the residual. When plugging  $\mathbf{u}^\epsilon(t, x, y) = \epsilon^2 \mathbf{U}^\epsilon(t, x, \epsilon y)$  with  $\mathbf{U}^\epsilon(t, x, Y)$  given by (4.34), we find a residual  $\mathbf{r}^\epsilon$  which reads

$$\mathbf{r}^\epsilon(t, x, y) = \mathbf{R}^\epsilon(t, x, \epsilon y) \quad \text{and} \quad \mathbf{R}^\epsilon(t, x, Y) = \sum_{j=1}^{11} \epsilon^j \mathbf{R}_j^\epsilon(t, x, Y).$$

In the uncoupled case we had  $\mathcal{R}_j = 0$  for  $j = 1 \dots 4$ , while we do not have here  $\mathbf{R}_j^\epsilon = 0$  for  $j = 1 \dots 4$  here since we have modified the profile equations in such a way that Eqs. (2.18)-(2.19) and (2.8) are not compatible. Hence,  $\mathbf{R}_2^\epsilon \neq 0$ . However, it can be proved as in [BC] that the *global* residual  $\mathbf{R}^\epsilon$  is smaller than in the uncoupled case. Indeed, the residual is  $O(\epsilon^5)$  here, while it was only  $o(\epsilon^4)$  in Prop. 3.3. Note also that we make a regularity assumption on  $\partial_x^{-1}\mathbf{U}_{01}^\epsilon$  and  $\partial_x^{-1}\mathbf{U}_{02}^\epsilon$ , which we had not made in the uncoupled case, and which allows us to compute  $\partial_t \mathbf{R}_5^\epsilon$  and all the other terms of the residual in Sobolev spaces, without using infrared cutoffs.

**Proposition 4.3** *Let  $s > 7$  and suppose that both  $\partial_x^{-1}\mathbf{U}_{01}^\epsilon$  and  $\partial_x^{-1}\mathbf{U}_{02}^\epsilon$  are in  $C^1([0, T_0]; H^{s-3}(\mathbb{R}^2)^N)$ .*

*Then the residual satisfies the following estimates*

$$\sup_{t \in [0, \frac{T_0}{\epsilon^2}]} \|\mathbf{R}^\epsilon(t, \cdot, \cdot)\|_{H^{s-6}(\mathbb{R}_{x,Y})^N} = O(\epsilon^5) \quad \text{and} \quad \|\mathbf{r}^\epsilon\|_{L^\infty([0, \frac{T_0}{\epsilon^2}] \times \mathbb{R}_{x,y})} = O(\epsilon^5).$$

**Proof.**

The idea of the proof of [BC] is as follows. First, note that we have

$$\mathbf{R}^\epsilon = \epsilon^2 \pi(0) \mathbf{R}_2^\epsilon + \epsilon^4 \pi(0) \mathbf{R}_4^\epsilon + O(\epsilon^5).$$

Next, technical computations yields that

$$\epsilon^2 \pi_1(0) \mathbf{R}_2^\epsilon + \epsilon^4 \pi_1(0) \mathbf{R}_4^\epsilon = \epsilon^4 \mathbf{P}_1(\partial) (\partial_t - \lambda'(0) \partial_x) \mathbf{U}_{01}^\epsilon,$$

where  $\mathbf{P}_1(\partial)$  is a constant coefficients matricial operator.

Since we know by Eqs. (4.37) that  $(\partial_t - \lambda'(0) \partial_x) \mathbf{U}_{01}^\epsilon = O(\epsilon^2)$ , we can conclude that the l.h.s. of the above equation is  $O(\epsilon^6)$ . We obtain similarly the same result for  $\epsilon^2 \pi_2(0) \mathbf{R}_2^\epsilon + \epsilon^4 \pi(0)_2 \mathbf{R}_4^\epsilon$ , which ends the proof. ■

This better estimates of the residual yields the better estimate of the error term stated in Theor. 4.1. The techniques used in the proof are exactly similar to those used in Theor. 3.2 for the uncoupled case, and are even simplified since we do not have to use the infrared cutoffs..

