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A GENERAL FRAMEWORK FOR DIFFRACTIVE OPTICS AND ITS APPLICATIONS TO LASERS WITH LARGE SPECTRUMS AND SHORT PULSES*

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Abstract. The aim of this article is to generalize the usual tools of diffractive optics in order to allow the study of phenomena which are out of their range. This generalization relies on the algebra of oscillations with a continuous oscillatory spectrum, which is wider than the usual spaces of periodic or almost-periodic functions. We perform the analysis for general nonlinear hyperbolic systems, both in the dispersive and in the nondispersive cases, and particularly focus on the behavior of the nonlinearities. Our tools yield considerable simplifications in these nonlinearities, which allows us to point out qualitative differences between the dispersive and the nondispersive cases. Finally, we study in detail two physical examples which can be modeled with the present tools: lasers with large spectrums, and those with ultrashort pulses.

Key words. nonlinear hyperbolic systems, Maxwell equations, diffractive optics, continuous oscillatory spectrum, large spectrum, short pulse, nonlinear Schrödinger equation

AMS subject classifications. 28B05, 35L40, 35Q55, 35Q60, 35S99

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1. General comment.

1.1. Introduction. Maxwell equations and many of the physical systems encountered in optics may be written in the form

$$\begin{cases} L^\varepsilon(\partial)\mathbf{u}^\varepsilon + f(\mathbf{u}^\varepsilon) = 0, \\ \mathbf{u}^\varepsilon|_{T=0}(X,Y,Z) = \mathbf{u}^0_\varepsilon(X,Y,Z), \end{cases} \tag{1.1}$$

where $\mathbf{u}^\varepsilon$ takes its values in R^n, and $L^\varepsilon(\partial)$ is a hyperbolic symmetric operator which one writes as

$$L^\varepsilon(\partial) = A_0\partial_T + A_1\partial_X + A_2\partial_Y + A_3\partial_Z + \frac{L_0}{\varepsilon},$$

the matrices A_i being symmetric and L_0 skew-symmetric.

In the study of the propagation of a diffractive laser beam with frequency ω_l and wavenumber $\overrightarrow{k_l} = (0,0,k_l)$, an approximate solution u^ε of $\mathbf{u}^\varepsilon$ is generally sought in the form

$$u^\varepsilon(T,X,Y,Z) = \varepsilon^p(\mathcal{U}_0(\varepsilon T,T,X,Y,Z)e^{i(\omega_l T - k_l Z)/\varepsilon} + \text{c.c.}), \tag{1.2}$$

the exponent p being chosen in order for the nonlinear and diffractive effects to come into play at the same time scale. The method of diffractive optics (see [8], for instance) consists of finding some equations which determine the *profile* $\mathcal{U}_0$.

The object of this article is to introduce a general framework for diffractive optics, which generalizes classical studies in both dispersive and nondispersive diffractive

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optics, as shown in [10] and [11], for instance, and allows us also, without added difficulty, to treat more pathological situations. Among these, we study here two physical problems which cannot be modeled by oscillations of type (1.2). These physical phenomena are *large-spectrum lasers* and *ultrashort pulses*.

Large spectrum. The oscillatory spectrum of a classical oscillation of type (1.2) is located at two points, $\{\pm(\omega_l, -\overrightarrow{k_l})\}$. Experimentally, such a localization for the spectrum is never realized. Physically, the spectrum is concentrated around $\{\pm(\omega_l, -\overrightarrow{k_l})\}$, but it never reduces to these two points. These variations are generally taken into account in the amplitude $\mathcal{U}_0$. However, modifying $\mathcal{U}_0$ can only make the spectrum of u^ε expand around $\{\pm(\omega_l, -\overrightarrow{k_l})\}$ in an $O(\varepsilon)$ range. Lasers with large spectrum typically have a spectrum of width $O(1)$ and therefore cannot be modeled with usual oscillations of type (1.2). Direct computations for lasers with large spectrums have been carried out by Morice [13]. Here, we choose another approach and seek an approximate solution of (1.1) in the form

$$\begin{aligned} u^\varepsilon(T,X,Y,Z) = \varepsilon^p \big(&\mathcal{U}_{0,I,1}(\varepsilon T, T, X, Y, Z) e^{i(\omega_l T - k_l Z)/\varepsilon} + \text{c.c.}\big) \\ &+ \varepsilon^p \,\mathcal{U}_{0,II}\left(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon}\right), \end{aligned}$$

where $\mathcal{U}_{0,II}$ is an oscillation with a *purely continuous spectrum*. We introduce this notion in Proposition 1.10 below; for the moment, just assume that $\mathcal{U}_{0,II}$ is smooth and decaying in its last two variables.

Considering a model of nonlinear Maxwell equations (system (M) in section 4.1), we prove that the amplitude $\mathcal{E}_{0,I,1}$ of the oscillating component of the electric field satisfies the usual nonlinear Schrödinger (NLS) equation

$$\partial_\tau \mathcal{E}_{0,I,1} + i\frac{\omega'(k_l)}{2k_l}(\partial_X^2 + \partial_Y^2)\mathcal{E}_{0,I,1} + i\frac{\omega''(k_l)}{2}\partial_Z^2 \mathcal{E}_{0,I,1} = \text{Cst}\ |\mathcal{E}_{0,I,1}|^2 \mathcal{E}_{0,I,1},$$

while the corrective term $\mathcal{E}_{0,II}$ satisfies a linear equation,

$$\partial_\tau \partial_{z_0} \mathcal{E}_{0,II} - \frac{\omega'(D_{z_0})}{2}(\partial_X^2 + \partial_Y^2)\mathcal{E}_{0,II} - \frac{D_{z_0}\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{E}_{0,II} = 0.$$

The main interest of this latter equation is that all the nonlinearities one would find by a direct computation have been dropped, making this equation linear when it was a priori nonlinear. This fact is a striking consequence of the general results proved thereafter.

Ultrashort pulses. Seeking an approximate solution in the form (1.2) supposes that the profile $\mathcal{U}_0$ varies little compared with the scale of an oscillation. This condition is satisfied for almost all lasers because the length of the pulse is great compared with the wavelength. For the ultrashort pulses, obtained by recent lasers, this is no longer the case (see Figure 1). We refer to [4] and the references therein for a brief history of the study of short pulses in geometric optics. For diffractive time scales, the most general studies we know have been performed by Alterman and Rauch; see [1], [2], [3], and [15]. In the *nondispersive* case, these authors proved rigorously, using asymptotic techniques, that the Schrödinger approximation used for wave trains must be replaced by another approximation for short pulses,

$$2\partial_{z_0}\partial_\tau \mathcal{V} = v(\partial_X^2 + \partial_Y^2)\mathcal{V} + \partial_{z_0} f(\mathcal{V}), \tag{1.3}$$

where $v\overrightarrow{e}_Z$ denotes the group velocity.

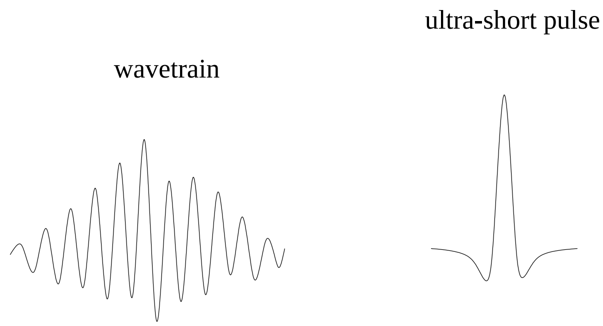

FIG. 1. *Example of a wave train and a short pulse.*

One of the main interests of [1], [3], and [15] is that these papers address pulses which may not have vanishing mean, as was the case in earlier papers [16], [17]. We use here Alterman's technique of infrared cutoffs to obtain this generality, but otherwise our approach is completely different since it is based on oscillations with continuous spectrum. We see three main interests in our method. First, it generalizes the usual methods of diffractive optics [8], [10], [11], so that short pulses do not appear as a pathological case, and "mixed" cases, such as the above lasers with large spectrums, can be addressed without added difficulty. The second interest resides in the study of the nonlinearities, since we prove that most of them can be dropped because their influence is negligible. Finally, we are able to address dispersive models, which are physically the most relevant.

More precisely, an approximate solution for the short pulse is sought in the form

$$u^\varepsilon(T,X,Y,Z) = \varepsilon^p \mathcal{U}_{0,II}\left(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon}\right),$$

where $\mathcal{U}_{0,II}$ is a profile with a purely continuous oscillatory spectrum. Indeed, the pulse here is too short for a sinusoidal oscillation to appear. In the nondispersive case, we find of course that $\mathcal{U}_{0,II}$ must satisfy Alterman and Rauch's equation (1.3). The dispersive case is both simpler and more complicated because if the nonlinearities of (1.3) can be neglected, the group velocity $v\overrightarrow{e}_Z$ then depends on the frequency.

Remark 1.1. As the common formalism suggests, short pulses and large spectrum corrections to wave trains are essentially the same. The former focus on the time domain, while the latter look at the Fourier domain.

1.2. The spaces. We seek approximate solutions of (1.1) for diffractive time scales. Therefore, three scales of variables are used in this study:

- the fast scale $O(\frac{1}{\varepsilon})$ of the oscillations,
- the intermediate scale $O(1)$ of geometrical optics,
- the slow scale $O(\varepsilon)$ related to diffractive effects.

In order to identify clearly the variations of the solutions in these scales, auxiliary functions named *profiles* are introduced as in [8], and we look for exact solutions of (1.1) in the form

$$\mathbf{u}^\varepsilon = \varepsilon^p \mathbf{U}^\varepsilon\left(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon}\right).$$

The factor ε^p is chosen to have both nonlinear and diffractive effects on the same time scale.

Before introducing the spaces associated to the profiles $\mathcal{V}(\tau, T, X, Y, Z, t_0, z_0)$ that we use to represent the exact and approximate solutions of (1.1), let us set some notation.

Notation. From now on, we write $\theta := (\omega_l t_0 - k_l z_0)$ and denote by $\xi := (\omega, k)$ and $\eta := (\eta_1, \eta_2, \eta_3)$ the Fourier dual variable of (t_0, z_0) and (X, Y, Z), respectively. The letter s will always denote a positive real number $s > 3/2$.

We also denote by $\mathcal{F}$ and by $\widehat{}$ the Fourier transforms with respect to the variables (t_0, z_0) and (X, Y, Z), respectively.

Throughout this paper constants are invariably denoted by C.

The space we choose for the profiles must contain oscillations with a discrete spectrum such as $U(\tau, T, X, Y, Z)e^{i\theta}$ and oscillations with a purely continuous spectrum. The spaces used here are a generalization to diffractive scales of those which have been introduced in [12] to describe Raman scattering.

DEFINITION 1.1. (i) *We denote by A_0^s the set of the functions defined on $\mathbb{R}^3_{X,Y,Z} \times \mathbb{R}^2_{t_0,z_0}$ with values in $\mathbb{C}^n$ whose Fourier transform with respect to (t_0, z_0) belongs to the set $\mathcal{BV}(\mathbb{R}^2_\xi, H^s(\mathbb{R}^3_{X,Y,Z})^n)$ of bounded variation Borel measures defined on $\mathbb{R}^2_\xi$ and with values in $H^s(\mathbb{R}^3)^n$. This space is endowed with the norm*

$$||\mathcal{V}||_{A_0^s} := |\mathcal{F}\mathcal{V}|_{\mathcal{BV}} \qquad \forall \mathcal{V} \in A_0^s;$$

(ii) *We denote by $E^s_{\tau*}$ the set of the functions defined on $[0, \tau^*] \times \mathbb{R}_T \times \mathbb{R}^3_{X,Y,Z} \times \mathbb{R}^2_{t_0,z_0}$ with values in $\mathbb{C}^n$ whose Fourier transform with respect to (t_0, z_0) belongs to $\mathcal{C}([0, \tau^*] \times \mathbb{R}_T, \mathcal{BV}(\mathbb{R}^2_\xi, H^s(\mathbb{R}^3_{X,Y,Z})^n))$. Moreover, for all $T \in \mathbb{R}$, we define*

$$||\mathcal{V}(T)||_{E^s_{\tau^*}} := \sup_{0 \le \tau \le \tau^*} |\mathcal{F}\mathcal{V}(\tau, T)|_{\mathcal{BV}} \qquad \forall \mathcal{V} \in E^s_{\tau^*};$$

(iii) *We denote by $A^s_{\tau^*}$ the subspace of $E^s_{\tau*}$ composed by all the functions of $E^s_{\tau*}$ bounded on $[0, \tau^*] \times \mathbb{R}_T$ and endow this space with the norm*

$$||\mathcal{V}||_{A^s_{\tau^*}} := \sup_{0 \le \tau \le \tau^*} \sup_{T \in \mathbb{R}} |\mathcal{F}\mathcal{V}(\tau, T)|_{\mathcal{BV}} = \sup_{T \in \mathbb{R}} ||\mathcal{V}(T)||_{E^s_{\tau^*}} \qquad \forall \mathcal{V} \in A^s_{\tau^*};$$

(iv) *We denote by $B^s_{\tau^*}$ the subspace of $A^s_{\tau^*}$ composed by all the functions of $A^s_{\tau^*}$ which do not depend on T.*

The well-known notion of the oscillatory spectrum of an (almost-)periodic function [9] can then be generalized as follows.

DEFINITION 1.2. *If $\mathcal{V} \in E^s_{\tau^*}$, then for all $(\tau, T) \in [0, \tau^*] \times \mathbb{R}$, the spectrum* Sp $\mathcal{V}(\tau, T)$ *of $\mathcal{V}(\tau, T)$ is the support of the Fourier transform $\mathcal{F}\mathcal{V}(\tau, T)$.*

We also define the spectrum of $\mathcal{V}$ as Sp $\mathcal{V} = \bigcup_{(\tau,T) \in [0,\tau^*] \times \mathbb{R}}$ Sp $\mathcal{V}(\tau, T)$.

The following proposition [12] states the main properties of these functional spaces.

PROPOSITION 1.3. (i) *The two normed spaces $(A_0^s, ||.||_{A_0^s})$ and $(A^s_{\tau^*}, ||.||_{A^s_{\tau^*}})$ are complete.*

(ii) *Any J-linear mapping G defined on $(\mathbb{C}^n)^J$ and with values in $\mathbb{C}^n$ extends to a continuous J-linear mapping defined on A_0^s (resp., $A^s_{\tau^*}$) and with values in A_0^s (resp., $A^s_{\tau^*}$). Moreover, there exists a constant $l > 0$ such that for all J-uplet $(\mathcal{V}_1, \dots, \mathcal{V}_J) \in A_0^s$ (resp., $A^s_{\tau^*}$), one has*

$$||G(\mathcal{V}_1, \dots, \mathcal{V}_J)|| \le l ||\mathcal{V}_1|| \dots ||\mathcal{V}_J||,$$

where $||.||$ represents the norm of A_0^s (resp., $A^s_{\tau^*}$).

(iii) *Let $\mathcal{V}$ be in A_0^s (resp., $A_{\tau^*}^s$). Then $\mathcal{V}$ is also in $\mathcal{C}(\mathbb{R}^5)^n$ (resp., $\mathcal{C}([0,\tau^*]\times\mathbb{R}^6)^n$). Moreover, $\mathcal{V}$ is bounded, and there exists a positive number l' such that*

$$||\mathcal{V}||_\infty \leq l'||\mathcal{V}||_{A_0^s} \quad (resp., \quad ||\mathcal{V}||_\infty \leq l'||\mathcal{V}||_{A_{\tau^*}^s}).$$

(iv) *Let $\mathcal{V} \in A_0^s$ (resp., $A_{\tau^*}^s$). Then the function v^ε defined on $\mathbb{R}^3$ (resp., $[0,\frac{\tau^*}{\varepsilon}]\times\mathbb{R}^3$) as $v^\varepsilon(X,Y,Z) = \mathcal{V}(X,Y,Z,0,\frac{Z}{\varepsilon})$ (resp., $v^\varepsilon(T,X,Y,Z) = \mathcal{V}(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon})$) belongs to $L^2(\mathbb{R}^3)^n$ (resp., $\mathcal{C}([0,\frac{\tau^*}{\varepsilon}], L^2(\mathbb{R}^3)^n)$). Moreover, one has*

$$||v^\varepsilon||_{L^2(\mathbb{R}^3)} \leq ||\mathcal{V}||_{A_0^s} \quad (resp., \sup_{0\leq T\leq \tau^*/\varepsilon} ||v^\varepsilon(T,.)||_{L^2(\mathbb{R}^3)} \leq ||\mathcal{V}||_{A_{\tau^*}^s}).$$

Examples.

Example 1. Oscillations with a discrete spectrum such as $U(\tau,T,X,Y,Z)e^{i\theta}$, with $\theta = \omega_l t_0 - k_l z_0$ and $U \in \mathcal{C}([0,\tau^*]\times\mathbb{R}_T, H^s(\mathbb{R}^3)^n)$, are in $E_{\tau^*}^s$. Indeed, taking the Fourier transform of such oscillations yields

$$\mathcal{F}(Ue^{i\theta}) = U\delta_{(\omega_l,-k_l)},$$

which belongs to $\mathcal{C}([0,\tau^*]\times\mathbb{R}_T, \mathcal{BV}(\mathbb{R}_\xi^2, H^s(\mathbb{R}_{X,Y,Z}^3)^n))$.

Example 2. Let $\mathcal{M}$ be a submanifold of $\mathbb{R}^2$ and α an L^1 function defined on $\mathcal{M}$ and with values in $\mathcal{C}([0,\tau^*]\times\mathbb{R}_T, H^s(\mathbb{R}^3)^n)$. Then the density function [12] defined as

$$\mathcal{V}(\tau,T,X,Y,Z,t_0,z_0) = \int_{\mathcal{M}} e^{i(t_0,z_0)\cdot(\omega,k)}\alpha(\omega,k)(\tau,T,X,Y,Z)\sigma(d\omega,dk),$$

where σ denotes the Lebesgue measure of $\mathcal{M}$, is in $E_{\tau^*}^s$, and its oscillatory spectrum is $\mathcal{M}$.

1.3. Solving the Cauchy problem (1.1). We recall that (1.1) is written as

$$\begin{cases} L^\varepsilon(\partial)\mathbf{u}^\varepsilon + f(\mathbf{u}^\varepsilon) = 0, \\ \mathbf{u}^\varepsilon|_{T=0}(X,Y,Z) = \mathbf{u}_\varepsilon^0(X,Y,Z). \end{cases}$$

We now make precise the assumptions we make on $L^\varepsilon(\partial)$.

Assumption 1.1. *The system* (1.1) *is symmetric hyperbolic. More accurately, the operator $L^\varepsilon(\partial)$ can be written*

$$L^\varepsilon(\partial) = A_0\partial_T + A_1\partial_X + A_2\partial_Y + A_3\partial_Z + \frac{L_0}{\varepsilon},$$

where the A_i are real symmetric matrices and A_0 is strictly positive. Moreover the system (1.1) *is conservative in the sense that $(L_0)^* = -L_0$.*

Remark 1.2. Since A_0 is strictly positive, we can take $A_0^{-1/2}\mathbf{u}^\varepsilon$ as a new unknown. Multiplying (1.1) by $A_0(0)^{-1/2}$, the resulting system has the same properties as system (1.1) and satisfies $A_0(0) = Id$. *Thus herein, we always consider that $A_0 = Id$.*

The following hypothesis gives the kind of nonlinearity f we study here.