## 5 Water waves

Although the Euler equations do not fall into the class of problems studied in the previous sections, the results proved in this latter framework are nevertheless of interest for the water-waves problem. First of all, we can control the growth of the correctors as for hyperbolic systems and use the same rigorous tools used there to derive the KP equation. We can also mimic the derivation of coupled KP systems for hyperbolic systems, and use the results on the growth of the correctors to obtain a system of coupled KP equations for the water-waves, with a symmetric nonlinearity. This new coupled system, which is the analogous in three dimensions of the coupled KdV system, is likely to furnish a better approximation than the usual uncoupled KP equations.

We consider here an irrotational and incompressible flow, associated to the appropriate boundary conditions at the bottom and no surface tension at the surface. Designating by  $\phi(x, y, z, t)$  the velocity potential, where  $(x, y)$  are the horizontal variables and  $z$  the vertical variable, and denoting by  $\zeta(x, y, t)$  the water elevation, the Euler equations with free boundary conditions read in their classical dimensionless form as,

$$\begin{cases} \beta\phi_{xx} + \beta\phi_{yy} + \phi_{zz} = 0 & 0 < z < 1 + \alpha\zeta \\ \phi_z = 0 & \text{at } z = 0 \\ \zeta_t + \alpha\phi_x\zeta_x + \alpha\phi_y\zeta_y - \frac{1}{\beta}\phi_z = 0 & \text{at } z = 1 + \alpha\zeta \\ \zeta + \phi_t + \frac{\alpha}{2}(\phi_x^2 + \phi_y^2) + \frac{\alpha}{2\beta}\phi_z^2 = 0 & \text{at } z = 1 + \alpha\zeta, \end{cases}$$

where  $\alpha = \frac{\text{amplitude}}{\text{depth}}$  and  $\beta = \left(\frac{\text{depth}}{\text{wavelength}}\right)^2$  that we suppose to be small pa-

rameters of the system. In our derivation, the small parameter appears to be unique, and we thus set  $\alpha = \beta := \epsilon^2$ . The Euler system thus writes

$$\begin{cases} \epsilon^2 \phi_{xx} + \epsilon^2 \phi_{yy} + \phi_{zz} = 0 & 0 < z < 1 + \epsilon^2 \zeta \\ \phi_z = 0 & \text{at } z = 0 \\ \zeta_t + \epsilon^2 \phi_x \zeta_x + \epsilon^2 \phi_y \zeta_y - \frac{1}{\epsilon^2} \phi_z = 0 & \text{at } z = 1 + \epsilon^2 \zeta \\ \zeta + \phi_t + \frac{\epsilon^2}{2} (\phi_x^2 + \phi_y^2) + \frac{1}{2} \phi_z^2 = 0 & \text{at } z = 1 + \epsilon^2 \zeta, \end{cases} \quad (5.42)$$

Introducing  $\varphi(x, y, t) := \phi(x, y, 1, t)$ , we can solve explicitly the first two equations of (5.42). Namely,

$$\widehat{\phi}(\Theta, z, t) = \frac{\cosh(\epsilon|\Theta|z)}{\cosh(\epsilon|\Theta|)} \widehat{\varphi}(\Theta, t),$$

where we recall that  $\Theta = (\xi, \eta)$  is the Fourier dual variable of  $(x, y)$ . From this expression, we can deduce the expansions

$$\begin{cases} \phi_z(x, y, 1, t) = -\epsilon^2(\varphi_{xx} + \varphi_{yy}) - \frac{\epsilon^4}{3}(\partial_x^2 + \partial_y^2)^2 \varphi + O(\epsilon^6) \\ \phi_{zz}(x, y, 1, t) = -\epsilon^2(\varphi_{xx} + \varphi_{yy}), \end{cases}$$

which are in turn used to compute the Taylor expansions

$$\begin{aligned} \phi(x, y, 1 + \epsilon^2 \zeta, t) &= \varphi + O(\epsilon^4) \\ \phi_z(x, y, 1 + \epsilon^2 \zeta, t) &= -\epsilon^2(\varphi_{xx} + \varphi_{yy}) - \epsilon^4 \left( \frac{1}{3} (\partial_x^2 + \partial_y^2)^2 \varphi + \zeta(\varphi_{xx} + \varphi_{yy}) \right) \\ &\quad + O(\epsilon^6) \\ \phi_{zz}(x, y, 1 + \epsilon^2 \zeta, t) &= -\epsilon^2(\varphi_{xx} + \varphi_{yy}) + O(\epsilon^4). \end{aligned}$$

Using these expansions, the last two equations of system (5.42) read

$$\begin{cases} \zeta_t + (\varphi_{xx} + \varphi_{yy}) + \epsilon^2(\varphi_x \zeta_x + \varphi_y \zeta_y + \frac{1}{3}(\partial_x^2 + \partial_y^2)^2 \varphi + \zeta(\varphi_{xx} + \varphi_{yy})) = O(\epsilon^4) \\ \varphi_t + \zeta + \frac{\epsilon^2}{2}(\varphi_x^2 + \varphi_y^2) = O(\epsilon^4). \end{cases} \quad (5.43)$$

We now seek approximate solutions  $\zeta^\epsilon$  and  $\varphi^\epsilon$  of the form

$$\zeta^\epsilon(x, y, t) = \zeta_0(t, x, \epsilon y, \epsilon^2 t) + \epsilon \zeta_1(t, x, \epsilon y, \epsilon^2 t) + \epsilon^2 \zeta_2(t, x, \epsilon y, \epsilon^2 t), \quad (5.44)$$

and

$$\varphi^\epsilon(x, y, t) = \varphi_0(t, x, \epsilon y, \epsilon^2 t) + \epsilon \varphi_1(t, x, \epsilon y, \epsilon^2 t) + \epsilon^2 \varphi_2(t, x, \epsilon y, \epsilon^2 t). \quad (5.45)$$

We now plug (5.44) and (5.45) into (5.43), and identify the terms at each order of  $\epsilon$ . This yields,

At order  $O(1)$ .

$$\begin{cases} \partial_t \zeta_0 + \partial_x^2 \varphi_0 = 0 \\ \partial_t \varphi_0 + \zeta_0 = 0. \end{cases} \quad (5.46)$$

At order  $O(\epsilon)$ .

$$\begin{cases} \partial_t \zeta_1 + \partial_x^2 \varphi_1 = 0 \\ \partial_t \varphi_1 + \zeta_1 = 0. \end{cases} \quad (5.47)$$

At order  $O(\epsilon^2)$ .

$$\begin{cases} \partial_T \zeta_0 + \partial_t \zeta_2 + \partial_x \varphi_0 \partial_x \zeta_0 + \partial_x^2 \varphi_2 + \partial_Y^2 \varphi_0 + \frac{1}{3} \partial_x^4 \varphi_0 + \zeta_0 \partial_x^2 \varphi_0 = 0 \\ \partial_T \varphi_0 + \partial_t \varphi_2 + \zeta_2 + \frac{1}{2} (\partial_x \varphi_0)^2 = 0. \end{cases} \quad (5.48)$$

First of all, remark that one can set  $\zeta_1$  and  $\varphi_1$  to zero, since these functions do not appear in the long time evolution equations (5.48).

These sets of equations suggest us to consider  $u^\epsilon := \varphi_x^\epsilon$  and  $v^\epsilon := \varphi_y^\epsilon$  as auxiliary unknowns. Differentiating the second equation of (5.46) with respect to  $x$  gives a coupled system on  $\zeta_0$  and  $u_0$  whose general solution writes

$$\begin{cases} \zeta_0(t, x, Y, T) = a(x - t, Y, T) + b(x + t, Y, T) \\ u_0(t, x, Y, T) = a(x - t, Y, T) - b(x + t, Y, T). \end{cases} \quad (5.49)$$

Moreover, (5.48) reads, after differentiating the second equation with respect to  $x$ ,

$$\begin{cases} \partial_T \zeta_0 + \partial_t \zeta_2 + u_0 \partial_x \zeta_0 + \partial_x u_2 + \partial_Y v_0 + \frac{1}{3} \partial_x^3 u_0 + \zeta_0 \partial_x u_0 = 0 \\ \partial_T u_0 + \partial_t u_2 + \partial_x \zeta_2 + u_0 \partial_x u_0 = 0. \end{cases}$$

Using the explicit expressions given by (5.49) and adding and subtracting the two equations of the above system yields the equivalent system

$$\begin{cases} (\partial_t + \partial_x)(\zeta_2 + u_2) + \partial_Y v_0 + 2u_0 \partial_x a + \frac{1}{3} \partial_x^3 u_0 + \zeta_0 \partial_x u_0 + 2\partial_T a = 0 \\ (\partial_t - \partial_x)(\zeta_2 - u_2) + \partial_Y v_0 + 2u_0 \partial_x b + \frac{1}{3} \partial_x^3 u_0 + \zeta_0 \partial_x u_0 + 2\partial_T b = 0 \end{cases}$$