Assumption 1.2. *There exists a trilinear mapping F such that for all $u \in \mathbb{C}^n$, $f(u) = F(u,u,u)$.*

Remark 1.3. In this paper, we consider nonlinearities of order 3 since the two examples we gave in the last section belong to this class. This limitation on the order of the nonlinearity is only due to technical reasons, and the interested reader could easily generalize our results to nonlinearities of different orders.

The initial conditions for (1.1) must be general enough to allow a model of both large spectrums and ultrashort pulses. The spaces introduced in the previous part are adapted to such a general point of view, and we therefore consider initial conditions of the form

$$\mathbf{u}_\varepsilon^0(X,Y,Z) = \varepsilon^p \mathbf{U}^0\left(X,Y,Z,0,\frac{Z}{\varepsilon}\right), \tag{1.4}$$

with $\mathbf{U}^0 \in A_0^s$.

Choice of the size of the solutions. The choice of p is given [8] by the order of the nonlinearity, $p = 1/2$. With this choice, nonlinear and diffractive effects occur simultaneously.

The following theorem proves that the unique solution L^2 of the Cauchy problem (1.1) with initial condition (1.4) can be written using profiles from $B^s_{\tau^*}$.

THEOREM 1.4. *Let $R > 0$ and $\mathbf{U}^0$ in A_0^s be such that $\|\mathbf{U}^0\|_{A_0^s} \le R$. There exists a positive real number $\tau_1^* > 0$, which depends on $\mathbb{R}$ but not on ε, such that for all $\varepsilon > 0$, the Cauchy problem*

$$\begin{cases} L^\varepsilon(\partial)\mathbf{u}^\varepsilon + f(\mathbf{u}^\varepsilon) = 0, \\ \mathbf{u}^\varepsilon|_{T=0}(X,Y,Z) = \varepsilon^{\frac{1}{2}}\mathbf{U}^0(X,Y,Z,0,Z/\varepsilon) \end{cases}$$

has a unique solution $\mathbf{u}^\varepsilon$ in $\mathcal{C}([0,\frac{\tau_1^*}{\varepsilon}]\times\mathbb{R}^3)^n \cap \mathcal{C}([0,\frac{\tau_1^*}{\varepsilon}], L^2(\mathbb{R}^3)^n)$.

Moreover, $\mathbf{u}^\varepsilon$ can be written $\mathbf{u}^\varepsilon(T,X,Y,Z) := \varepsilon^{\frac{1}{2}}\mathbf{U}^\varepsilon(\varepsilon T, X,Y,Z,\frac{T}{\varepsilon},\frac{Z}{\varepsilon})$, where $\mathbf{U}^\varepsilon \in B^s_{\tau_1^*}$ is uniquely determined by the so-called singular equation*,*

$$\begin{cases} \partial_\tau \mathbf{U}^\varepsilon + \varepsilon^{-1}(A_1\partial_X + A_2\partial_Y + A_3\partial_Z)\mathbf{U}^\varepsilon + \varepsilon^{-2}(\partial_{t_0} + A_3\partial_{z_0} + L_0)\mathbf{U}^\varepsilon + f(\mathbf{U}^\varepsilon) = 0, \\ \mathbf{U}^\varepsilon_{|\tau=0} = \mathbf{U}^0, \end{cases} \tag{1.5}$$

and for all $\varepsilon \in (0,1)$, $\mathbf{U}^\varepsilon$ satisfies the uniform bound $\|\mathbf{U}^\varepsilon\|_{B^s_{\tau_1^*}} \le 2R$.

Proof. The proof of this theorem is similar to the proof of the existence theorem of [12], and we give only a sketch of it. First, we prove that the existence of $\mathbf{u}^\varepsilon$ is a consequence of the existence of a profile $\mathbf{U}^\varepsilon$ satisfying (1.5). This latter result is obtained by Picard iterates using the following lemma, which gives linear estimates.

LEMMA 1.5. *Let $\mathcal{V}^0 \in A_0^s$ and $\mathcal{W} \in B^s_{\tau_1^*}$. The linear problem*

$$\begin{cases} \partial_\tau \mathcal{V} + \varepsilon^{-1}(A_1\partial_X + A_2\partial_Y + A_3\partial_Z)\mathcal{V} + \varepsilon^{-2}(\partial_{t_0} + A_3\partial_{z_0} + L_0)\mathcal{V} = \mathcal{W}, \\ \mathcal{V}_{|\tau=0} = \mathcal{V}^0 \end{cases}$$

has a unique solution in $B^s_{\tau_1^*}$. Moreover, one has

$$||\mathcal{V}||_{B^s_{\tau_1^*}} = ||\mathcal{V}^0||_{A_0^s} + \tau_1^*||\mathcal{W}||_{B^s_{\tau_1^*}}.$$

The existence of $\mathbf{u}^\varepsilon$ being established, one proves uniqueness using a classical L^2-uniqueness argument. ∎

1.4. General method. We seek an approximate solution u^ε of the exact solution $\mathbf{u}^\varepsilon$ of (1.1) using the tools of diffractive optics. The approximate solution u^ε is sought in the form

$$u^\varepsilon = \varepsilon^{\frac{1}{2}}\mathcal{U}^\varepsilon\left(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon}\right), \qquad \text{with} \quad \mathcal{U}^\varepsilon = \mathcal{U}_0 + \varepsilon\mathcal{U}_1 + \varepsilon^2\mathcal{U}_2, \tag{1.6}$$

and $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2 \in E^s_{\tau^*}$.

The expansion of $L^\varepsilon(\partial)u^\varepsilon + f(u^\varepsilon)$ in powers of ε yields

$$L^\varepsilon(\partial)u^\varepsilon + f(u^\varepsilon) = \sum_{j=-1}^{7} \varepsilon^{\frac{1}{2}+j}\mathcal{R}_j(\tau,T,X,Y,Z,t_0,z_0)|_{\tau=\varepsilon T, t_0=T/\varepsilon, z_0=Z/\varepsilon}, \tag{1.7}$$

where

$$\begin{aligned}
\mathcal{R}_{-1}(\tau,T,X,Y,Z,t_0,z_0) &= i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_0,\\
\mathcal{R}_{0}(\tau,T,X,Y,Z,t_0,z_0) &= i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_1 + L_1(\partial)\mathcal{U}_0,\\
\mathcal{R}_{1}(\tau,T,X,Y,Z,t_0,z_0) &= i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_2 + L_1(\partial)\mathcal{U}_1 + \partial_\tau\mathcal{U}_0 + f(\mathcal{U}_0),\\
\mathcal{R}_{2}(\tau,T,X,Y,Z,t_0,z_0) &= L_1(\partial)\mathcal{U}_2 + \partial_\tau\mathcal{U}_1 + \langle f(\mathcal{U}^\varepsilon)\rangle_{1/2+2},\\
\mathcal{R}_{3}(\tau,T,X,Y,Z,t_0,z_0) &= \partial_\tau\mathcal{U}_2 + \langle f(\mathcal{U}^\varepsilon)\rangle_{1/2+3},\\
\mathcal{R}_{j\geq 4}(\tau,T,X,Y,Z,t_0,z_0) &= \langle f(\mathcal{U}^\varepsilon)\rangle_{1/2+j},
\end{aligned} \tag{1.8}$$

with the notation

$$\begin{aligned}
&\mathcal{L}(D_{t_0,z_0}) := D_{t_0} + A_3 D_{z_0} + L_0/i, \qquad D_{t_0} = -i\partial_{t_0}, \quad D_{z_0} = -i\partial_{z_0}\\
&L_1(\partial) := \partial_T + A_1\partial_X + A_2\partial_Y + A_3\partial_Z := \partial_T + A(\partial_{X,Y,Z}),
\end{aligned}$$

while $\langle f(\mathcal{U}^\varepsilon)\rangle_k$ denotes the coefficient of the monomial ε^k in the expansion into powers of ε of $f(\mathcal{U}^\varepsilon)$.

Notation. We used the pseudodifferential notation $D_{t_0} = -i\partial_{t_0}$ and $D_{z_0} = -i\partial_{z_0}$ to define the operator $\mathcal{L}(D_{t_0,z_0})$. This explains the factor i which appears in front of it in expansion (1.8). Recalling that (ω, k) denote the dual variables of (t_0, z_0), the symbol of this operator reads $\mathcal{L}(\omega,k) = \omega Id + kA_3 + L_0/i$.

The strategy of diffractive optics consists of seeking $\mathcal{U}_0$, $\mathcal{U}_1$, and $\mathcal{U}_2$ in order to cancel the profiles $\mathcal{R}_m(\tau,T,X,Y,Z,t_0,z_0)$, $m = -1, 0, 1$. We then prove that the associated function u^ε given by (1.6) is indeed an approximate solution of (1.1) and give a stability theorem.

1.5. A few tools. The following definition introduces some concepts of diffractive optics.

Definition 1.6. (i) *The characteristic variety associated to the operator $\mathcal{L}$ is the set*

$$\mathcal{C}_\mathcal{L} = \{(\omega,k) \in \mathbb{R}^2, \det(\mathcal{L}(\omega,k)) = \det(\omega Id + kA_3 + L_0/i) = 0\}.$$

(ii) *We denote by $\pi(\omega,k)$ the orthogonal projector onto* $\ker \mathcal{L}(\omega,k)$ *and by $\mathcal{L}^{-1}(\omega,k)$ the partial inverse of $\mathcal{L}(\omega,k)$ defined as*

$$\mathcal{L}^{-1}(\omega,k)\pi(\omega,k) = 0 \quad \text{and} \quad \mathcal{L}^{-1}(\omega,k)\mathcal{L}(\omega,k) = Id - \pi(\omega,k).$$

(iii) *Near every smooth point $(\underline{\omega}, \underline{k})$ of $\mathcal{C}_\mathcal{L}$, we denote by $\omega(k)$ a local parameterization of $\mathcal{C}_\mathcal{L}$.*

The following lemma expresses the resolubility condition of a linear equation with the tools introduced in the previous definition.

Lemma 1.7. *Let $a, b \in \mathbb{C}^n$. Then the following two assertions are equivalent:*
(i) $\mathcal{L}(\omega,k)a = b$*;*
(ii) $\pi(\omega,k)b = 0$ *and* $(Id - \pi(\omega,k))a = \mathcal{L}^{-1}(\omega,k)b$*.*

We want to generalize these resolubility conditions to equations of type $\mathcal{L}(D_{t_0,z_0})\mathcal{V} = \mathcal{W}$, where $\mathcal{V}$ and $\mathcal{W}$ are in $E^s_{\tau^*}$. Following [12], we first have to introduce the notion of $\mathcal{L}^{-1}$-regularity.

Definition 1.8. *Let $\mathcal{V} \in E^s_{\tau^*}$ and $\mu(\tau,T) := \mathcal{F}\mathcal{V}(\tau,T)$. We say that $\mathcal{V}$ is $\mathcal{L}^{-1}$-regular if for all $(\tau,T) \in [0,\tau^*]\times\mathbb{R}_T$, $\mathcal{L}^{-1}$ is $\mu(\tau,T)$-integrable.*

We can now generalize Lemma 1.7 in the following way.

Lemma 1.9. *Let $\mathcal{V}$ and $\mathcal{W}$ be in $E^s_{\tau^*}$. The following assertions are equivalent:*

(i) *$\mathcal{L}(D_{t_0,z_0})\mathcal{V} = \mathcal{W}$;*

(ii) *$\pi(D_{t_0,z_0})\mathcal{W} = 0$, $\mathcal{W}$ is $\mathcal{L}^{-1}$-regular, and $(Id - \pi(D_{t_0,z_0}))\mathcal{V} = \mathcal{L}^{-1}(D_{t_0,z_0})\mathcal{W}$.*

Remark 1.4. The $\mathcal{L}^{-1}$-regularity condition may not be satisfied in the physical phenomena we are interested in here. Indeed, the mapping $(\omega,k)\to\mathcal{L}^{-1}(\omega,k)$ is not bounded at the neighborhood of the origin. When low frequencies are excluded, as in most applications in optics, $\mathcal{L}^{-1}$ remains bounded for the frequencies considered, and the $\mathcal{L}^{-1}$-regularity condition is easily satisfied. But when low frequencies are allowed, as they have to be here, $\mathcal{L}^{-1}$ effectively blows up and the $\mathcal{L}^{-1}$-regularity condition is in general not satisfied. In that case, we have to use tools similar to those introduced by Alterman [1].

It is also interesting to decompose the profiles of $E^s_{\tau^*}$ into a discrete spectrum (sinusoidal) component and a purely continuous one. Such a decomposition is assured by the foregoing proposition.

Proposition 1.10. *Let $\mathcal{V} \in A^s_0$ (resp., $E^s_{\tau^*}$). The profile $\mathcal{V}$ is written uniquely as $\mathcal{V} = \mathcal{V}_I + \mathcal{V}_{II}$, with $\mathcal{V}_I$, $\mathcal{V}_{II} \in A^s_0$ (resp., $E^s_{\tau^*}$) such that*

(i) *$\mathcal{V}_I$ (resp., $\mathcal{V}_I(\tau,T,.)$ for all $(\tau,T)\in[0,\tau^*]\times\mathbb{R}_T$) has a discrete spectrum;*

(ii) *$\mathcal{V}_{II}$ has a purely continuous spectrum, i.e., every point of $\mathbb{R}^2$ has zero measure for $\mathcal{F}\mathcal{V}_{II}$ (resp., for $\mathcal{F}\mathcal{V}_{II}(\tau,T,.)$ for all $(\tau,T)\in[0,\tau^*]\times\mathbb{R}_T$).*

Notation. From now on, and for every profile $\mathcal{V}$ of A^s_0 or $E^s_{\tau^*}$, we denote by $\mathcal{V}_I$ the component with a discrete spectrum and by $\mathcal{V}_{II}$ the component with a purely continuous one.

Proof. First, the existence of the decomposition of a profile $\mathcal{V}\in E^s_{\tau^*}$ is proved. We introduce

$$\mu(\tau,T) := \mathcal{F}\mathcal{V}(\tau,T) \qquad \forall(\tau,T)\in[0,\tau^*]\times\mathbb{R}_T,$$

and $S_{\tau,T}$ the set of points with nonzero measure for $\mu(\tau,T)$,

$$S_{\tau,T} = \{p\in\mathbb{R}^2, \mu(\tau,T)(\{p\})\neq 0\}.$$

We decompose $\mu(\tau,T)$ in the form

$$\mu(\tau,T) = \mathbb{I}_{S_{\tau,T}}\mu(\tau,T) + (1-\mathbb{I}_{S_{\tau,T}})\mu(\tau,T),$$

and the proposition will be proved if we can show that $\mathcal{F}^{-1}(\mathbb{I}_{S_{\tau,T}}\mu(\tau,T))$ and $\mathcal{F}^{-1}[(1-\mathbb{I}_{S_{\tau,T}})\mu(\tau,T)]$ are in $E^s_{\tau^*}$. Indeed, one would then have for all (τ,T),

$$\mathcal{V}_I(\tau,T) = \mathcal{F}^{-1}(\mathbb{I}_{S_{\tau,T}}\mu(\tau,T)) \quad \text{and} \quad \mathcal{V}_{II}(\tau,T) = \mathcal{F}^{-1}([1-\mathbb{I}_{S_{\tau,T}}]\mu(\tau,T)).$$

It is clear that for all $(\tau,T)\in[0,\tau^*]\times\mathbb{R}_T$, the measures $\mathbb{I}_{S_{\tau,T}}\mu(\tau,T)$ and $(1-\mathbb{I}_{S_{\tau,T}})\mu(\tau,T)$ belong to $\mathcal{BV}(\mathbb{R}^2_\xi, H^s(\mathbb{R}^3)^n)$. The main difficulty is to prove the continuous dependence on (τ,T) of these two measures. As the mapping $(\tau,T)\mapsto\mu(\tau,T)$ is continuous by assumption, it is sufficient to prove that the mapping

$$\begin{aligned}[0,\tau^*]\times\mathbb{R}_T &\longrightarrow \mathcal{BV}(\mathbb{R}^2, H^s(\mathbb{R}^3)^n),\\ (\tau,T) &\longmapsto \mathbb{I}_{S_{\tau,T}}\mu(\tau,T)\end{aligned}$$

is continuous, i.e., that $\mathbb{I}_{S_{\tau,T}}\mu(\tau,T) - \mathbb{I}_{S_{\tau',T'}}\mu(\tau',T')$ tends to 0 in $\mathcal{BV}(\mathbb{R}^2_\xi, H^s(\mathbb{R}^3)^n)$ when (τ',T') tends to (τ,T). One has

$$\mathbb{I}_{S_{\tau,T}}\mu(\tau,T) - \mathbb{I}_{S_{\tau',T'}}\mu(\tau',T') = (\mathbb{I}_{S_{\tau,T}} - \mathbb{I}_{S_{\tau',T'}})\mu(\tau,T) + \mathbb{I}_{S_{\tau',T'}}(\mu(\tau,T) - \mu(\tau',T')).$$

The second term of the right-hand side of this equation tends to 0 when (τ',T') tends to (τ,T) thanks to the continuity of the mapping $(\tau,T) \mapsto \mu(\tau,T)$. Moreover, the first term of the right-hand side of the equation reads

$$(\mathbb{I}_{S_{\tau,T}} - \mathbb{I}_{S_{\tau',T'}})\mu(\tau,T) = \mathbb{I}_{S_{\tau,T}\setminus S_{\tau',T'}}\mu(\tau,T) - \mathbb{I}_{S_{\tau',T'}\setminus S_{\tau,T}}\mu(\tau,T).$$

But one has $\mathbb{I}_{S_{\tau',T'}\setminus S_{\tau,T}}\mu(\tau,T) = 0$, because if $p \in S_{\tau',T'} \setminus S_{\tau,T}$, then $\mu(\tau,T)(\{p\}) = 0$. On the other hand, one has $\mathbb{I}_{S_{\tau,T}\setminus S_{\tau',T'}}(\{p\}) \to 0$ for all $p \in \mathbb{R}^2$ when $(\tau',T') \to (\tau,T)$. Indeed, for all $p \in S_{\tau,T}$, $\mu(\tau,T)(\{p\}) \neq 0$. By continuity of the mapping $(\tau,T) \mapsto \mu(\tau,T)$, one has $\mu(\tau',T')(\{p\}) \longrightarrow \mu(\tau,T)(\{p\})$ in $H^s(\mathbb{R}^3)^n$ when $(\tau',T') \to (\tau,T)$. Consequently, $\mu(\tau',T')(\{p\}) \neq 0$ if (τ',T') is close enough to (τ,T). In other words, $p \in S_{\tau',T'}$, and hence $p \notin S_{\tau,T} \setminus S_{\tau',T'}$. The proof of the continuous time dependence is then achieved by a dominated convergence argument.