Since  $v_0 = \partial_Y \varphi_0$  and  $u_0 = \partial_x \varphi_0$ , we will denote the term  $\partial_Y v_0$  of the above system by  $\partial_x^{-1} \partial_Y^2 u_0$ , and the system thus writes

$$\begin{cases} (\partial_t + \partial_x)(\zeta_2 + u_2) + \partial_x^{-1} \partial_Y^2 u_0 + 2u_0 \partial_x a + \frac{1}{3} \partial_x^3 u_0 + \zeta_0 \partial_x u_0 + 2\partial_T a = 0 \\ (\partial_t - \partial_x)(\zeta_2 - u_2) + \partial_x^{-1} \partial_Y^2 u_0 + 2u_0 \partial_x b + \frac{1}{3} \partial_x^3 u_0 + \zeta_0 \partial_x u_0 + 2\partial_T b = 0 \end{cases} \quad (5.50)$$

### Derivation of the uncoupled system.

If we apply here the general method described in the previous sections, we obtain the usual uncoupled KP equations. This method, which corresponds to the classical way of deriving those equations, consists in applying the average projectors to these equations, which gives two uncoupled equations on  $a$  and  $b$ ,

$$\begin{cases} \partial_T a + \frac{1}{6} \partial_x^3 a + \frac{1}{2} \partial_x^{-1} \partial_Y^2 a + \frac{3}{2} a \partial_x a = 0 \\ \partial_T b - \frac{1}{6} \partial_x^3 b - \frac{1}{2} \partial_x^{-1} \partial_Y^2 b - \frac{3}{2} b \partial_x b = 0, \end{cases}$$

while the correctors  $u_2$  and  $\zeta_2$  must satisfy

$$\begin{cases} (\partial_t + \partial_x)(\zeta_2 + u_2) = \partial_x^{-1} \partial_Y^2 b + 2b \partial_x a + \frac{1}{3} \partial_x^3 b + a \partial_x b - b \partial_x a + b \partial_x b \\ (\partial_t - \partial_x)(\zeta_2 - u_2) = -\partial_x^{-1} \partial_Y^2 a - 2a \partial_x b - \frac{1}{3} \partial_x^3 a - b \partial_x a + a \partial_x b - a \partial_x a. \end{cases}$$

As we have set  $\zeta_1 = 0$ , we have

$$\zeta^\epsilon(x, y, t) = a(x - t, \epsilon y, \epsilon^2 t) + b(x + t, \epsilon y, \epsilon^2 t) + \epsilon^2 \zeta_2(t, x, \epsilon y, \epsilon^2 t),$$

and the general results of Section 3 can be used here to see that the term  $\epsilon^2 \zeta_2$  found with the above equations is  $o(1)$  and therefore is indeed a corrective term to the leading term  $\zeta_0 = a + b$  of the formal ansatz.

### Derivation of the coupled system.

We now follow the techniques of Section 4 in order to obtain two coupled KP equations. We have seen that in order to do so, we must consider functions depending on  $(t, x, Y)$  instead of  $(t, x, Y, T)$ . More precisely, this means that we seek  $\zeta^\epsilon$  under the form

$$\zeta^\epsilon(x, y, t) = a^\epsilon(t, x, \epsilon y) + b^\epsilon(t, x, \epsilon y) + \epsilon^2 \zeta_2^\epsilon(t, x, \epsilon y),$$

where  $a^\epsilon$  and  $b^\epsilon$  must solve the following system, which is the analogous in the framework of water-waves of the coupled system (4.37) of Section 4.