The existence of the decomposition for every profile of $E^s_{\tau^*}$, and a fortiori of A^s_0, is thus proved. The proof of the uniqueness of the decomposition is straightforward. ◻

1.6. Organization of the paper. We first address in section 2 general dispersive hyperbolic systems. Section 2.1 is devoted to the derivation of the profile equations under a simplifying assumption of absence of low frequencies. In section 2.2, we perform a sharp analysis of the nonlinearities found for the profile equations. We show that many of these nonlinearities vanish, which is of crucial importance for the resolubility theorems given in section 2.3. The fact that the approximate solution associated to the profiles found in the previous sections converges towards the exact solution of (1.1) is then proved in section 2.4. The general case (presence of low frequencies) is addressed in section 2.5. Using Alterman's technique of infrared cutoffs, we use the results of the previous sections to give profile equations in the general case, as well as a stability theorem which generalizes the one given in section 2.4.

In section 3, we treat the nondispersive case. Since the methods used in this section are the same as in the dispersive case, we do not extend the proofs. However, the main difference between both cases is by itself interesting enough to justify this section: *nonlinear interactions between components with a purely continuous spectrum can be observed in the nondispersive case*, while they are negligible in the dispersive case.

The two physical examples used as guidelines throughout this paper are studied in section 4. These examples are lasers with large spectrums and short pulses and illustrate the notable simplifications yielded by our general theory with respect to direct computations.

Finally, an intermediate case between dispersive and nondispersive systems, called weakly dispersive, is briefly commented on in section 5. In particular, we find that for large-spectrum lasers, equations for the continuous spectrum component are still linear but, as opposed to the dispersive case, coupled with the discrete spectrum component.

2. Dispersive case. In this part, we consider problems of type (1.1) which are dispersive. More precisely, we suppose that the following assumption is satisfied.

ASSUMPTION 2.1. *One has* $\{0, \pm(\omega_l, -k_l)\} \subset \mathcal{C}_{\mathcal{L}}$*, but for all* $j \in Z \setminus \{0, \pm 1\}$*, the point* $j(\omega_l, -k_l)$ *is not on* $\mathcal{C}_{\mathcal{L}}$*. We also assume that* $\mathcal{C}_{\mathcal{L}}$ *is a union of smooth curves which are never parallel, asymptotic, nor tangent to one another, and which intersect only on the vertical axis* $(O\omega)$*.*

Remark 2.1. The different nonlinear models whose linearization gives the Maxwell–Lorentz equations (see the last section for an example) do not exactly satisfy this assumption since $\mathcal{C}_{\mathcal{L}}$ then contains three horizontal lines, which are a fortiori parallel. Moreover, two of these lines are also tangent to curved sheets of the characteristic variety. However, this is not important since the horizontal lines are excluded by the divergence-free conditions one has to add to these systems. Therefore, the Maxwell–Lorentz model, and its nonlinear versions, fall into the range of this assumption.

2.1. The ansatz in the absence of infrared frequencies. As we have said in Remark 1.4, low frequencies make the analysis far more difficult because $\mathcal{L}^{-1}$-regularity of the profiles may fail. That is why in this section we focus on profiles $\mathcal{U}_0$ whose spectrum is outside the band $\{(\omega, k), |k| \leq \delta\}$. More precisely, we assume throughout this section that the continuous spectrum component of the leading term $\mathcal{U}_0$ satisfies the following assumption.

ASSUMPTION 2.2. *The spectrum* $\mathrm{Sp}\,\mathcal{U}_{0,II}$ *of the continuous spectrum component of* $\mathcal{U}_0$ *is in* $\{(\omega, k), |k| > \delta\}$*, where* $\delta > 0$*.*

The following lemma makes the link between absence of low frequencies and $\mathcal{L}^{-1}$-regularity.

LEMMA 2.1. *Let* $\mathcal{V}_{II}$ *be a profile of* $E^s_{\tau^*}$ *such that* $\mathrm{Sp}\ \mathcal{V}_{II} \subset \mathcal{C}_{\mathcal{L}}$*.*

If, moreover, $\mathrm{Sp}\ \mathcal{V}_{II} \subset \{(\omega, k), |k| > \delta\}$*, then* $\mathcal{V}_{II}$ *is* $\mathcal{L}^{-1}$*-regular, and for all* $T \in \mathbb{R}$*,*

$$\|\mathcal{L}^{-1}\mathcal{V}_{II}(T)\|_{E^s_{\tau^*}} \leq \frac{C}{\delta}\|\mathcal{V}_{II}(T)\|_{E^s_{\tau^*}}.$$

In particular, if $\mathcal{V}_{II} \in A^s_{\tau^*}$*, one has*

$$\|\mathcal{L}^{-1}\mathcal{V}_{II}\|_{A^s_{\tau^*}} \leq \frac{C}{\delta}\|\mathcal{V}_{II}\|_{A^s_{\tau^*}}.$$

Proof. For all $(\omega, k) \in \mathbb{R}^2$, $\mathcal{L}^{-1}(\omega, k)$ reads

$$\mathcal{L}^{-1}(\omega, k) = \sum_{j, \omega_j(k) \neq \omega} \frac{1}{\omega - \omega_j(k)}\pi(\omega_j(k), k),$$

where the ω_j are parameterizations of the different sheets of $\mathcal{C}_{\mathcal{L}}$. If $(\omega, k) \in \mathcal{C}_{\mathcal{L}}$, i.e., if there exists j_0 such that $(\omega, k) = (\omega_{j_0}(k), k)$, then

$$\mathcal{L}^{-1}(\omega, k) = \mathcal{L}^{-1}(\omega_{j_0}(k), k) = \sum_{j \neq j_0} \frac{1}{\omega_{j_0}(k) - \omega_j(k)}\pi(\omega_j(k), k). \tag{2.1}$$

Saying that $\mathcal{V}_{II}$ is $\mathcal{L}^{-1}$-regular means that for all $(\tau, T) \in [0, \tau^*] \times \mathbb{R}$, $\mathcal{L}^{-1}(\omega, k)$ is integrable for $\mu(\tau, T) := \mathcal{F}\mathcal{V}_{II}(\tau, T)$. As we have supposed that $\mathrm{Sp}\ \mathcal{V}_{II} \subset \mathcal{C}_{\mathcal{L}}$, we must prove that the expression given by (2.1) is integrable for $\mu(\tau, T)$.

Since we supposed in Assumption 2.1 that the different sheets of $\mathcal{C}_{\mathcal{L}}$ are not asymptotic, (2.1) is bounded for large $|k|$. The only points where this expression is not bounded are those where the different sheets intersect. Thanks to Assumption 2.1, these points are all on the axis $(O\omega)$. We now consider what happens in the neighborhood of such a point. Consider j and j_0 such that $\lim_{k\to 0}\omega_{j_0}(k) = \lim_{k\to 0}\omega_j(k) = \omega_0$. Near 0, one has

$$\frac{1}{\omega_{j_0}(k)-\omega_j(k)} = \frac{1}{(\omega_{j_0}(k)-\omega_0)-(\omega_j(k)-\omega_0)} \sim \frac{1}{k(v_0-v)},$$

where $v_0 = \lim_{k\to 0}\omega'_{j_0}(k)$ and $v = \lim_{k\to 0}\omega'_j(k)$. We know that $v_0 - v \neq 0$ since Assumption 2.1 assures us that two different sheets are never tangent.

Therefore, the expression given by (2.1) can be bounded for $|k| > \delta$ by C/δ, which yields both the $\mathcal{L}^{-1}$-regularity result for $\mathcal{V}_{II}$ and the estimate of the lemma. □

2.1.1. Annihilating $\mathcal{R}_{-1}$. Annihilating the $\varepsilon^{-\frac{1}{2}}$ term in expansion (1.7) is equivalent to $\mathcal{L}(D_{t_0,z_0})\mathcal{U}_0 = 0$. Thanks to Lemma 1.7, this equation is equivalent to the *polarization condition*

$$\pi(D_{t_0,z_0})\mathcal{U}_0 = \mathcal{U}_0.$$

Moreover, thanks to Proposition 1.10, $\mathcal{U}_0$ can be decomposed in the form $\mathcal{U}_0 = \mathcal{U}_{0,I} + \mathcal{U}_{0,II}$, where $\mathcal{U}_{0,I}$ has a discrete spectrum and $\mathcal{U}_{0,II}$ a purely continuous one.

Looking for $\mathcal{U}_{0,I}$ of the form $\mathcal{U}_{0,I,1}(\tau,T,X,Y,Z)e^{i\theta}$ + c.c., the polarization condition $\pi(D_{t_0,z_0})\mathcal{U}_0 = \mathcal{U}_0$ gives

$$\pi(\omega_l,-k_l)\mathcal{U}_{0,I,1} = \mathcal{U}_{0,I,1} \quad\text{and}\quad \pi(D_{t_0,z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}. \tag{2.2}$$

Remark 2.2. The notation *c.c.* used above denotes the complex conjugate of the preceding expression. As we are concerned with real-valued solutions of (1.1) and (1.5), we always assume that in the Fourier expansion of the discrete spectrum components, one has $\mathcal{U}_{j,I,k} = \overline{\mathcal{U}_{j,I,-k}}$.

2.1.2. Annihilating $\mathcal{R}_0$. Annihilating the $\varepsilon^{\frac{1}{2}}$ term in expansion (1.7) reads $i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_1 + L_1(\partial)\mathcal{U}_0 = 0$.

As for $\mathcal{U}_0$, decompose $\mathcal{U}_1$ in the form $\mathcal{U}_1 = \mathcal{U}_{1,I} + \mathcal{U}_{1,II}$, where $\mathcal{U}_{1,I}$ has a discrete spectrum and $\mathcal{U}_{1,II}$ a purely continuous one. We also look for $\mathcal{U}_{1,I}$ in the form $\mathcal{U}_{1,I,1}(\tau,T,X,Y,Z)e^{i\theta} + c.c.$, so that the equation $\mathcal{R}_0 = 0$ may read

$$\begin{cases} i\mathcal{L}(\omega_l,-k_l)\mathcal{U}_{1,I,1} + L_1(\partial)\mathcal{U}_{0,I,1} = 0, \\ i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_{1,II} + L_1(\partial)\mathcal{U}_{0,II} = 0. \end{cases} \tag{2.3}$$

With the polarization condition (2.2) and Lemma 1.7, the first equation of (2.3) is equivalent to

$$\begin{cases} \pi(\omega_l,-k_l)L_1(\partial)\pi(\omega_l,-k_l)\mathcal{U}_{0,I,1} = 0, \\ (Id-\pi(\omega_l,-k_l))\mathcal{U}_{1,I,1} = i\mathcal{L}^{-1}(\omega_l,-k_l)A(\partial_{X,Y,Z})\mathcal{U}_{0,I,1}. \end{cases} \tag{2.4}$$

Since Assumption 2.2 and Lemma 2.1 assure us that $A(\partial_{X,Y,Z})\mathcal{U}_{0,II}$ is $\mathcal{L}^{-1}$-regular, we know, thanks to Lemma 1.9, that the second equation of (2.3) is equivalent to

$$\begin{cases} \pi(D_{t_0,z_0})L_1(\partial)\pi(D_{t_0,z_0})\mathcal{U}_{0,II} = 0, \\ (Id-\pi(D_{t_0,z_0}))\mathcal{U}_{1,II} = i\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{U}_{0,II}. \end{cases} \tag{2.5}$$

Remark 2.3. At this stage, only the component $(Id-\pi(\omega_l,-k_l))\mathcal{U}_{1,I,1}$ is determined. We can therefore choose to take the other component equal to zero, i.e.,

$$\pi(\omega_l,-k_l)\mathcal{U}_{1,I,1} = 0. \tag{2.6}$$

We cannot do the same thing for $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ because this component will play an important role in the solvability of the profile equations.

2.1.3. Annihilating $\mathcal{R}_1$. Annihilating the $\varepsilon^{\frac{3}{2}}$ term in expansion (1.7) yields $i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_2+L_1(\partial)\mathcal{U}_1+\partial_\tau\mathcal{U}_0+f(\mathcal{U}_0)=0$. Thanks to Proposition 1.10, this equation can be decomposed into a discrete spectrum component,

$$-i\mathcal{L}(\omega_l,-k_l)\mathcal{U}_{2,I}=L_1(\partial)\mathcal{U}_{1,I}+\partial_\tau\mathcal{U}_{0,I}+f(\mathcal{U}_0)_I, \tag{2.7}$$

and a purely continuous one,

$$-i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_{2,II}=L_1(\partial)\mathcal{U}_{1,II}+\partial_\tau\mathcal{U}_{0,II}+f(\mathcal{U}_0)_{II}. \tag{2.8}$$

The nonlinearity $f(\mathcal{U}_0)_I$ in (2.7) is given by $f(\mathcal{U}_0)_I=f(\mathcal{U}_{0,I})$, which is a trigonometric polynomial,

$$f(\mathcal{U}_{0,I})=f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})e^{i\theta}+f(\mathcal{U}_{0,I,1})e^{i3\theta}+\text{c.c.} \tag{2.9}$$

Since the third harmonic is created by the nonlinearity, we must seek $\mathcal{U}_{2,I}$ in the form

$$\mathcal{U}_{2,I}(\tau,T,X,Y,Z,\theta)=\mathcal{U}_{2,I,1}(\tau,T,X,Y,Z)e^{i\theta}+\mathcal{U}_{2,I,3}(\tau,T,X,Y,Z)e^{i3\theta}+\text{c.c.}$$

According to Assumption 2.1, the component $\mathcal{U}_{2,I,3}$ can be found by elliptic inversion since $\mathcal{L}(3\omega_l,-3k_l)$ is then nonsingular,

$$\mathcal{U}_{2,I,3}=\mathcal{L}(3\omega_l,-3k_l)^{-1}f(\mathcal{U}_{0,I,1}). \tag{2.10}$$

The remaining component $\mathcal{U}_{2,I,1}$ satisfies

$$i\mathcal{L}(\omega_l,-k_l)\mathcal{U}_{2,I,1}+L_1(\partial)\mathcal{U}_{1,I,1}+\partial_\tau\mathcal{U}_{0,I,1}+f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})=0,$$

and thanks to Lemma 1.7, we get

$$\begin{cases}\pi(\omega_l,-k_l)\big(L_1(\partial)\mathcal{U}_{1,I,1}+\partial_\tau\mathcal{U}_{0,I,1}+f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})\big)=0,\\ (Id-\pi(\omega_l,-k_l))\mathcal{U}_{2,I,1}=i\mathcal{L}^{-1}(\omega_l,-k_l)\big(L_1(\partial)\mathcal{U}_{1,I,1}+\partial_\tau\mathcal{U}_{0,I,1}+f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})\big).\end{cases}$$

Using the polarization condition (2.2) and equations (2.4) and (2.6), the first equation of the previous system reads

$$\begin{aligned}&\partial_\tau\mathcal{U}_{0,I,1}+i\pi(\omega_l,-k_l)A(\partial_{X,Y,Z})\mathcal{L}^{-1}(\omega_l,-k_l)A(\partial_{X,Y,Z})\pi(\omega_l,-k_l)\mathcal{U}_{0,I,1}\\&\qquad+\pi(\omega_l,-k_l)f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})=0,\end{aligned} \tag{2.11}$$

while the second equation gives $(Id-\pi(\omega_l,-k_l))\mathcal{U}_{2,I,1}$ in terms of $\mathcal{U}_{0,I,1}$,

$$\begin{aligned}&(I-\pi(\omega_l,-k_l))\mathcal{U}_{2,I,1}\\&=i\mathcal{L}^{-1}(\omega_l,-k_l)\big(iL_1(\partial)\mathcal{L}^{-1}(\omega_l,-k_l)A(\partial_{X,Y,Z})\mathcal{U}_{0,I,1}+f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})\big).\end{aligned} \tag{2.12}$$

While the analysis of the discrete and the continuous spectrum components was formally the same in section 2.1.2, this is no longer true here. This is due to the fact that Assumption 2.2 cannot ensure that the right-hand side of (2.8) is $\mathcal{L}^{-1}$-regular. Therefore, Lemma 1.9 cannot be invoked to solve this equation.

Because of this difficulty, we cannot in general find $\mathcal{U}_2$ in $E^s_{\tau^*}$ such that $\mathcal{R}_1=0$. However, $\mathcal{U}_2\in E^s_{\tau^*}$ may be chosen in such a way that $\mathcal{R}_1$ is very small. We first introduce Alterman's infrared cutoff filter.

DEFINITION 2.2. *Let ψ be defined as follows: $\psi(k)=1$ for $|k|>1$ and $\psi(k)=0$ otherwise. For all $k\in\mathbb{R}$, $\psi^\delta(k)$ is defined as $\psi^\delta(k)=\psi(k/\delta)$, where $\delta>0$ is the same as in Assumption* 2.2.

Remark 2.4. The function ψ introduced above is not smooth, while Alterman's filter is smooth. Since our framework allows it, we have made this choice in order to lighten a few equations. In particular, we have the equivalence Sp $\mathcal{V} \subset \{(\omega,k), |k| > \delta\} \iff \psi^\delta(D_{z_0})\mathcal{V} = \mathcal{V}$.

Since no condition has been found at this stage on $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$, we can impose a condition of absence of low frequencies on this component and, more precisely, that

$$\psi^\delta(D_{z_0})\pi(D_{t_0,z_0})\mathcal{U}_{1,II} = \pi(D_{t_0,z_0})\mathcal{U}_{1,II}. \tag{2.13}$$

Instead of solving (2.8), we solve the approximate equation $(2.8)_\delta$ defined as

$$-i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_{2,II} = L_1(\partial)\mathcal{U}_{1,II} + \partial_\tau \mathcal{U}_{0,II} + \psi^\delta(D_{z_0})f(\mathcal{U}_0)_{II}.$$

Using (2.5), this equation reads

$$\begin{aligned} -i\mathcal{L}(D_{t_0,z_0})\mathcal{U}_{2,II} = {}& iL_1(\partial)\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{U}_{0,II} + L_1(\partial)\pi(D_{t_0,z_0})\mathcal{U}_{1,II} \\ & +\partial_\tau \mathcal{U}_{0,II} + \psi^\delta(D_{z_0})f(\mathcal{U}_0)_{II}. \end{aligned} \tag{2.14}$$

Thanks to the presence of the filter ψ^δ, to (2.13), and to Assumption 2.2, the right-hand side of this equation is $\mathcal{L}^{-1}$-regular, so that (2.14) is equivalent to

$$\begin{aligned} \partial_\tau \mathcal{U}_{0,II} &+ i\pi(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\pi(D_{t_0,z_0})\mathcal{U}_{0,II} \\ &+ \pi(D_{t_0,z_0})L_1(\partial)\pi(D_{t_0,z_0})\mathcal{U}_{1,II} \\ &+ \pi(D_{t_0,z_0})\psi^\delta(D_{z_0})[f(\mathcal{U}_0)]_{II} = 0 \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} (Id - \pi(D_{t_0,z_0}))\mathcal{U}_{2,II} = {}& -\mathcal{L}^{-1}(D_{t_0,z_0})L_1(\partial)\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{U}_{0,II} \\ & +i\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\pi(D_{t_0,z_0})\mathcal{U}_{1,II} \\ & +i\mathcal{L}^{-1}(D_{t_0,z_0})\psi^\delta(D_{z_0})f(\mathcal{U}_0)_{II}. \end{aligned} \tag{2.16}$$

Remark 2.5. (i) As said above, $\mathcal{R}_1 = 0$ is not solved exactly. If we can find $\mathcal{U}_0, \mathcal{U}_1$, and $\mathcal{U}_2$ solutions of (2.2), (2.4)–(2.6), (2.10), (2.11)–(2.12), (2.13), and (2.15)–(2.16), we then have $\mathcal{R}_{-1} = \mathcal{R}_0 = 0$ but only

$$\mathcal{R}_1 = (1 - \psi^\delta(D_{z_0}))f(\mathcal{U}_0)_{II}. \tag{2.17}$$

We will see later that this term tends towards zero as $\delta \to 0$.

(ii) Only the components $(Id - \pi(\omega_l, -k_l))\mathcal{U}_{2,I,1}$ and $(Id - \pi(D_{t_0,z_0}))\mathcal{U}_{2,II}$ of $\mathcal{U}_{2,I,1}$ and $\mathcal{U}_{2,II}$ are determined. We can therefore choose to take

$$\pi(\omega_l, -k_l)\mathcal{U}_{2,I,1} = 0 \quad \text{and} \quad \pi(D_{t_0,z_0})\mathcal{U}_{2,II} = 0. \tag{2.18}$$

2.1.4. Simplification of the profile equations. According to the previous results, one has

$$\mathcal{U}_0(\tau,T,X,Y,Z,t_0,z_0) = \mathcal{U}_{0,I,1}(\tau,T,X,Y,Z)e^{i\theta} + c.c. + \mathcal{U}_{0,II}(\tau,T,X,Y,Z,t_0,z_0),$$

with

$$\pi(\omega_l, -k_l)\mathcal{U}_{0,I,1} = \mathcal{U}_{0,I,1} \quad \text{and} \quad \pi(D_{t_0,z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}. \tag{2.19}$$

Moreover $\mathcal{U}_{0,I,1}$ must satisfy

$$(2.20)\quad \begin{cases} \pi(\omega_l,-k_l)L_1(\partial)\pi(\omega_l,-k_l)\mathcal{U}_{0,I,1}=0,\\ \partial_\tau\mathcal{U}_{0,I,1}+i\pi(\omega_l,-k_l)A(\partial_{X,Y,Z})\mathcal{L}^{-1}(\omega_l,-k_l)A(\partial_{X,Y,Z})\pi(\omega_l,-k_l)\mathcal{U}_{0,I,1}\\ \qquad +\pi(\omega_l,-k_l)f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})=0,\end{cases}$$

and the purely continuous spectrum component $\mathcal{U}_{0,II}$ must satisfy

$$(2.21)\quad \begin{cases} \pi(D_{t_0,z_0})L_1(\partial)\pi(D_{t_0,z_0})\mathcal{U}_{0,II}=0,\\ \partial_\tau\mathcal{U}_{0,II}+i\pi(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\pi(D_{t_0,z_0})\mathcal{U}_{0,II}\\ \qquad +\pi(D_{t_0,z_0})L_1(\partial)\pi(D_{t_0,z_0})\mathcal{U}_{1,II}+\pi(D_{t_0,z_0})\psi^\delta(D_{z_0})[f(\mathcal{U}_0)]_{II}=0,\end{cases}$$

where we recall that $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ must satisfy the condition of absence of low frequencies (2.13).

Before these systems are simplified, a new symbol, $\mathcal{M}$, is introduced.

DEFINITION 2.3. *We denote by $\mathcal{M}(\omega,K)$ the symbol defined for all $(\omega,K)\in\mathbb{R}^{1+3}$ as*

$$\mathcal{M}(\omega,K)=wId+K_1A_1+K_2A_2+K_3A_3+L_0/i,$$

where $K=(K_1,K_2,K_3)$.

We also define the orthogonal projector $\pi_{\mathcal{M}}(\omega,K)$ onto $\ker\mathcal{M}(\omega,K)$*, the partial inverse $\mathcal{M}^{-1}(\omega,K)$ of $\mathcal{M}(\omega,k)$, and denote by $\mathcal{C}_{\mathcal{M}}$ the characteristic variety associated to $\mathcal{M}$, i.e., the set of points (ω,K) where $\mathcal{M}(\omega,K)$ is singular.*

We also make the following assumption, satisfied, for instance, by Maxwell systems in isotropic media.

ASSUMPTION 2.3. *$\mathcal{C}_{\mathcal{M}}$ is axisymmetric around $(O\omega)$.*

We can now state the following proposition.

PROPOSITION 2.4. *Suppose that Assumption* 2.3 *is satisfied and that $\{(\omega_l,-k_l)\}$ is a smooth point of $\mathcal{C}_{\mathcal{L}}$. One then has the following:*

(i) $\pi(\omega_l,-k_l)L_1(\partial)\pi(\omega_l,-k_l)=\pi(\omega_l,-k_l)(\partial_T+\omega'(k_l)\partial_Z)$*;*

(ii)
$$\pi(\omega_l,-k_l)A(\partial_{X,Y,Z})\mathcal{L}^{-1}(\omega_l,-k_l)A(\partial_{X,Y,Z})\pi(\omega_l,-k_l) = \tfrac{\omega'(k_l)}{2k_l}\pi(\omega_l,-k_l)(\partial_X^2+\partial_Y^2)+\tfrac{\omega''(k_l)}{2}\pi(\omega_l,-k_l)\partial_Z^2.$$

(iii) *If $\mathcal{V}_{II}$ is a profile with a purely continuous spectrum, then one also has*

$$\pi(D_{t_0,z_0})L_1(\partial)\pi(D_{t_0,z_0})\mathcal{V}_{II}=\pi(D_{t_0,z_0})\big(\partial_T-\omega'(D_{z_0})\partial_Z\big)\mathcal{V}_{II}.$$

(iv) *If, moreover, $A(\partial_{X,Y,Z})\pi(D_{t_0,z_0})\mathcal{V}_{II}$ is $\mathcal{L}^{-1}$-regular, then*

$$\pi(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\pi(D_{t_0,z_0})\mathcal{V}_{II} = \frac{\omega'(D_{z_0})}{2D_{z_0}}\pi(D_{t_0,z_0})(\partial_X^2+\partial_Y^2)\mathcal{V}_{II}+\frac{\omega''(D_{z_0})}{2}\partial_Z^2\mathcal{V}_{II}.$$

Proof. If $(\underline{\omega},\underline{K})$ is a smooth point of $\mathcal{C}_{\mathcal{M}}$, then we can define a local parameterization $\omega_{\mathcal{M}}(K)$ of $\mathcal{C}_{\mathcal{M}}$ near $(\underline{\omega},\underline{K})$. We know [8] that

$$\pi_{\mathcal{M}}(\underline{\omega},\underline{K})A_j\pi_{\mathcal{M}}(\underline{\omega},\underline{K})=-\partial_j\omega_{\mathcal{M}}(\underline{K})\pi_{\mathcal{M}}(\underline{\omega},\underline{K}),\qquad j=1,2,3.$$

As $(\omega_l,-k_l)\in\mathcal{C}_{\mathcal{L}}$, it is easy to see that $(\omega_l,(0,0,-k_l))\in\mathcal{C}_{\mathcal{M}}$. Moreover, since $(\omega_l,-k_l)$ is a smooth point of $\mathcal{C}_{\mathcal{L}}$, $(\omega_l,(0,0,-k_l))$ is a smooth point of $\mathcal{C}_{\mathcal{M}}$ thanks to

Assumption 2.3. Thanks to the same assumption, a local parameterization may be used to write $\omega_{\mathcal{M}}(K) = \omega(|K|)$, where $\omega(\cdot)$ is a local parameterization of $\mathcal{C}_{\mathcal{L}}$ near $(\omega_l, -k_l)$.

Taking $\underline{K} = (0, 0, -k_l)$, one therefore has

$$\pi_{\mathcal{M}}(\omega_l, (0,0,-k_l))A_j\pi_{\mathcal{M}}(\omega_l, (0,0,-k_l)) = -\frac{\underline{K}_j}{k_l}\omega'(k_l).$$

Since $\underline{K}_1 = \underline{K}_2 = 0$, $\underline{K}_3 = -k_l$, and $\pi_M(\omega_l, \underline{K}) = \pi(\omega_l, -k_l)$, we obtain

$$\begin{aligned}\pi(\omega_l, -k_l)A_1\pi(\omega_l, -k_l) &= 0,\\ \pi(\omega_l, -k_l)A_2\pi(\omega_l, -k_l) &= 0,\\ \pi(\omega_l, -k_l)A_3\pi(\omega_l, -k_l) &= \omega'(k_l),\end{aligned}$$

which proves point (i).

The same proof shows that $\pi(\omega, k)A_1\pi(\omega, k) = \pi(\omega, k)A_2\pi(\omega, k) = 0$ and similarly $\pi(\omega, k)A_3\pi(\omega, k) = -\omega'(k)$ for every smooth point (ω, k) of $\mathcal{C}_{\mathcal{L}}$. Since we know by Assumption 2.1 that the set of the singular points of $\mathcal{C}_{\mathcal{L}}$ is discrete and hence has zero measure for $\mathcal{F}\mathcal{V}_{II}$ (because the spectrum of $\mathcal{V}_{II}$ is purely continuous), we can conclude that

$$\pi(D_{t_0,z_0})A_1\pi(D_{t_0,z_0})\mathcal{V}_{II} = \pi(D_{t_0,z_0})A_2\pi(D_{t_0,z_0})\mathcal{V}_{II} = 0$$

and that $\pi(D_{t_0,z_0})A_3\pi(D_{t_0,z_0})\mathcal{V}_{II} = -\omega'(D_{z_0})\mathcal{V}_{II}$, which proves point (iii).

For (ii), we still write $(0, 0, -k_l) = \underline{K}$. Thanks to [8], we know that

$$\begin{aligned}&\pi_{\mathcal{M}}(\omega_l, \underline{K})A_i\mathcal{M}^{-1}(\omega_l, \underline{K})A_j\pi_{\mathcal{M}}(\omega_l, \underline{K}) = \frac{1}{2}\pi_{\mathcal{M}}(\omega_l, \underline{K})\partial^2_{ij}\omega_{\mathcal{M}}(\underline{K})\\ &= \frac{1}{2}\pi_{\mathcal{M}}(\omega_l, \underline{K})\left(-\frac{\underline{K}_i\underline{K}_j}{|\underline{K}|^3}\omega'(|\underline{K}|) + \frac{\underline{K}_i\underline{K}_j}{|\underline{K}|^2}\omega''(|\underline{K}|) + \frac{\delta_{ij}}{|\underline{K}|}\omega'(|\underline{K}|)\right),\end{aligned}$$

where δ_{ij} denotes Kronecker's symbol, $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

Since $\underline{K}_1 = \underline{K}_2 = 0$, one has

$$\begin{aligned}\pi_{\mathcal{M}}(\omega_l, \underline{K})A_1\mathcal{M}^{-1}(\omega_l, \underline{K})A_2\pi_{\mathcal{M}}(\omega_l, \underline{K}) &= 0,\\ \pi_{\mathcal{M}}(\omega_l, \underline{K})A_2\mathcal{M}^{-1}(\omega_l, \underline{K})A_1\pi_{\mathcal{M}}(\omega_l, \underline{K}) &= 0,\end{aligned}$$

and since $\underline{K}_3 = -k_l$,

$$\begin{aligned}\pi_{\mathcal{M}}A_1\mathcal{M}^{-1}(\omega_l, \underline{K})A_1\pi_{\mathcal{M}} &= \pi_{\mathcal{M}}A_2\mathcal{M}^{-1}(\omega_l, \underline{K})A_2\pi_{\mathcal{M}}\\ &= \frac{\omega'(k_l)}{2k_l}\pi_{\mathcal{M}}(\omega_l, \underline{K})\end{aligned}$$

and $\pi_{\mathcal{M}}A_3\mathcal{M}^{-1}(\omega_l, \underline{K})A_3\pi_{\mathcal{M}} = \frac{\omega''(k_l)}{2}\pi_{\mathcal{M}}$.

Since $\pi_{\mathcal{M}}(\omega_l, \underline{K}) = \pi(\omega_l, -k_l)$ and $\mathcal{M}^{-1}(\omega_l, \underline{K}) = \mathcal{L}^{-1}(\omega_l, -k_l)$ we obtain (ii).

The same reasoning as in (iii) yields (iv). □

Therefore, according to Proposition 2.4 and systems (2.20), (2.21), the leading term $\mathcal{U}_0(\tau, T, X, Y, Z, t_0, z_0) = \mathcal{U}_{0,I,1}(\tau, T, X, Y, Z)e^{i\theta} + \text{c.c.} + \mathcal{U}_{0,II}(\tau, T, X, Y, Z, t_0, z_0)$ is found by solving (if possible)

$$\left\{\begin{aligned}&\pi(\omega_l, -k_l)\mathcal{U}_{0,I,1} = \mathcal{U}_{0,I,1},\\ &(\partial_T + \omega'(k_l)\partial_Z)\mathcal{U}_{0,I,1} = 0,\\ &\partial_\tau\mathcal{U}_{0,I,1} + i\frac{\omega'(k_l)}{2k_l}(\partial^2_X + \partial^2_Y)\mathcal{U}_{0,I,1} + i\frac{\omega''(k_l)}{2}\partial^2_Z\mathcal{U}_{0,I,1}\\ &\qquad + \pi(\omega_l, -k_l)f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}}) = 0\end{aligned}\right. \tag{2.22}$$

and

$$(2.23)\quad \begin{cases} \pi(D_{t_0,z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}, \\ (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau \mathcal{U}_{0,II} + (\partial_T - \omega'(D_{z_0})\partial_Z)\pi(D_{z_0})\mathcal{U}_{1,II} + i\dfrac{\omega'(D_{z_0})}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} \\ \qquad + i\dfrac{\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{U}_{0,II} + \psi^\delta(D_{z_0})\pi(D_{t_0,z_0})[f(\mathcal{U}_0)]_{II} = 0. \end{cases}$$

Remark 2.6. (i) The notation $\frac{\omega'(D_{z_0})}{2D_{z_0}}$ is ambiguous since one could think that this Fourier multiplier does not depend on D_{t_0}. In fact, for any $k \in \mathbb{R}$, there are several ω_j such that $(\omega_j, k) \in \mathcal{C}_{\mathcal{L}}$. In Fourier variables, $\omega'(D_{z_0})$ therefore reads $\mathcal{F}\omega'(D_{z_0})(\omega, k) = \omega_j'(k)$, where the subscript j is such that $(\omega_j, k) = (\omega, k)$ and thus depends on w.

(ii) In general $\frac{\omega'(D_{z_0})}{2D_{z_0}}$ is not a Fourier multiplier of $E^s_{\tau^*}$. However, it is well defined here since it is applied to $\mathcal{U}_{0,II}$, which satisfies Assumption 2.2.

2.2. Analysis of the nonlinearity. We have to work more on these profile equations before trying to solve them. In particular, it is essential to simplify the nonlinearity $[f(\mathcal{U}_0)]_{II}$ which appears in (2.23). In this section, we show a striking result. *The nonlinearity* $[f(\mathcal{U}_0)]_{II}$ *is in fact linear.*

We recall that there exists a trilinear mapping F such that $F(u,u,u) = f(u)$ for all $u \in C^n$. The nonlinearity $f(\mathcal{U}_0)_{II}$ which appears in the evolution equation of $\mathcal{U}_{0,II}$ therefore reads

$$\begin{aligned} &F(\mathcal{U}_{0,I} + \mathcal{U}_{0,II}, \mathcal{U}_{0,I} + \mathcal{U}_{0,II}, \mathcal{U}_{0,I} + \mathcal{U}_{0,II})]_{II} = F(\mathcal{U}_{0,II}, \mathcal{U}_{0,II}, \mathcal{U}_{0,II}) \\ &+ F(\mathcal{U}_{0,I}, \mathcal{U}_{0,II}, \mathcal{U}_{0,II}) + F(\mathcal{U}_{0,II}, \mathcal{U}_{0,I}, \mathcal{U}_{0,II}) + F(\mathcal{U}_{0,II}, \mathcal{U}_{0,II}, \mathcal{U}_{0,I}) \\ &+ \big(F(\mathcal{U}_{0,I,1}, \mathcal{U}_{0,I,1}, \mathcal{U}_{0,II}) + F(\mathcal{U}_{0,II}, \mathcal{U}_{0,I,1}, \mathcal{U}_{0,I,1}) + F(\mathcal{U}_{0,I,1}, \mathcal{U}_{0,II}, \mathcal{U}_{0,I,1})\big)e^{2i\theta} \\ &+ \big(F(\overline{\mathcal{U}_{0,I,1}}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}) + F(\mathcal{U}_{0,II}, \overline{\mathcal{U}_{0,I,1}}, \overline{\mathcal{U}_{0,I,1}}) + F(\overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}, \overline{\mathcal{U}_{0,I,1}})\big)e^{-2i\theta} \\ &+ F(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}) + F(\overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,I,1}, \mathcal{U}_{0,II}) + F(\mathcal{U}_{0,I,1}, \mathcal{U}_{0,II}, \overline{\mathcal{U}_{0,I,1}}) \\ &+ F(\overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}, \mathcal{U}_{0,I,1}) + F(\mathcal{U}_{0,II}, \mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}) + F(\mathcal{U}_{0,II}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,I,1}). \end{aligned}$$

This expression can be simplified a lot because the role of the components with a purely continuous spectrum in the nonlinearity is not often efficient. This is the object of the following lemma.

LEMMA 2.5. *Suppose Assumption* 2.1 *is satisfied and let* $\mathcal{V}_{II} \in E^s_{\tau^*}$ *be a profile with purely continuous spectrum and such that* Sp $\mathcal{V}_{II} \subset \mathcal{C}_{\mathcal{L}}$. *Take also* $a, b \in \mathbb{C}^n$. *Then one has*

(i) $\pi(D_{t_0,z_0})F(\mathcal{V}_{II}, \mathcal{V}_{II}, \mathcal{V}_{II}) = 0$;

(ii) $\pi(D_{t_0,z_0})F(ae^{i\theta}, \mathcal{V}_{II}, \mathcal{V}_{II}) = \pi(D_{t_0,z_0})F(ae^{-i\theta}, \mathcal{V}_{II}, \mathcal{V}_{II}) = 0$;

(iii) $\pi(D_{t_0,z_0})F(ae^{i\theta}, be^{i\theta}, \mathcal{V}_{II}) = \pi(D_{t_0,z_0})F(ae^{-i\theta}, be^{-i\theta}, \mathcal{V}_{II}) = 0$.