$$\begin{cases} \partial_t a^\epsilon + \partial_x a^\epsilon + \epsilon^2 \left( \frac{1}{6} \partial_x^3 a^\epsilon + \frac{1}{2} \partial_x^{-1} \partial_Y^2 a^\epsilon + \frac{1}{2} (3a^\epsilon - b^\epsilon) \partial_x a^\epsilon - \frac{1}{2} (a^\epsilon + b^\epsilon) \partial_x b^\epsilon \right) = 0 \\ \partial_t b^\epsilon - \partial_x b^\epsilon - \epsilon^2 \left( \frac{1}{6} \partial_x^3 b^\epsilon + \frac{1}{2} \partial_x^{-1} \partial_Y^2 b^\epsilon - \frac{1}{2} (a^\epsilon + b^\epsilon) \partial_x a^\epsilon - \frac{1}{2} (a^\epsilon - 3b^\epsilon) \partial_x b^\epsilon \right) = 0, \end{cases} \quad (5.51)$$

while the correctors  $u_2$  and  $\zeta$  are found solving

$$\begin{cases} (\partial_t + \partial_x)(\zeta_2^\epsilon + u_2^\epsilon) = \partial_x^{-1} \partial_Y^2 b^\epsilon + \frac{1}{3} \partial_x^3 b^\epsilon \\ (\partial_t - \partial_x)(\zeta_2^\epsilon - u_2^\epsilon) = -\partial_x^{-1} \partial_Y^2 a^\epsilon - \frac{1}{3} \partial_x^3 a^\epsilon. \end{cases} \quad (5.52)$$

Here again, we can use the result proved in the general theory, namely in Section 4 (and Prop. 4.1 of [BC]), to see that  $\epsilon^2 \zeta_2^\epsilon$  is indeed a corrective term to the leading term  $a^\epsilon + b^\epsilon$  of the formal ansatz.

However, Prop. 4.1 does not allow us to find  $a^\epsilon$  and  $b^\epsilon$  since the coupled system (5.51) does not fall into its range because of the lack of symmetry of the nonlinearities. The trick consists here, as in [BC] for the coupled KdV equations, in modifying the nonlinearities of (5.51) in such a way that they become symmetric, and that  $\epsilon^2 \zeta_2^\epsilon$  remains a corrector.

Thanks to Prop. 4.1 of [BC], we know that adding terms of the form  $b^\epsilon \partial_x b^\epsilon$  to the r.h.s. of the first equation of (5.52) only causes a bounded perturbation of  $\zeta_2^\epsilon$  and  $u_2^\epsilon$ . Similarly, one can add terms of the form  $a^\epsilon \partial_x a^\epsilon$  to the r.h.s. of the second equation of (5.52). Because of the factor  $\epsilon^2$  in front of  $\zeta_2^\epsilon$ , these contributions are only  $O(\epsilon^2)$  in the ansatz, and hence do not affect the leading term. This procedure introduced in [BC] for the KdV equations can easily be

extended to the present case.

We therefore add  $b^\epsilon \partial_x b^\epsilon$  to the first equation of (5.51) and  $-a^\epsilon \partial_x a^\epsilon$  to the second, and modify subsequently the correctors (5.52), which gives

$$\begin{cases} \partial_t a^\epsilon + \partial_x a^\epsilon + \epsilon^2 \left( \frac{1}{6} \partial_x^3 a^\epsilon + \frac{1}{2} \partial_x^{-1} \partial_Y^2 a^\epsilon + \frac{1}{2} (3a^\epsilon - b^\epsilon) \partial_x a^\epsilon - \frac{1}{2} (a^\epsilon - b^\epsilon) \partial_x b^\epsilon \right) = 0 \\ \partial_t b^\epsilon - \partial_x b^\epsilon - \epsilon^2 \left( \frac{1}{6} \partial_x^3 b^\epsilon + \frac{1}{2} \partial_x^{-1} \partial_Y^2 b^\epsilon + \frac{1}{2} (a^\epsilon - b^\epsilon) \partial_x a^\epsilon - \frac{1}{2} (a^\epsilon - 3b^\epsilon) \partial_x b^\epsilon \right) = 0, \end{cases} \quad (5.53)$$

while the correctors  $u_2^\epsilon$  and  $\zeta^\epsilon$  are found solving

$$\begin{cases} (\partial_t + \partial_x)(\zeta_2^\epsilon + u_2^\epsilon) = \partial_x^{-1} \partial_Y^2 b^\epsilon + \frac{1}{3} \partial_x^3 b^\epsilon - 2b^\epsilon \partial_x b^\epsilon \\ (\partial_t - \partial_x)(\zeta_2^\epsilon - u_2^\epsilon) = -\partial_x^{-1} \partial_Y^2 a^\epsilon - \frac{1}{3} \partial_x^3 a^\epsilon + 2a^\epsilon \partial_x a^\epsilon. \end{cases}$$

As we have just said, we can solve (5.53) with Prop. 4.1, and  $\epsilon^2 \zeta_2^\epsilon$  remains a corrective term. The comparison made between the coupled and uncoupled systems in the general theory makes us suspect that the approximation given by (5.53) is better than the usual approximation obtained by two uncoupled KP equations.