Proof. Let $\mathcal{V}_{II} \in E^s_{\tau^*}$ be as in the lemma. For (τ, T) fixed, we introduce $\mu := \mathcal{F}\mathcal{V}_{II}(\tau, T)$ and denote by $v(\mu)$ the total variation of μ. Thanks to the Radon–Nikodým property, we can write, for all Borel sets E of $\mathbb{R}^2$,

$$\mu(E) = \int_E r_\mu(\xi) v(\mu)(d\xi),$$

where r_μ is an $H^s(\mathbb{R}^3)^n$-valued integrable function such that $||r_\mu(\xi)||_{H^s} = 1$ for $v(\mu)$ for almost all ξ.

Introducing $\nu := \mathcal{F}(\pi(D_{t_0,z_0})F(\mathcal{V}_{II},\mathcal{V}_{II},\mathcal{V}_{II}))$, the first point of the lemma will be proved if we can show that $\nu = 0$, i.e., that $v(\nu)(\mathbb{R}^2) = 0$. One has

$$v(\nu)(\mathbb{R}^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} ||\pi(\xi_1+\xi_2+\xi_3)F(r_\mu(\xi_1),r_\mu(\xi_2),r_\mu(\xi_3))||_{H^s(\mathbb{R}^3)^n} \times v(\mu)(d\xi_1)v(\mu)(d\xi_2)v(\mu)(d\xi_3).$$

Since Sp $\mathcal{V}_{II} \subset \mathcal{C}_{\mathcal{L}}$, one can take $r_\mu(\xi) = 0$ if $\xi \notin \mathcal{C}_{\mathcal{L}}$. Hence,

$$\begin{aligned} v(\nu)(\mathbb{R}^2) &= \int_{\mathcal{C}_{\mathcal{L}}}\int_{\mathcal{C}_{\mathcal{L}}}\int_{\mathcal{C}_{\mathcal{L}}} ||\pi(\xi_1+\xi_2+\xi_3)F(r_\mu(\xi_1),r_\mu(\xi_2),r_\mu(\xi_3))||_{H^s(\mathbb{R}^3)^n} \\ &\qquad \times v(\mu)(d\xi_1)v(\mu)(d\xi_2)v(\mu)(d\xi_3) \\ &= \int_{\mathcal{C}_{\mathcal{L}}}\int_{\mathcal{C}_{\mathcal{L}}}\left[\int_{\mathcal{C}_{\mathcal{L}}} ||\pi(\xi_1+\xi_2+\xi_3)F(r_\mu(\xi_1),r_\mu(\xi_2),r_\mu(\xi_3))||_{H^s(\mathbb{R}^3)^n} v(\mu)(d\xi_3)\right] \\ &\qquad \times v(\mu)(d\xi_1)v(\mu)(d\xi_2). \end{aligned}$$

Notice that if $\xi_1+\xi_2 \neq 0$, then the set of $\xi \in \mathcal{C}_{\mathcal{L}}$ such that $\xi_1+\xi_2+\xi \in \mathcal{C}_{\mathcal{L}}$ is discrete. Indeed following [6] and [5], the set of $\xi \in \mathcal{C}_{\mathcal{L}}$ such that $\xi_1+\xi_2+\xi \in \mathcal{C}_{\mathcal{L}}$ is either a sheet of $\mathcal{C}_{\mathcal{L}}$ or an algebraic submanifold of $\mathcal{C}_{\mathcal{L}}$ of strictly lower dimension. Since $\mathcal{C}_{\mathcal{L}}$ is an algebraic manifold of dimension one, the set of $\xi \in \mathcal{C}_{\mathcal{L}}$ such that $\xi_1+\xi_2+\xi \in \mathcal{C}_{\mathcal{L}}$ is then either a sheet of $\mathcal{C}_{\mathcal{L}}$ or a discrete set. The first case is not possible thanks to Assumption 2.1 because we would have two parallel sheets of $\mathcal{C}_{\mathcal{L}}$. Therefore, the set of points $\xi \in \mathcal{C}_{\mathcal{L}}$ such that $\xi_1+\xi_2+\xi \in \mathcal{C}_{\mathcal{L}}$ is discrete if $\xi_1+\xi_2 \neq 0$. Since $v(\mu)(\{\xi\}) = 0$ for all $\xi \in \mathbb{R}^2$, we can conclude that

$$\int_{\mathcal{C}_{\mathcal{L}}} ||\pi(\xi_1+\xi_2+\xi_3)F(r_\mu(\xi_1),r_\mu(\xi_2),r_\mu(\xi_3))||_{H^s(\mathbb{R}^3)^n} v(\mu)(d\xi_3) = 0,$$

when $\xi_1+\xi_2 \neq 0$.

Therefore, one has

$$v(\nu)(\mathbb{R}^2) = \int_{\mathcal{C}_{\mathcal{L}}}\int_{\mathcal{C}_{\mathcal{L}}} ||\pi(\xi_3)F(r_\mu(\xi_1),r_\mu(-\xi_1),r_\mu(\xi_3))||_{H^s(\mathbb{R}^3)^n} v(\mu)(\{-\xi_1\}) \times v(\mu)(d\xi_1)v(\mu)(d\xi_3).$$

Since $v(\mu)(\{\xi\}) = 0$ for all $\xi \in \mathbb{R}^2$, the above quantity is equal to 0, i.e., $v(\nu)(\mathbb{R}^2) = 0$, which proves point (i).

For (ii), introduce $\lambda := \mathcal{F}(\pi(D_{t_0,z_0})F(ae^{i\theta},\mathcal{V}_{II},\mathcal{V}_{II}))$. One has

$$v(\lambda)(\mathbb{R}^2) = \int_{\mathcal{C}_{\mathcal{L}}}\int_{\mathcal{C}_{\mathcal{L}}} ||\pi((\omega_l,-k_l)+\xi_1+\xi_2)F(a,r_\mu(\xi_1),r_\mu(\xi_2))||_{H^s(\mathbb{R}^3)^n} \times v(\mu)(d\xi_1)v(\mu)(d\xi_2).$$

With the same reasoning as in (i), one can prove that this expression vanishes, thus yielding point (ii).

Point (iii) is a direct consequence of Assumption 2.1 and, more precisely, of the fact that two different sheets of $\mathcal{C}_{\mathcal{L}}$ are never parallel. □

Remark 2.7. Lemma 2.5 can easily be generalized to N-linear nonlinear functions F. When $\mathcal{V}_{II}$ appears twice or more in the arguments of F, the term vanishes. When it appears once only, the spectrum of this nonlinear term is a translation of the spectrum of $\mathcal{V}_{II}$, and hence this nonlinear term vanishes unless the intersection of this spectrum with $\mathcal{C}_{\mathcal{L}}$ has the same dimension as $\mathcal{C}_{\mathcal{L}}$.

Thanks to Lemma 2.5, many terms vanish when we apply the operator $\pi(D_{t_0,z_0})$ to the nonlinearity. One then finds $\pi(D_{t_0,z_0})[f(\mathcal{U}_0)]_{II} = \pi(D_{t_0,z_0})F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II})$, where the symmetrized function F^S associated to F is defined as

$$F^S(a,b,c) = F(a,b,c) + F(a,c,b) + F(b,a,c) + F(c,a,b) + F(b,c,a) + F(c,b,a)$$

for all $a, b, c \in \mathbb{C}^n$.

The equations for the profile $\mathcal{U}_{0,II}$ are thus

$$\begin{cases} \pi(D_{t_0,z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}, \\ (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau \mathcal{U}_{0,II} + (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{1,II} + i\dfrac{\omega'(D_{z_0})}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} \\ \qquad + i\frac{\omega''(D_{z_0})}{2}\partial_Z^2\mathcal{U}_{0,II} + \psi^\delta(D_{z_0})\pi(D_{t_0,z_0})F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}) = 0, \end{cases} \tag{2.24}$$

and are therefore coupled with the equations found above for the amplitude $\mathcal{U}_{0,I,1}$,

$$\begin{cases} \pi(\omega_l, -k_l)\mathcal{U}_{0,I,1} = \mathcal{U}_{0,I,1}, \\ (\partial_T + \omega'(k_l)\partial_Z)\mathcal{U}_{0,I,1} = 0, \\ \partial_\tau \mathcal{U}_{0,I,1} + i\dfrac{\omega'(k_l)}{2k_l}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,I,1} + i\dfrac{\omega''(k_l)}{2}\partial_Z^2\mathcal{U}_{0,I,1} \\ \qquad + \pi(\omega_l, -k_l)f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}}) = 0. \end{cases} \tag{2.25}$$

Remark 2.8. One can see that the evolution equation for the oscillating mode is the usual *uncoupled* NLS equation. The evolution equation of $\mathcal{U}_{0,II}$ is in turn *linear* but *coupled* with $\mathcal{U}_{0,I,1}$. The next step is to prove that this coupling is not efficient. This is the object of the next section.

2.3. Solving the profile equations in absence of low frequencies. If system (2.25) can easily be solved by standard Picard iterates (see [8], for instance), this is not the case for system (2.24), which deals with the continuous spectrum component of $\mathcal{U}_0$. Moreover, one can see that it is not possible to take $\pi(D_{t_0,z_0})\mathcal{U}_{1,II} = 0$ as for the discrete spectrum component, since the term $F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II})$ makes the last equation of (2.24) obviously incompatible with the transport equation $(\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II} = 0$.

In fact, as in the papers where various group velocities are studied [10], [11], only a good choice of $\pi(D_{t_0,z_0})\mathcal{U}_{1,II} = 0$ can allow the solubility for (2.24). Inspired by the method and arguments of [10] and [11] (i.e., interactions between components traveling at different speeds do not affect the main profile), we decompose (2.24) as follows:

$$\begin{cases} \pi(D_{t_0,z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}, \\ (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau \mathcal{U}_{0,II} + i\dfrac{\omega'(D_{z_0})}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} + i\dfrac{\omega''(D_{z_0})}{2}\partial_Z^2\mathcal{U}_{0,II} = 0, \\ (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{1,II} = -\psi^\delta(D_{z_0})\pi(D_{t_0,z_0})F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}). \end{cases} \tag{2.26}$$

We can now state a solubility result, provided that the function $\mathbf{U}^0_{II}$ has no low frequencies.

PROPOSITION 2.6. *Let $\sigma \geq s$, and let $R > 0$ be such that $\mathbf{U}^0 = \mathbf{U}^0_I + \mathbf{U}^0_{II} \in A^\sigma_0$ and $\|\mathbf{U}^0\|_{A^\sigma_0} \leq R$. Suppose, moreover, that $\mathbf{U}^0_I = \mathbf{U}^0_{I,1} e^{i\theta} + c.c.$ and*

$$\pi(\omega_l, -k_l)\mathbf{U}^0_{I,1} = \mathbf{U}^0_{I,1}, \qquad \pi(D_{t_0,z_0})\mathbf{U}^0_{II} = \mathbf{U}^0_{II},$$

and that there exists $\delta > 0$ such that Sp $\mathbf{U}^0_{II} \subset \{(\omega, k), |k| > \delta\}$.

Then there exists $\tau_2^* > 0$, which depends on R but not on ε nor on δ, such that there exists

- *a unique $\mathcal{U}_{0,I,1} = \pi(\omega_l, -k_l)\mathcal{U}_{0,I,1} \in \mathcal{C}_b([0, \tau_2^*] \times \mathbb{R}_T, H^\sigma(\mathbb{R}^3)^n)$ solving*

$$\begin{cases} (\partial_T + \omega'(k_l)\partial_Z)\mathcal{U}_{0,I,1} = 0, \\ \partial_\tau \mathcal{U}_{0,I,1} + i\dfrac{\omega'(k_l)}{2k_l}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,I,1} + i\dfrac{\omega''(k_l)}{2}\partial_Z^2 \mathcal{U}_{0,I,1} \\ \qquad\qquad + \pi(\omega_l, -k_l) f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}}) = 0, \\ \mathcal{U}_{0,I,1}|_{\tau=T=0} = \mathbf{U}^0_{I,1}; \end{cases} \tag{2.27}$$

- *a unique $\mathcal{U}_{0,II} = \pi(D_{t_0,z_0})\mathcal{U}_{0,II} \in A^\sigma_{\tau_2^*}$ solving*

$$\begin{cases} (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau \mathcal{U}_{0,II} + i\dfrac{\omega'(D_{z_0})}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} + i\dfrac{\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{U}_{0,II} = 0, \\ \mathcal{U}_{0,II}|_{\tau=T=0} = \mathbf{U}^0_{II}; \end{cases} \tag{2.28}$$

- *a unique $\mathcal{U}_{1,II} \in E^\sigma_{\tau_2^*}$ solving*

$$\begin{cases} (Id - \pi(D_{t_0,z_0}))\mathcal{U}_{1,II} = i\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{U}_{0,II}, \\ \psi^\delta(D_{z_0})\pi(D_{t_0,z_0})\mathcal{U}_{1,II} = \pi(D_{t_0,z_0})\mathcal{U}_{1,II}, \\ (\partial_T - \omega'(D_{z_0})\partial_Z)\pi(D_{z_0,t_0})\mathcal{U}_{1,II} \\ \qquad\qquad = -\psi^\delta(D_{z_0})\pi(D_{t_0,z_0})F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}), \\ \mathcal{U}_{1,II}|_{\tau=T=0} = 0. \end{cases} \tag{2.29}$$

Moreover $\mathcal{U}_{0,II}$ satisfies Assumption 2.2 with the same δ as above, and we have the upper bound $\|\mathcal{U}_0\|_{A^\sigma_{\tau_2^*}} \leq 2R$.

Proof. The proof of the existence and the uniqueness of the solution of (2.27) is done by standard Picard iterates, as in [8], for instance. Since $\mathcal{U}_{0,I,1}$ must solve the transport equation $(\partial_T + \omega'(k_l)\partial_Z)\mathcal{U}_{0,I,1} = 0$, there exists $\mathtt{U}_{0,I,1} \in \mathcal{C}([0, \tau_2^*], H^s(\mathbb{R}^3)^n)$ such that

$$\mathcal{U}_{0,I,1}(\tau, T, X, Y, Z) = \mathtt{U}_{0,I,1}(\tau, Z - \omega'(k_l)T, X, Y). \tag{2.30}$$

Moreover, the Schrödinger equation that $\mathcal{U}_{0,I,1}$ must satisfy implies, for $\mathtt{U}_{0,I,1}$,

$$\partial_\tau \mathtt{U}_{0,I,1} + i\frac{\omega'(k_l)}{2k_l}(\partial_X^2 + \partial_Y^2)\mathtt{U}_{0,I,1} + i\frac{\omega''(k_l)}{2}\partial_Z^2 \mathtt{U}_{0,I,1} + \pi(\omega_l, -k_l) f'(\mathtt{U}_{0,I,1})(\overline{\mathtt{U}_{0,I,1}}) = 0.$$

Existence and uniqueness of such a $\mathtt{U}_{0,I,1} \in \mathcal{C}([0, \tau_2^*], H^s(\mathbb{R}^3)^n)$ can be proved by standard Picard iterates, and we thus obtain point (i) of the theorem.

In order to prove (ii), introduce $\lambda := \mathcal{F}\mathcal{U}_{0,II}$ and $\lambda_0 := \mathcal{F}\mathbf{U}^0_{II}$. There is a unique solution of (2.28) in the sense of distributions, given by

$$\widehat{\lambda}(\tau, T) = e^{-i\tau(\frac{\omega'(k)}{2k}(\eta_1^2 + \eta_2^2) + \frac{\omega''(k)}{2}\eta_3^2)} e^{iT\omega'(k)\eta_3} \widehat{\lambda}_0, \tag{2.31}$$

where $\widehat{}$ denotes the Fourier transform with respect to the variables (X, Y, Z), and the measure $\widehat{\lambda}$ is defined for all Borel sets E of $\mathbb{R}^2$ by $\widehat{\lambda}(E) := \widehat{\lambda(E)}$.

The distribution λ defined above is in $\mathcal{C}_b([0, \tau_2^*] \times \mathbb{R}_T, \mathcal{BV}(\mathbb{R}^2_\xi, H^\sigma(\mathbb{R}^3)^n))$, as one can prove with the same arguments as in Theorem 4 of [12], which proves point (ii) of the theorem.

By the explicit expression (2.31) and under the assumption made on $\mathbf{U}^0_{II}$, we also know that $\mathcal{U}_{0,II}$ satisfies Assumption 2.2 and that $||\mathcal{U}_{0,II}||_{A^\sigma_{\tau_2^*}} \leq \mathbb{R}$.

Before solving the next equation, notice that $\pi(D_{t_0,z_0})F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}) \in A^\sigma_{\tau_2^*}$, thanks to the above results and to Proposition 1.3 . We denote by μ the Fourier transform of this profile and also denote by ν the Fourier transform of the profile $\mathcal{U}_{1,II}$. The last equation of system (2.29) has a unique solution in the sense of the distributions, which reads

$$\widehat{\nu}(\tau, T) = -\int_0^T e^{i(T-u)\omega'(k)\eta_3}\psi^\delta(k)\widehat{\mu}(\tau, u)du. \tag{2.32}$$

Here again, we can prove that ν is in $\mathcal{C}([0, \tau_2^*] \times \mathbb{R}_T, \mathcal{BV}(\mathbb{R}^2_\xi, H^\sigma(\mathbb{R}^3)^n))$ so that $\mathcal{U}_{1,II} \in E^\sigma_{\tau_2^*}$. From this expression it is also clear that Sp $\mathcal{U}_{1,II} \subset \{(\omega, k), |k| > \delta\}$, i.e., that the first equation of (2.29) is also satisfied. ∎

Remark 2.9. (i) The profile equations are solved in $E^\sigma_{\tau_2^*}$ for all $\sigma \geq s$, not only in $E^s_{\tau_2^*}$, because if one wants the residual term $\mathcal{R}^\varepsilon$ to be in $E^s_{\tau_2^*}$, $\mathcal{U}_0$ must be in $E^{s+4}_{\tau_2^*}$, as we will see in the next section.

(ii) The remaining profiles $(Id - \pi(\omega_l, -k_l))\mathcal{U}_{1,I,1}$, $(Id - \pi(D_{t_0,z_0}))\mathcal{U}_{1,II}$, and $\mathcal{U}_2$ can be found in terms of the profiles given by the above theorem, thanks to the expressions given by (2.4)–(2.5), (2.12), (2.16), and (2.18).

2.4. Stability in the absence of low frequencies. In this part, we want to prove that the solution of diffractive optics gives a good approximation of the exact solution of (1.1), provided that low frequencies are excluded. To get this result, we prove that the residual associated to the approximate solution is small and that the approximate solution is close to the exact solution $\mathbf{u}^\varepsilon$ of (1.1).

Suppose $\mathcal{U}_0$ and $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ are given by Proposition 2.6 with initial condition $\mathbf{U}^0$ without frequencies lower than $\delta > 0$. $\mathcal{U}_{0,II}$ then satisfies Assumption 2.2, $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ fulfills condition (2.13), and $\mathcal{U}_{2,I}$ and $\mathcal{U}_{2,II}$ can then be constructed with the results of section 2.1. Before proving that $\mathcal{U}^\varepsilon = \mathcal{U}_0 + \varepsilon\mathcal{U}_1 + \varepsilon^2\mathcal{U}_2$ is an approximate solution of the singular equation (1.5) in the sense that the residual remains small, we need estimates of the profiles $\mathcal{U}_j$, $j = 0, 1, 2$, and of the residual terms $\mathcal{R}_j$, $j \geq 2$.

LEMMA 2.7. *Let $\sigma \geq s + 4$, $\sigma' \geq s$, and $\delta \in (0, 1)$ and suppose $\mathcal{U}_0 \in A^\sigma_{\tau_2^*}$ and $\pi(D_{t_0,z_0})\mathcal{U}_{1,II} \in E^\sigma_{\tau_2^*}$ are given by Proposition* 2.6. *Take $\mathcal{U}_{1,I}$, $(Id - \pi(D_{t_0,z_0}))\mathcal{U}_{1,II}$ and $\mathcal{U}_2$ as computed in section* 2.1. *Then*

(i) *the two components $\mathcal{U}_{1,I}$ and $\mathcal{U}_{1,II}$ of $\mathcal{U}_1$ are controlled as follows:*

$$\|\mathcal{U}_{1,I}\|_{A^{\sigma'}_{\tau_2^*}} \leq C\|\mathcal{U}_{0,I}\|_{A^{\sigma'+1}_{\tau_2^*}}, \quad \|(Id - \pi(D_{t_0,z_0}))\mathcal{U}_{1,II}\|_{A^{\sigma'}_{\tau_2^*}} \leq \frac{C}{\delta}\|\mathcal{U}_{0,II}\|_{A^{\sigma'+1}_{\tau_2^*}},$$

and

$$\|\pi(D_{t_0,z_0})\mathcal{U}_{1,II}(T)\|_{E^{\sigma'}_{\tau_2^*}} \leq CT\|\mathcal{U}_{0,I}\|^2_{A^{\sigma'}_{\tau_2^*}}\|\mathcal{U}_{0,II}\|_{A^{\sigma'}_{\tau_2^*}} \qquad \forall T \geq 0;$$

(ii) *the discrete spectrum component* $\mathcal{U}_{2,I}$ *of* $\mathcal{U}_2$ *is controlled as follows:*

$$\|\mathcal{U}_{2,I}\|_{A^{\sigma'}_{\tau_2^*}} \leq C\left(\|\mathcal{U}_{0,I}\|_{A^{\sigma'+2}_{\tau_2^*}} + \|\mathcal{U}_{0,I}\|^3_{A^{\sigma'}_{\tau_2^*}}\right),$$

while $\mathcal{U}_{2,II}$ *satisfies*

$$\|\mathcal{U}_{2,II}\|_{A^{\sigma'}_{\tau_2^*}} \leq \frac{C}{\delta}\left(\frac{1}{\delta}\|\mathcal{U}_{0,II}\|_{A^{\sigma'+2}_{\tau_2^*}} + \|\mathcal{U}_0\|^3_{A^{\sigma'}_{\tau_2^*}} + T\|\mathcal{U}_{0,I}\|^2_{A^{\sigma'+1}_{\tau_2^*}}\|\mathcal{U}_{0,II}\|_{A^{\sigma'+1}_{\tau_2^*}}\right);$$

(iii) *rough estimates of the profiles* $\mathcal{R}_{j\geq 2}$ *are given by*

$$\|\mathcal{R}_{j,I}\|_{A^{\sigma'}_{\tau_2^*}} \leq h_1\left(\|\mathcal{U}_{0,I}\|_{A^{\sigma'+4}_{\tau_2^*}}\right) \quad \text{and} \quad \|\mathcal{R}_{j,II}(T)\|_{E^{\sigma'}_{\tau_2^*}} \leq \frac{T}{\delta^6}h_2\left(\|\mathcal{U}_0\|_{A^{\sigma'+4}_{\tau_2^*}}\right) \quad \forall T\geq 0,$$

where h_1 *and* h_2 *are smooth positive functions defined on* $\mathbb{R}^+$ *and independent of* $\delta\in(0,1)$ *and of* $T\geq 0$.

Proof. The estimate of $\mathcal{U}_{1,I}$ is easily deduced from (2.4) and (2.6), while Lemma 2.1 and (2.5)–(2.6) yield the estimate of $(Id-\pi(D_{t_0,z_0}))\mathcal{U}_{1,II}$. The estimate of $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ is a consequence of (2.32).

The modes ± 3 of $\mathcal{U}_{2,I}$ are controlled using (2.10) together with the algebra properties of $A^{\sigma'}_{\tau_2^*}$. The modes ± 1 are controlled as stated in the lemma as a consequence of (2.12) and (2.18). Lemma 2.1 and (2.16), (2.18) are used to estimate $\mathcal{U}_{2,II}$.

The estimates of the profiles $\mathbb{R}_j$ follow directly from the explicit formulae of these profiles given in (1.8), from Lemma 2.1 and the equations satisfied by $\mathcal{U}_{1\leq j\leq 3}$, and from the estimates of these profiles computed above. ∎

Proposition 2.8. *Let* $\sigma\geq s+4$ *and* $\delta\in(0,1)$ *and suppose* $\mathcal{U}_0$ *and* $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ *are as given by Proposition* 2.6. *Take* $\mathcal{U}_{1,I}$, $(Id-\pi(D_{t_0,z_0}))\mathcal{U}_{1,II}$ *and* $\mathcal{U}_2$ *as computed in section* 2.1 *and let* $\mathcal{U}^\varepsilon=\mathcal{U}_0+\varepsilon\mathcal{U}_1+\varepsilon^2\mathcal{U}_2\in E^{s+2}_{\tau_2^*}\subset E^s_{\tau_2^*}$.

Then the profile $\underline{\mathcal{U}}^\varepsilon$ *defined as* $\underline{\mathcal{U}}^\varepsilon(\tau,X,Y,Z,t_0,z_0)=\mathcal{U}^\varepsilon(\tau,\tau/\varepsilon,X,Y,Z,t_0,z_0)$ *is in* $B^{s+2}_{\tau_2^*}$ *and is bounded uniformly in* $\varepsilon\in(0,1)$.

If δ *is small enough,* $\underline{\mathcal{U}}^\varepsilon$ *is an approximate solution of the singular equation* (1.5). *More precisely, for any* $\mu>0$ *there exists* $\delta(\mu)>0$ *such that if* $0<\delta<\delta(\mu)$, $\underline{\mathcal{U}}^\varepsilon$ *satisfies*

$$\limsup_{\varepsilon\to 0}\|\partial_\tau\underline{\mathcal{U}}^\varepsilon+\varepsilon^{-1}(A_1\partial_X+A_2\partial_Y+A_3\partial_Z)\underline{\mathcal{U}}^\varepsilon+\varepsilon^{-2}(\partial_{t_0}+A_3\partial_{z_0}+L_0)\underline{\mathcal{U}}^\varepsilon+f(\underline{\mathcal{U}}^\varepsilon)\|<\mu/3,$$

where the norm is taken in $B^s_{\tau_2^*}$.

Proof. Define, for $j=1,2,3$, $\underline{\mathcal{U}}^\varepsilon_j(\tau,X,Y,Z,t_0,z_0)=\mathcal{U}_j(\tau,\tau/\varepsilon,X,Y,Z,t_0,z_0)$. One has $\underline{\mathcal{U}}^\varepsilon_0\in B^{s+2}_{\tau_2^*}$ since $\mathcal{U}_0\in A^{s+2}_{\tau_2^*}$. Similarly, $\underline{\mathcal{U}}^\varepsilon_{1,I}$ and $(Id-\pi(D_{t_0,z_0}))\underline{\mathcal{U}}^\varepsilon_{1,II}$ are in $B^{s+2}_{\tau_2^*}$. Their norm in this space is obviously uniformly bounded in $\varepsilon\in(0,1)$.

Since $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}\notin A^{s+2}_{\tau_2^*}$, we cannot apply the same reasoning for this component. However, point (i) of Lemma 2.7 asserts that $\varepsilon\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ satisfies, for all $T\geq 0$,

$$\|\varepsilon\pi(D_{t_0,z_0})\mathcal{U}_{1,II}(T)\|_{E^{s+2}_{\tau_2^*}} \leq \varepsilon T\|\mathcal{U}_{0,I}\|^2_{A^{s+2}_{\tau_2^*}}\|\mathcal{U}_{0,II}\|_{A^{s+2}_{\tau_2^*}},$$

and since one has

$$\|\pi(D_{t_0,z_0})\underline{\mathcal{U}}^\varepsilon_{1,II}\|_{B^{s+2}_{\tau_2^*}} \leq \sup_{T\in[0,\tau_2^*/\varepsilon]}\|\pi(D_{t_0,z_0})\mathcal{U}_{1,II}(T)\|_{E^{s+2}_{\tau_2^*}},$$

we can conclude that

$$\|\varepsilon\pi(D_{t_0,z_0})\underline{\mathcal{U}}^{\varepsilon}_{1,II}\|_{B^{s+2}_{\tau_2^*}} \leq \tau_2^*\|\mathcal{U}_{0,I}\|^2_{A^{s+2}_{\tau_2^*}}\|\mathcal{U}_{0,II}\|_{A^{s+2}_{\tau_2^*}}.$$

Thus, $\varepsilon\pi(D_{t_0,z_0})\underline{\mathcal{U}}^{\varepsilon}_{1,II}$ is in $B^{s+2}_{\tau_2^*}$ and is uniformly bounded with respect to $\varepsilon \in (0,1)$. The same thing can be said about $\underline{\mathcal{U}}^{\varepsilon}_{2,II}$, which proves that $\underline{\mathcal{U}^{\varepsilon}} \in B^{s+2}_{\tau_2^*}$, with uniformly bounded norm.

We recall that

$$\partial_\tau\underline{\mathcal{U}}^{\varepsilon}+\varepsilon^{-1}(A_1\partial_X+A_2\partial_Y+A_3\partial_Z)\underline{\mathcal{U}}^{\varepsilon}+\varepsilon^{-2}(\partial_{t_0}+A_3\partial_{z_0}+L_0)\underline{\mathcal{U}}^{\varepsilon}+f(\underline{\mathcal{U}}^{\varepsilon}) = \sum_{j=-1}^{7}\varepsilon^{j-1}\underline{\mathcal{R}}^{\varepsilon}_j,$$

where the profiles $\underline{\mathcal{R}}^{\varepsilon}_j$ are defined as $\underline{\mathcal{R}}^{\varepsilon}_j(\tau,X,Y,Z,t_0,z_0) = \mathcal{R}_j(\tau,\tau/\varepsilon,X,Y,Z,t_0,z_0)$ with $\mathcal{R}_j$ given in (1.8).

Thanks to the results of the previous section, we know that $\underline{\mathcal{R}}^{\varepsilon}_{-1} = \underline{\mathcal{R}}^{\varepsilon}_0 = 0$ and that $\underline{\mathcal{R}}^{\varepsilon}_1 = (1-\psi^\delta(D_{z_0}))F^S(\mathcal{U}_{0,I,1},\overline{\mathcal{U}_{0,I,1}},\mathcal{U}_{0,II})$. Therefore,

$$\|\underline{\mathcal{R}}^{\varepsilon}_1\|_{B^s_{\tau_2^*}} \leq \|\mathcal{U}_{0,I}\|^2_{A^s_{\tau_2^*}}\|(1-\psi^\delta(D_{z_0}))\underline{\mathcal{U}}^{\varepsilon}_{0,II}\|_{B^s_{\tau_2^*}}.$$

The following lemma says that this term goes to zero as $\delta \to 0$ uniformly in $\varepsilon \in (0,1)$.

LEMMA 2.9. *Let $\sigma \geq s$ and $\mathcal{U}_{0,II} \in A^\sigma_{\tau_2^*}$ be as given by Proposition* 2.6; *let $\underline{\mathcal{U}}^{\varepsilon}_{0,II} \in B^\sigma_{\tau_2^*}$ be defined as $\underline{\mathcal{U}}^{\varepsilon}_{0,II}(\tau,X,Y,Z,t_0,z_0) = \mathcal{U}_{0,II}(\tau,\tau/\varepsilon,X,Y,Z,t_0,z_0)$.*

Then one has $(1-\psi^\delta(D_{z_0}))\underline{\mathcal{U}}^{\varepsilon}_{0,II} \to 0$ in $B^\sigma_{\tau_2^*}$ as $\delta \to 0$ uniformly in $\varepsilon \in (0,1)$.

Proof. Let $\lambda^\varepsilon := \mathcal{F}\underline{\mathcal{U}}^{\varepsilon}_{0,II} \in \mathcal{C}([0,\tau_2^*],\mathcal{BV}(\mathbb{R}^2_\xi,H^\sigma(\mathbb{R}^3_{X,Y,Z})^n))$ and $\lambda_0 := \mathcal{F}\mathbf{U}^0_{II}$. Thanks to (2.31), we have

$$\widehat{\lambda}^\varepsilon(\tau) = e^{-i\tau(\frac{\omega'(k)}{2k}(\eta_1^2+\eta_2^2)+\frac{\omega''(k)}{2}\eta_3^2)}e^{i\frac{\tau}{\varepsilon}\omega'(k)\eta_3}\widehat{\lambda}_0.$$

We also know by the Radon–Nikodým property that there exists an H^σ-valued integral function r_0 such that $\|r_0(\xi)\| = 1$ for $v(\lambda_0)$ for almost all $\xi = (\omega,k)$ and such that for all Borel sets $E \subset \mathbb{R}^2$,

$$\lambda_0(E) = \int_E r_0(\xi)v(\lambda_0)(d\xi),$$

where $v(\lambda_0)$ denotes the total variation measure associated to λ_0. Hence,

$$\widehat{\lambda}^\varepsilon(\tau)(E) = \int_E e^{-i\tau(\frac{\omega'(k)}{2k}(\eta_1^2+\eta_2^2)+\frac{\omega''(k)}{2}\eta_3^2)}e^{i\frac{\tau}{\varepsilon}\omega'(k)\eta_3}\widehat{r}_0(\xi)v(\lambda_0)(d\xi), \tag{2.33}$$

and also

$$(1-\psi^\delta(k))\widehat{\lambda}^\varepsilon(\tau)(E) = \int_E (1-\psi^\delta(k))e^{-i\tau(\frac{\omega'(k)}{2k}(\eta_1^2+\eta_2^2)+\frac{\omega''(k)}{2}\eta_3^2)}e^{i\frac{\tau}{\varepsilon}\omega'(k)\eta_3} \times\widehat{r}_0(\xi)v(\lambda_0)(d\xi).$$

Therefore

$$\|(1-\psi^\delta(D_{z_0}))\underline{\mathcal{U}}^{\varepsilon}_{0,II}\|_{B^\sigma_{\tau_2^*}} = \sup_{\tau\in[0,\tau_2^*]}|(1-\psi^\delta(k))\widehat{\lambda}^\varepsilon|_{\mathcal{B}V} \leq \int_E (1-\psi^\delta(k))v(\lambda_0)(d\xi)$$

and hence tends to zero as $\delta \to 0$ uniformly in $\varepsilon \in (0,1)$ as a consequence of the dominated convergence theorem. ∎

We now turn to investigating the residual terms $\varepsilon^{j-1}\underline{\mathcal{R}}_j^\varepsilon$ for $j \geq 2$.

Using point (iii) of Lemma 2.7 and with the same techniques used above to prove that $\varepsilon\pi(D_{t_0,z_0})\underline{\mathcal{U}}_{1,II}^\varepsilon$ is uniformly bounded in $B_{\tau_2^*}^{s+2}$, one can prove that $\varepsilon\underline{\mathcal{R}}_j^\varepsilon$ is uniformly bounded in $B_{\tau_2^*}^s$ with respect to $\varepsilon \in (0,1)$. Therefore, if $j \geq 3$, then the residual terms $\varepsilon^{j-1}\underline{\mathcal{R}}_j^\varepsilon$ tend to 0 in $B_{\tau_2^*}^s$ when $\varepsilon \to 0$ (and with $\delta > 0$ being fixed).

The only remaining term to treat is therefore $\varepsilon\underline{\mathcal{R}}_2^\varepsilon$. In fact, only its continuous spectrum component needs care; so far, we know only that it is uniformly bounded in $B_{\tau_2^*}^s$. We recall that $\mathcal{R}_{2,II}$ is given by

$$\begin{aligned}\mathcal{R}_{2,II} = &-L_1(\partial)\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\mathcal{U}_{0,II} + L_1(\partial)\mathcal{L}^{-1}(D_{t_0,z_0})\psi^\delta(D_{z_0})f(\mathcal{U}_0)_{II}\\ &+iL_1(\partial)\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\pi(D_{t_0,z_0})\mathcal{U}_{1,II} + i\mathcal{L}^{-1}(D_{t_0,z_0})A(\partial_{X,Y,Z})\partial_\tau\mathcal{U}_{0,II}\\ &+\partial_\tau\pi(D_{t_0,z_0})\mathcal{U}_{1,II} + \big(f'(\mathcal{U}_0)(\mathcal{U}_1)\big)_{II}.\end{aligned}$$

In this expression, all the terms which do not involve $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$ are in $A_{\tau_2^*}^s$, and their contribution to $\underline{\mathcal{R}}_2^\varepsilon$ is therefore in $B_{\tau_2^*}^s$. Hence, the only possible problems come from the terms which involve $\pi(D_{t_0,z_0})\mathcal{U}_{1,II}$. We need the following lemma.

LEMMA 2.10. *Let $\sigma \geq s+4$ and $\pi(D_{t_0,z_0})\mathcal{U}_{1,II} \in E_{\tau_2^*}^\sigma$ be as given by Proposition* 2.6.

Then one has

$$\lim_{\varepsilon\to 0} \varepsilon\|\underline{\mathcal{U}}_{1,II}^\varepsilon\|_{B_{\tau_2^*}^\sigma} = 0 \quad \textit{and} \quad \lim_{\varepsilon\to 0} \varepsilon\|\partial_\tau\underline{\mathcal{U}}_{1,II}^\varepsilon\|_{B_{\tau_2^*}^{\sigma-2}} = 0.$$

Proof. For all $\tau \in [0,\tau_2^*]$, let $\nu^\varepsilon(\tau) := \mathcal{F}\underline{\mathcal{U}}_{1,II}^\varepsilon(\tau)$. Thanks to (2.32), one then has

$$\widehat{\nu^\varepsilon}(\tau) = -\int_0^{\tau/\varepsilon} e^{-(\frac{\tau}{\varepsilon}-t)\omega'(k)\eta_3}\psi^\delta(k)\widehat{\mu}(\tau,t)dt, \tag{2.34}$$

where, for all (τ,T), $\mu(\tau,T)$ is defined as $\mu(\tau,T) := \mathcal{F}\big(F^S(\mathcal{U}_{0,I,1},\overline{\mathcal{U}_{0,I,1}},\mathcal{U}_{0,II})(\tau,T)\big)$. We also recall that $\widehat{\ }$ denotes the Fourier transform with respect to the variables (X,Y,Z).