## ACKNOWLEDGEMENTS

The authors want to thank T. Colin for his very interesting comments and insight about this work. They also thank B. Texier for his remarks on a previous version of this work.

## REFERENCES

- [A] D. Alterman, Diffractive Nonlinear Geometric Optics for Short Pulses, *PhD thesis*, University of Michigan, 1999.
- [AG] S. Alinhac and P. Gerard, Opérateurs pseudo-différentiels et opérateurs de Nash-Moser, *Savoirs actuels*, InterEditions/Editions du CNRS, 1991.

- [AR] D. Alterman and J. Rauch, Diffractive nonlinear geometric optics for short pulses, *in preparation* (2001).
- [BC] W. Ben Youssef and T. Colin, Rigorous derivation of Korteweg-de Vries type systems from a general class of nonlinear hyperbolic systems, *Math. Modelling and Num. Anal.* **34**, 4, (2000), 873-911.
- [BL] K. Barrailh and D. Lannes, A general framework for diffractive optics and its applications to lasers with large spectrum and short pulses, *Preprint Université Bordeaux I*, (2000).
- [Cr] W. Craig, An existence theory for water waves and the Boussinesq and the Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations* **10** (1985), 787-1003.
- [GS] T. Gallay and G. Schneider, KP description of unidirectional long waves -The model case -. *Preprint Université Paris Sud, Orsay.* (2000).
- [JMR1] J.L. Joly, G. Métivier and J. Rauch, Generic rigorous expansions for weakly nonlinear multidimensional oscillatory waves. *Duke Math. J.* **70** (1993), 373-404.
- [JMR2] J.L. Joly, G. Métivier and J. Rauch, Diffractive nonlinear geometric optics with rectification. *Indiana Univ. Math. J.* **47** (1998), 1167-1241.
- [Kan] T. Kano, L'équation de Kadomtsev-Petviashvili approchant les ondes longues de surface de l'eau en écoulement trois-dimensionnel. *Patterns and waves, Stud. Math. Appl., North-Holland, Amsterdam* **18** (1986), 431-444.
- [Kat1] T. Kato, Perturbation theory for linear operators. *Grundlehren Math. Wiss.* **132** (1996).
- [Kat2] T. Kato, The Cauchy problem for quasilinear symmetric hyperbolic systems. *Arch. Rational Mech. Anal.* **58** (1975), 181-205.
- [Kat3] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation. *Advances in Math. Supplementary Studies* **8** (1983), 273-294.
- [KN] T. Kano and T. Nishida, A mathematical justification for Korteweg-

- de Vries equation and Boussinesq equation of water surface waves. *Osaka J. Math.* **23** (1986), 389-413.
- [KIM] S. Klainerman and A. Majda, Compressible and incompressible fluids. *Comm. Pure Appl. Math.* **35** (1982), 629-653.
- [L] D. Lannes, Dispersive effects for nonlinear geometrical optics with rectification. *Asymptot. Anal.* **18** (1998), 111-146.
- [M] A. Majda, Compressible fluid flows and Studies of Conservation laws in several space variables. *Appl. Math. Sci. vol. 53* Springer-Verlag, New-York-Berlin-Heidelberg-Tokyo, 1984.
- [Sa] J.-C. Saut, Remarks on the generalized Kadomtsev-Petviashvili equations. *Indiana University Mathematics Journal* **42** (1993), 1011-1026.
- [S] G. Schneider, The long wave limit for a Boussinesq equation. *SIAM J. Appl. Math.* **58** (1998), 1237-1245.
- [SW] G. Schneider and C.E. Wayne, The long wave limit for the water wave problem. I. the case of zero surface tension. *Comm. Pure Appl. Math.* **53** (2000), 1475-1535.
- [T] B. Texier, The short wave limit for a general class of nonlinear hyperbolic systems, *In preparation*, (2001).
- [U] S. Ukai, Local solutions of the Kadomtsev-Petvishvili equation, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **36** (1989), 193-209.