As with (2.33), we can write $\lambda(\tau,T) := \mathcal{F}\mathcal{U}_{0,II}(\tau,T)$ in the form

$$\widehat{\lambda}(\tau,T) = \int_E e^{-i\tau(\frac{\omega'(k)}{2k}(\eta_1^2+\eta_2^2)+\frac{\omega''(k)}{2}\eta_3^2)}e^{iT\omega'(k)\eta_3}\widehat{r_0(\xi)}v(\lambda_0)(d\xi)$$

for all Borel sets $E \subset \mathbb{R}^2$.

Hence, $\widehat{\mu}(\tau,T)(E) \in H^\sigma(\mathbb{R}^3)$ is given, for all Borel sets $E \subset \mathbb{R}^2$ and $\eta \in \mathbb{R}^3$, by

$$\begin{aligned}\widehat{\mu}(\tau,T)(E)(\eta) = \int_E\int_{\mathbb{R}^3\times\mathbb{R}^3} F^S\Big(&\widehat{\mathcal{U}_{0,I,1}}(\eta-\eta'),\widehat{\overline{\mathcal{U}_{0,I,1}}}(\eta''-\eta'),\\ &e^{-i\tau(\frac{\omega'(k)}{2k}(\eta_1'^2+\eta_2'^2)+\frac{\omega''(k)}{2}\eta_3'^2)}e^{iT\omega'(k)\eta_3'}\widehat{r_0(\xi)}(\eta')\Big)\,d\eta' d\eta'' v(\lambda_0)(d\xi).\end{aligned}$$

Combining this equation with (2.34) then yields

$$\begin{aligned}|\widehat{\nu^\varepsilon}(\tau)|_{\mathcal{BV}} \leq &\int_{\mathbb{R}^2}\|\int_{\mathbb{R}^3\times\mathbb{R}^3}\int_0^{\tau/\varepsilon} F^S\Big(\Big(\widehat{\mathcal{U}_{0,I,1}}(\cdot-\eta'),\widehat{\overline{\mathcal{U}_{0,I,1}}}(\eta''-\eta'),\\ &e^{-i\tau(\frac{\omega'(k)}{2k}(\eta_1'^2+\eta_2'^2)+\frac{\omega''(k)}{2}\eta_3'^2)}e^{it\omega'(k)\eta_3'}\widehat{r_0(\xi)}(\eta')\Big)\,dt d\eta' d\eta''\|_{\mathcal{F}(H^\sigma)}v(\lambda_0)(d\xi)\\ := &\int_{\mathbb{R}^2} G^\varepsilon(\xi)v(\lambda_0)(d\xi).\end{aligned}$$

The family $\varepsilon G^\varepsilon(\xi)$ can be bounded by a constant (and constants are $v(\lambda_0)$-integrable); thanks to Lemma 6 of [11], we also know that $\varepsilon G^\varepsilon(\xi) \to 0$ as $\varepsilon \to 0$, provided that $\omega'(k) \neq \omega'(k_l)$. Since this equality occurs only when $k = k_l$, and since $v(\lambda_0)(\{k_l\}) = 0$, we thus have $\varepsilon G^\varepsilon(\xi) \to 0$ $v(\lambda_0)$ almost everywhere, so that the first part of the lemma follows from the dominated convergence theorem.

The second part of the lemma can be established with the same techniques. □

From the above lemma, it is clear that $\varepsilon \underline{\mathcal{R}}_2^\varepsilon = o(1)$, which achieves the last step of the proof of the proposition. □

We know that u^ε almost solves (1.1), but we have not yet proved that the difference $\mathbf{u}^\varepsilon - u^\varepsilon$ remains small. This is what the following theorem shows.

THEOREM 2.11. *Suppose the characteristic variety* $\mathcal{C}_\mathcal{L}$ *is as in Assumptions* 2.1 *and* 2.3.

Let $\mathbf{U}^0 = \mathbf{U}_I^0 + \mathbf{U}_{II}^0 \in A_0^{s+4}$ *such that* $\mathbf{U}_I^0 = \mathbf{U}_{I,1}^0 e^{i\theta} + c.c.$ *and suppose, moreover, that*

$$\pi(\omega_l, -k_l)\mathbf{U}_{I,1}^0 = \mathbf{U}_{I,1}^0 \quad \textit{and} \quad \pi(D_{t_0,z_0})\mathbf{U}_{II}^0 = \mathbf{U}_{II}^0,$$

and that $\mathrm{Sp}\ \mathbf{U}_{II}^0 \subset \{(\omega,k), |k| > \delta\}$ *for a given* $\delta > 0$.

Then, for $0 < \tau^* \le \inf(\tau_1^*, \tau_2^*)$, *the following holds:*

(i) *The profile* $\mathcal{U}_0 = \mathcal{U}_{0,I} + \mathcal{U}_{0,II}$ *given by Propostion* 2.6 *satisfies Assumption* 2.2, *and the associated profile* $\underline{\mathcal{U}}_0^\varepsilon \in B_{\tau^*}^s$ *approximates the singular equation* (1.5) *in the sense that for all* $\mu > 0$ *there exists a* $\delta(\mu)$ *such that if* $0 < \delta < \delta(\mu)$, *then*

$$\|\mathbf{U}_I^\varepsilon - \underline{\mathcal{U}}_{0,I}^\varepsilon\|_{B_{\tau^*}^s} = O(\varepsilon) \quad \textit{and} \quad \limsup_{\varepsilon\to 0} \|\mathbf{U}_{II}^\varepsilon - \underline{\mathcal{U}}_{0,II}^\varepsilon\|_{B_{\tau^*}^s} < \mu/3,$$

where we have decomposed the profile $\mathbf{U}^\varepsilon$ *of the exact solution* $\mathbf{u}^\varepsilon$, *given by Theorem* 1.4, *into* $\mathbf{U}^\varepsilon = \mathbf{U}_I^\varepsilon + \mathbf{U}_{II}^\varepsilon$.

(ii) *We also have stability of the approximate solution defined with* $\mathcal{U}_0$,

$$\|\mathbf{u}_I^\varepsilon - u_{0,I}^\varepsilon\| = O(\varepsilon^{3/2}) \quad \textit{and} \quad \limsup_{\varepsilon\to 0} \frac{1}{\sqrt{\varepsilon}} \|\mathbf{u}_{II}^\varepsilon - u_{0,II}^\varepsilon\| < \mu/3,$$

where the norm can be taken either in $\mathcal{C}([0,\frac{\tau^*}{\varepsilon}] \times \mathbb{R}^3)^n$ *or in* $\mathcal{C}([0,\frac{\tau^*}{\varepsilon}], L^2(\mathbb{R}^3)^n)$.

Notation. We have used in this theorem the notation

$$\mathbf{u}_I^\varepsilon = \sqrt{\varepsilon}\mathbf{U}_I^\varepsilon(\varepsilon T, X, Y, Z, T/\varepsilon, Z/\varepsilon), \quad u_{0,I}^\varepsilon = \sqrt{\varepsilon}\underline{\mathcal{U}}_{0,I}^\varepsilon(\varepsilon T, X, Y, Z, T/\varepsilon, Z/\varepsilon),$$

and similarly

$$\mathbf{u}_{II}^\varepsilon = \sqrt{\varepsilon}\mathbf{U}_{II}^\varepsilon(\varepsilon T, X, Y, Z, T/\varepsilon, Z/\varepsilon), \quad u_{0,II}^\varepsilon = \sqrt{\varepsilon}\underline{\mathcal{U}}_{0,II}^\varepsilon(\varepsilon T, X, Y, Z, T/\varepsilon, Z/\varepsilon).$$

Proof. (i) Since $\underline{\mathcal{R}}_{-1}^\varepsilon = \underline{\mathcal{R}}_0^\varepsilon = 0$, the error profile $\mathcal{W}^\varepsilon = \mathbf{U}^\varepsilon - \underline{\mathcal{U}}^\varepsilon$ satisfies

$$\begin{aligned}\partial_\tau \mathcal{W}^\varepsilon &+ \varepsilon^{-1}(A_1\partial_X + A_2\partial_Y + A_3\partial_Z)\mathcal{W}^\varepsilon + \varepsilon^{-2}(\partial_{t_0} + A_3\partial_{z_0} + L_0)\mathcal{W}^\varepsilon \\ &= f(\underline{\mathcal{U}}^\varepsilon) - f(\mathbf{U}^\varepsilon) + \underline{\mathcal{R}}_1^\varepsilon + \sum_{j=2}^{7} \varepsilon^{j-1}\underline{\mathcal{R}}_j^\varepsilon.\end{aligned}$$

Thanks to Taylor's theorem, there exists a regular function G such that $f(\underline{\mathcal{U}}^\varepsilon) - f(\mathbf{U}^\varepsilon) = G(\underline{\mathcal{U}}^\varepsilon, \mathbf{U}^\varepsilon)\mathcal{W}^\varepsilon$. Therefore, the profile $\mathcal{W}^\varepsilon$ satisfies

$$\begin{aligned}\partial_\tau \mathcal{W}^\varepsilon &+ \varepsilon^{-1}(A_1\partial_X + A_2\partial_Y + A_3\partial_Z)\mathcal{W}^\varepsilon + \varepsilon^{-2}(\partial_{t_0} + A_3\partial_{z_0} + L_0)\mathcal{W}^\varepsilon \\ &- G(\underline{\mathcal{U}}^\varepsilon, \mathbf{U}^\varepsilon)\mathcal{W}^\varepsilon = \underline{\mathcal{R}}_1^\varepsilon + \sum_{j=2}^{7} \varepsilon^{j-1}\underline{\mathcal{R}}_j^\varepsilon.\end{aligned} \tag{2.35}$$

Let $R > 0$ such that $\|\mathbf{U}^0\|_{A^{s+4}_{\tau^*}} \leq R$. We recall that τ_1^* and τ_2^* are chosen (in Theorem 1.4 and Proposition 2.6, respectively) in such a way that $\|\mathbf{U}^\varepsilon\|_{A^s_{\tau_1^*}} \leq 2R$ and $\|\mathcal{U}_0\|_{A^{s+4}_{\tau_2^*}} \leq 2R$. Since $\tau^* \leq \inf(\tau_1^*, \tau_2^*)$, we can replace $\tau_{1,2}^*$ by τ^* in these inequalities. Hence, we can deduce from Lemma 2.7 that

$$\|\underline{\mathcal{U}}^\varepsilon\|_{B^s_{\tau^*}} \leq CR\left(R^2 + \varepsilon\left(1 + \frac{1+R^2}{\delta}\right) + \varepsilon^2(1+R^2)\left(\frac{1}{\delta} + \frac{1}{\delta^2}\right)\right).$$

Thus

$$\|G(\underline{\mathcal{U}}^\varepsilon, \mathbf{U}^\varepsilon)\|_{B^s_{\tau^*}} \leq h\left(R\left(R^2 + \varepsilon\left(1 + \frac{1+R^2}{\delta}\right) + \varepsilon^2(1+R^2)\left(\frac{1}{\delta} + \frac{1}{\delta^2}\right)\right), R\right),$$

where $h(\cdot,\cdot)$ is a smooth positive function which does not depend on R, δ, nor ε.

By a Gronwall-type argument, we can therefore deduce from (2.35) that

$$\|\mathcal{W}^\varepsilon\|_{B^s_{\tau^*}} \leq \tau^*\left(\|\underline{\mathcal{R}}_1^\varepsilon\|_{B^s_{\tau^*}} + \sum_{j=2}^{7}\varepsilon^{j-1}\|\underline{\mathcal{R}}_j^\varepsilon\|_{B^s_{\tau^*}}\right)e^{h\left(R(R^2+\varepsilon(1+\frac{1+R^2}{\delta})+\varepsilon^2(1+R^2)(\frac{1}{\delta}+\frac{1}{\delta^2})),R\right)\tau^*}. \tag{2.36}$$

It is now easy to deduce from (2.36), Lemma 2.7(iii), and Proposition 2.8 that $\underline{\mathcal{U}}^\varepsilon = \underline{\mathcal{U}}_0^\varepsilon + \varepsilon\underline{\mathcal{U}}_1^\varepsilon + \varepsilon^2\underline{\mathcal{U}}_2^\varepsilon$ satisfies the asymptotic properties of point (i) of the theorem. This point will be proved if we can replace $\underline{\mathcal{U}}^\varepsilon$ by $\underline{\mathcal{U}}_0^\varepsilon$. This is obviously the case since, as a consequence of Lemmas 2.7 and 2.10, $\varepsilon\underline{\mathcal{U}}_1^\varepsilon + \varepsilon^2\underline{\mathcal{U}}_2^\varepsilon$ goes to 0 in $B^s_{\tau^*}$ as $\varepsilon \to 0$.

Taking the discrete spectrum component of (2.35) yields the usual equations of diffractive optics (in particular, $\mathcal{R}_{1,I} = 0$) so that the techniques of [8], [10], and [11] can be used to obtain a better estimate $O(\varepsilon)$ of the error term.

(ii) This point is a direct consequence of (i) and of the embedding results of Proposition 1.3. ∎

2.5. Stability in the general case. In this section, we consider the general case, i.e., we allow low frequencies. Therefore, we consider initial conditions with profile $\mathbf{U}^0 = \mathbf{U}_I^0 + \mathbf{U}_{II}^0 \in A_0^{s+4}$ without making any assumption on the spectrum of $\mathbf{U}_{II}^0$. Alterman's methods [1] are used to relax this assumption. We first introduce the following notation.

Notation. We denote by $\mathbf{U}_{II}^{0,\delta}$ and $\mathbf{U}^{0,\delta}$ the "filtered" profiles

$$\mathbf{U}_{II}^{0,\delta} = \psi^\delta(D_{z_0})\mathbf{U}_{II}^0 \quad \text{and} \quad \mathbf{U}^{0,\delta} = \mathbf{U}_I^0 + \mathbf{U}_{II}^{0,\delta}, \tag{2.37}$$

where $0 < \delta < 1$.

The exact solution of (1.1) with initial condition $\sqrt{\varepsilon}\mathbf{U}^{0,\delta}(X,Y,Z,0,Z/\varepsilon)$ determined by Theorem 1.4 is denoted by $\mathbf{u}^{\varepsilon,\delta}$ and its associated profile by $\mathbf{U}^{\varepsilon,\delta}$, so that one has $\mathbf{u}^{\varepsilon,\delta}(T,X,Y,Z) = \sqrt{\varepsilon}\mathbf{U}^{\varepsilon,\delta}(\varepsilon T, X, Y, Z, T/\varepsilon, Z/\varepsilon)$.

The dominated convergence theorem shows that $\mathbf{U}_{II}^{0,\delta} \to \mathbf{U}_{II}^0$ in A_0^s. We also have convergence of the exact solutions of (1.1) associated to these initial conditions, as the following proposition shows.

PROPOSITION 2.12. *Let* $\mathbf{U}^0 = \mathbf{U}_I^0 + \mathbf{U}_{II}^0 \in A_0^s$ *and* $\mathbf{U}^{0,\delta} = \mathbf{U}_I^0 + \psi^\delta(D_{z_0})\mathbf{U}_{II}^0$.

There exists $\tau_1^* > 0$*, independent of* ε *and* δ*, such that the exact solutions* $\mathbf{U}^\varepsilon$ *and* $\mathbf{U}^{\varepsilon,\delta}$ *of the singular equation* (1.5) *with initial conditions* $\mathbf{U}^0$ *and* $\mathbf{U}^{0,\delta}$*, respectively, exist in* $B^s_{\tau_1^*}$*. Moreover, one has*

$$\mathbf{U}^\varepsilon - \mathbf{U}^{\varepsilon,\delta} \to 0 \quad \text{in } B^s_{\tau_1^*} \quad \text{as } \delta \to 0$$

uniformly in $\varepsilon \in (0,1)$.

Proof. It is easy to see that $\|\mathbf{U}^{0,\delta}\|_{B^s_{\tau_1^*}} \le \|\mathbf{U}^0\|_{B^s_{\tau_1^*}}$. Hence, if R is such that $\|\mathbf{U}^0\|_{B^s_{\tau_1^*}} \le R$, one also has $\|\mathbf{U}^{0,\delta}\|_{B^s_{\tau_1^*}} \le R$. Therefore, Theorem 1.4 implies that the profile $\mathbf{U}^{\varepsilon,\delta}$ also exists on the existence interval $[0,\tau_1^*]$ of $\mathbf{U}^\varepsilon$, since this interval depends only on R.

Moreover, on $[0,\tau_1^*]$, the difference $\mathbf{W}^\delta = \mathbf{U}^\varepsilon - \mathbf{U}^{\varepsilon,\delta}$ satisfies

$$\begin{aligned}\partial_\tau \mathbf{W}^\delta + \varepsilon^{-1}(A_1\partial_X + A_2\partial_Y + A_3\partial_Z)\mathbf{W}^\delta + \varepsilon^{-2}(\partial_{t_0} + A_3\partial_{z_0} + L_0)\mathbf{W}^\delta \\ = G(\mathbf{U}^{\varepsilon,\delta},\mathbf{U}^\varepsilon)\mathbf{W}^\delta,\end{aligned}$$

where, as for (2.35), G is a regular function satisfying $G(\mathbf{U}^{\varepsilon,\delta},\mathbf{U}^\varepsilon)\mathbf{W}^\delta = f(\mathbf{U}^{\varepsilon,\delta}) - f(\mathbf{U}^\varepsilon)$.

As in the proof of Theorem 2.11, we obtain

$$\|G(\mathbf{U}^{\varepsilon,\delta},\mathbf{U}^\varepsilon)\|_{B^s_{\tau_1^*}} \le h(R,R),$$

where $h(\cdot,\cdot)$ is a smooth positive function independent of δ and ε.

A Gronwall-type argument then yields

$$\|\mathbf{W}^\delta\|_{B^s_{\tau_1^*}} \le \|\mathbf{W}^{0,\delta}\|_{A^s_0} e^{h(R,R)\tau_1^*},$$

so that the desired result is now a consequence of the dominated convergence theorem. □

We now study the convergence of the approximate solutions. If we take $\mathbf{U}^{0,\delta}$ as the initial condition, all the results of sections 2.1–2.4 remain valid. In particular, we can construct an approximate profile $\underline{\mathcal{U}}^{\varepsilon,\delta} = \underline{\mathcal{U}}_0^{\varepsilon,\delta} + \varepsilon\underline{\mathcal{U}}_1^{\varepsilon,\delta} + \varepsilon^2\underline{\mathcal{U}}_2^{\varepsilon,\delta}$ of $\mathbf{U}^{\varepsilon,\delta}$. The leading term $\underline{\mathcal{U}}_0^{\varepsilon,\delta}$ satisfies

$$\underline{\mathcal{U}}_0^{\varepsilon,\delta}(\tau,X,Y,Z,t_0,z_0) = \mathcal{U}_0^\delta(\tau,\tau/\varepsilon,X,Y,Z,t_0,z_0),$$

with $\mathcal{U}_0^\delta = \mathcal{U}_{0,I} + \mathcal{U}_{0,II}^\delta$. The discrete spectrum component $\mathcal{U}_{0,I}$ (which does not depend on δ) is given as before by system (2.27), while $\mathcal{U}_{0,II}^\delta = \pi(D_{t_0,z_0})\mathcal{U}_{0,II}^\delta$ is found solving

$$(2.38)\qquad \begin{cases} (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II}^\delta = 0, \\ \partial_\tau \mathcal{U}_{0,II}^\delta + i\frac{\omega'(D_{z_0})}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II}^\delta + i\frac{\omega''(D_{z_0})}{2}\partial_Z^2\mathcal{U}_{0,II}^\delta = 0, \\ \mathcal{U}_{0,II}^\delta|_{\tau=T=0} = \mathbf{U}_{II}^{0,\delta}. \end{cases}$$

The following proposition shows that when $\delta \to 0$, the profile $\mathcal{U}_{0,II}^\delta$ tends to the profile $\mathcal{U}_{0,II}$ obtained formally by taking $\delta = 0$ in (2.38).

PROPOSITION 2.13. *Let $\sigma \ge s$ such that $\mathbf{U}_{II}^0 \in A_0^\sigma$. Suppose, moreover, that $\pi(D_{t_0,z_0})\mathbf{U}_{II}^0 = \mathbf{U}_{II}^0$.*

Then there exists $\tau_2^* > 0$ such that the solution $\mathcal{U}_{0,II}^\delta$ of (2.38) exists in $A^\sigma_{\tau_2^*}$ for all $0 < \delta < 1$ and such that the limit system

$$(2.39)\qquad \begin{cases} (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau\partial_{z_0}\mathcal{U}_{0,II} - \frac{\omega'(D_{z_0})}{2}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} - \frac{D_{z_0}\omega''(D_{z_0})}{2}\partial_Z^2\mathcal{U}_{0,II} = 0, \\ \mathcal{U}_{0,II}|_{\tau=T=0} = \mathbf{U}_{II}^0 \end{cases}$$

admits a unique solution in $A^\sigma_{\tau_2^*}$.

Moreover, $\mathcal{U}^{\delta}_{0,II} \to \mathcal{U}_{0,II}$ *in* $A^{\sigma}_{\tau_2^*}$ *as* $\delta \to 0$.

Proof. The results of the proposition are easily obtained from the explicit expression of $\mathcal{U}^{\delta}_{0,II}$ given by (2.31) and from the dominated convergence theorem. ◻

Remark 2.10. (i) Since $\omega'(k)$ is an even function and $\omega''(k)$ an odd function of (ω, k), the Fourier multipliers $\omega'(D_{z_0})$ and $D_{z_0}\omega''(D_{z_0})$ transform real functions of the variable (t_0, z_0) into real functions.

(ii) System (2.39) is formally obtained by differentiating (2.38) and letting $\delta \to 0$. In (2.39), there is no D_{z_0} inverse (which is not a Fourier multiplier), and therefore this system can be solved explicitly in the Fourier domain.

We are now ready to state our main theorem.

THEOREM 2.14. *Suppose the characteristic variety* $\mathcal{C}_{\mathcal{L}}$ *is as in Assumptions* 2.1 *and* 2.3.

Let $\mathbf{U}^0 = \mathbf{U}^0_I + \mathbf{U}^0_{II} \in A^{s+4}_0$ *such that* $\mathbf{U}^0_I = \mathbf{U}^0_{I,1}e^{i\theta} + c.c.$ *and suppose, moreover, that*

$$\pi(\omega_l, -k_l)\mathbf{U}^0_{I,1} = \mathbf{U}^0_{I,1} \quad \textit{and} \quad \pi(D_{t_0,z_0})\mathbf{U}^0_{II} = \mathbf{U}^0_{II}.$$

Then for $\tau_3^* = \min\{\tau_1^*, \tau_2^*\}$ *we have*

(i) *the exact solution* $\mathbf{u}^\varepsilon$ *of* (1.1) *exists on* $[0, \tau_3^*/\varepsilon]$ *and can be written* $\mathbf{u}^\varepsilon(T, X, Y, Z) = \sqrt{\varepsilon}\mathbf{U}^\varepsilon(\varepsilon T, X, Y, Z, T/\varepsilon, Z/\varepsilon)$, *with* $\mathbf{U}^\varepsilon = \mathbf{U}^\varepsilon_I + \mathbf{U}^\varepsilon_{II} \in B^{s+4}_{\tau_3^*}$;

(ii) $\mathcal{U}_{0,I,1}$ *is defined in* $\mathcal{C}_b([0, \tau_3^*] \times \mathbb{R}_T, H^{s+4}(\mathbb{R}^3)^n)$ *as the unique solution of* (2.27);

(iii) $\mathcal{U}_{0,II}$ *is defined in* $A^{s+4}_{\tau_3^*}$ *as the unique solution of* (2.39);

(iv) *the profile* $\underline{\mathcal{U}}^\varepsilon_0 \in B^s_{\tau_3^*}$ *associated to* $\mathcal{U}_0 = \mathcal{U}_{0,I} + \mathcal{U}_{0,II} \in A^s_{\tau_3^*}$, *with* $\mathcal{U}_{0,I} = \mathcal{U}_{0,I,1}e^{i\theta} + c.c.$, *approximates the singular equation* (1.5) *in the sense that*

$$\|\mathbf{U}^\varepsilon_I - \underline{\mathcal{U}}^\varepsilon_{0,I}\|_{B^s_{\tau_3^*}} = O(\varepsilon) \quad \textit{and} \quad \|\mathbf{U}^\varepsilon_{II} - \underline{\mathcal{U}}^\varepsilon_{0,II}\|_{B^s_{\tau_3^*}} = o(1) \quad \textit{as } \varepsilon \to 0;$$

(v) *we also have stability of the approximate solution* u^ε_0 *defined with* $\mathcal{U}_0$,

$$\|\mathbf{u}^\varepsilon_I - u^\varepsilon_{0,I}\| = O(\varepsilon^{3/2}) \quad \textit{and} \quad \|\mathbf{u}^\varepsilon_{II} - u^\varepsilon_{0,II}\| = o(\sqrt{\varepsilon}) \quad \textit{as } \varepsilon \to 0,$$

where the norm can be taken either in $\mathcal{C}([0, \frac{\tau_3^*}{\varepsilon}] \times \mathbb{R}^3)^n$ *or in* $\mathcal{C}([0, \frac{\tau_3^*}{\varepsilon}], L^2(\mathbb{R}^3)^n)$.

Notation. We have used the same notation $\mathbf{u}^\varepsilon_I$, $\mathbf{u}^\varepsilon_{II}$, $u^\varepsilon_{0,I}$, and $u^\varepsilon_{0,II}$ as in Theorem 2.11.

Proof. (i)–(iii) The first three points have been proved in Theorem 1.4, Proposition 2.6, and Proposition 2.13.

(iv) Convergence of the discrete spectrum components is exactly the same as in Theorem 2.11 since the assumption of the absence of low frequencies only affects the continuous spectrum components.

We now want to prove that $\mathbf{U}^\varepsilon_{II} \to \underline{\mathcal{U}}^\varepsilon_{0,II}$ in $B^s_{\tau_3^*}$, i.e., for all $\mu > 0$, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, $\|\mathbf{U}^\varepsilon_{II} - \underline{\mathcal{U}}^\varepsilon_{0,II}\|_{B^s_{\tau_3^*}} < \mu$.

Now, write

$$\|\mathbf{U}^\varepsilon_{II} - \underline{\mathcal{U}}^\varepsilon_{0,II}\|_{B^s_{\tau_3^*}} \le \|\mathbf{U}^\varepsilon_{II} - \mathbf{U}^{\varepsilon,\delta}_{II}\|_{B^s_{\tau_3^*}} + \|\mathbf{U}^{\varepsilon,\delta}_{II} - \underline{\mathcal{U}}^{\varepsilon,\delta}_{0,II}\|_{B^s_{\tau_3^*}} + \|\underline{\mathcal{U}}^{\varepsilon,\delta}_{0,II} - \underline{\mathcal{U}}^\varepsilon_{0,II}\|_{B^s_{\tau_3^*}},$$

where $\delta > 0$ and $\mathbf{U}^{\varepsilon,\delta}_{II}$ and $\underline{\mathcal{U}}^{\varepsilon,\delta}_{0,II}$ are defined as usual.

Thanks to Propositions 2.12–2.13, we know that for $\delta \le \delta_0$ small enough and for all $\varepsilon \in (0, 1)$, one has

$$\|\mathbf{U}^\varepsilon_{II} - \mathbf{U}^{\varepsilon,\delta}_{II}\|_{B^s_{\tau_3^*}} < \mu/3 \quad \text{and} \quad \|\underline{\mathcal{U}}^{\varepsilon,\delta}_{0,II} - \underline{\mathcal{U}}^\varepsilon_{0,II}\|_{B^s_{\tau_3^*}} \le \|\mathcal{U}^\delta_{0,II} - \mathcal{U}_{0,II}\|_{A^s_{\tau_3^*}} < \mu/3.$$

Moreover, the profile $\underline{\mathcal{U}}_{0,II}^{\varepsilon,\delta}$ satisfies all the assumptions required to apply Theorem 2.11. Therefore, taking $0 < \delta < \inf\{\delta_0, \delta(\mu)\}$, we know that for ε small enough,

$$\|\mathbf{U}_{II}^{\varepsilon,\delta} - \underline{\mathcal{U}}_{0,II}^{\varepsilon,\delta}\|_{B^s_{\tau_3^*}} < \mu/3.$$

The above three inequalities thus yield

$$\|\mathbf{U}_{II}^{\varepsilon} - \underline{\mathcal{U}}_{0,II}^{\varepsilon}\|_{B^s_{\tau_3^*}} < \mu,$$

which proves the result.

(v) This point is a direct consequence of point (iv) and of the embedding properties of Proposition 1.3. □

3. Nondispersive case. In section 2, we have considered dispersive systems. If most of the physical applications fall into this class, nondispersive systems are also physically relevant. Ultrashort pulses, for instance, are often modeled with such a framework. There is also a mathematical reason why we study nondispersive problems in this section: We have seen in the previous section that interactions between oscillations with a purely continuous spectrum are not possible. The proof of this result relies strongly on the dispersive properties of the characteristic variety. We show in this section that when these properties do not hold, nonlinearities can be observed on the continuous spectrum components.

As already mentioned, this nondispersive framework has already been investigated by Alterman and Rauch in [1], [2], [3], and [15]. Of course, our results coincide with theirs, but our method is completely different, and the nondispersive case appears to be a particular case of the general framework presented in this paper and does not require an ad hoc analysis.

The systems we consider here are in the form

$$\left\{ \; L(\partial)\mathbf{u}^\varepsilon + f(\mathbf{u}^\varepsilon) = 0, \mathbf{u}^\varepsilon|_{T=0}(X,Y,Z) = \mathbf{u}^0_\varepsilon(X,Y,Z), \right.$$

with $L(\partial) = A_0\partial_T + A_1\partial_X + A_2\partial_Y + A_3\partial_Z$. We thus consider problems of type (1.1) with $L_0 = 0$. As we have seen in Remark 1.2, we can suppose that $A_0 = Id$.

The symbol $\mathcal{L}(\omega,k)$ then reads $\mathcal{L}(\omega,k) = \omega Id + A_3 k$ and is therefore homogeneous of degree one in (ω,k) so that Assumption 2.1 is never realized. Nevertheless without any additional hypothesis on $L(\partial)$, we know some properties on the characteristic variety $\mathcal{C}_{\mathcal{L}}$. Since $\mathcal{L}(\omega,k)$ is homogeneous of degree one, $\mathcal{C}_{\mathcal{L}}$ is a union of lines which all go through the origin. Moreover, if (ω,k) and (ω',k') are on the same line of $\mathcal{C}_{\mathcal{L}}$, then one has $\pi(\omega,k) = \pi(\omega',k')$. From this point onward, we use the following notation.

Notation. We denote by $\mathcal{D}_1,\dots,\mathcal{D}_N$ the lines such that $\mathcal{C}_{\mathcal{L}} = \mathcal{D}_1 \cup \cdots \cup \mathcal{D}_N$. We denote by $-v_j$ the slope of line $\mathcal{D}_j$. If $(\omega,k) \in \mathcal{D}_j$, write $\pi_j := \pi(\omega,k)$ and $\mathcal{L}_j^{-1} := \mathcal{L}^{-1}(\omega,k)$. Up to a renumbering, we can also suppose that $(\omega_l, -k_l) \in \mathcal{D}_1$.

As the study of the nondispersive case does not raise many other difficulties than in the dispersive case, most of the following results are given without proof.

3.1. The profile equations. As in section 2, an approximate solution is sought in the form

$$u^\varepsilon(T,X,Y,Z) = \sqrt{\varepsilon}\,\mathcal{U}^\varepsilon\left(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon}\right),$$

where the profile $\mathcal{U}^\varepsilon$ is written

$$\mathcal{U}^\varepsilon = \mathcal{U}_0 + \varepsilon\mathcal{U}_1 + \varepsilon^2\mathcal{U}_2,$$

with $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2 \in E^s_{\tau^*}$.

Thanks to Proposition 1.10, these profiles are decomposed into a component with a discrete spectrum and a component with a purely continuous one,

$$\mathcal{U}_j = \mathcal{U}_{j,I} + \mathcal{U}_{j,II}, \qquad j = 0, 1, 2.$$

We recall that the profiles which are labeled I always have a discrete spectrum and the profiles which are labeled II always have a purely continuous one.

The analysis of the discrete spectrum components slightly differs from the analysis performed for the dispersive case. Indeed, the superior harmonics created by the nonlinearity are noncharacteristic in the dispersive case and thus do not play any important role because they are not propagated; conversely, in the nondispersive case, the superior harmonics are characteristic, and so we must seek $\mathcal{U}_{0,I}, \mathcal{U}_{1,I}$, and $\mathcal{U}_{2,I}$ as periodic functions. Since the nonlinearity is odd, only odd harmonics are created by the nonlinearity, if no even harmonic is present initially. Therefore, we look for profiles of the form

$$\mathcal{U}_{l,I}(\tau, T, X, Y, Z, \theta) = \sum_{j \in Z} \mathcal{U}_{l,I,2j+1}(\tau, T, X, Y, Z) e^{i(2j+1)\theta}, \qquad l = 0, 1, 2.$$

In order for these profiles to be in $E^s_{\tau^*}$, one must have normal convergence of the harmonics, and that is why we introduce the following spaces.

Definition 3.1. *We denote by D^s_0 (resp., $D^s_{\tau^*}$) the set of the sequences of profiles $(\mathcal{V}_{2j+1})_{j\in Z}$, with $\mathcal{V}_{2j+1} \in H^s(\mathbb{R}^3)^n$ (resp., $\mathcal{C}_b([0,\tau^*] \times \mathbb{R}_T, H^s(\mathbb{R}^3)^n)$) and such that*

$$\sum_{j\in Z} ||\mathcal{V}_{2j+1}|| < \infty,$$

where $||.||$ represents the norm of $H^s(\mathbb{R}^3)^n$ (resp., $\mathcal{C}_b([0,\tau^*] \times \mathbb{R}_T, H^s(\mathbb{R}^3)^n)$).

This finite positive number endows D^s_0 (resp., $D^s_{\tau^*}$) with a norm, denoted by $\|\cdot\|_{D^s_0}$ (resp., $\|\cdot\|_{D^s_{\tau^*}}$).

Annihilating $\mathcal{R}_{-1,I}$ yields as usual the polarization condition

$$\pi_1 \mathcal{U}_{0,I,2j+1} = \mathcal{U}_{0,I,2j+1} \quad \forall j \in Z. \tag{3.1}$$

As in section 2.1.2 and thanks to Lemma 1.7, the annihilation of $\mathcal{R}_{0,I}$ is equivalent to

$$\begin{cases} \pi_1 L_1(\partial)\pi_1 \mathcal{U}_{0,I,2j+1} = 0, \\ (Id - \pi_1)\mathcal{U}_{1,I,2j+1} = \dfrac{i}{2j+1}\mathcal{L}_1^{-1} A(\partial_{X,Y,Z})\mathcal{U}_{0,I,2j+1}, \end{cases} \tag{3.2}$$

and as in the dispersive case, we can impose

$$\pi_1 \mathcal{U}_{1,I,2j+1} = 0 \quad \forall j \in Z. \tag{3.3}$$

When we annihilate $\mathcal{R}_{1,I}$ the need for periodic functions appears clearly. Indeed, in the dispersive case, all harmonics different from $\pm\theta$ are solved by elliptic inversion.

This kind of inversion is not possible in the nondispersive case because all harmonics are characteristic. Since all harmonics are odd, the nonlinearity $f(\mathcal{U}_0)_I$ can be written

$$f(\mathcal{U}_0)_I = \Lambda(\mathcal{U}_{0,I,\cdot}) = \sum_{j\in Z} \Lambda_{2j+1}(\mathcal{U}_{0,I,\cdot})e^{i(2j+1)\theta}, \tag{3.4}$$

where the notation $\mathcal{U}_{0,I,\cdot}$ stands for the sequence $(\mathcal{U}_{0,I,2j+1})_{j\in Z}$. The annihilation of $\mathcal{R}_{1,I}$ is then equivalent to

$$i(2j+1)\mathcal{L}(\omega_l, -k_l)\mathcal{U}_{2,I,2j+1} + L_1(\partial)\mathcal{U}_{1,I,2j+1} + \partial_\tau \mathcal{U}_{0,I,2j+1} \\ + \Lambda_{2j+1}(\mathcal{U}_{0,I,\cdot}) = 0$$

for all $j \in Z$. These equations are decomposed, thanks to Lemma 1.7 and (3.2)–(3.3), into

$$(Id - \pi_1)\mathcal{U}_{2,I,2j+1} = -\frac{1}{(2j+1)^2}\mathcal{L}_1^{-1}L_1(\partial)\mathcal{L}_1^{-1}A(\partial_{X,Y,Z})\mathcal{U}_{0,I,2j+1} \\ + \frac{i}{2j+1}\mathcal{L}_1^{-1}\big(\partial_\tau \mathcal{U}_{0,I,2j+1} + \Lambda_{2j+1}(\mathcal{U}_{0,I,\cdot})\big)$$

and

$$\partial_\tau \mathcal{U}_{0,I,2j+1} + i\pi_1 A(\partial_{X,Y,Z})\mathcal{L}_1^{-1}A(\partial_{X,Y,Z})\pi_1\mathcal{U}_{0,I,2j+1} + \pi_1\Lambda_{2j+1}(\mathcal{U}_{0,I,\cdot}) = 0.$$

Under Assumption 2.3, the same simplifications as in section 2.1.4 can be made using (3.1)–(3.3) and Proposition 2.4, so that $\mathcal{U}_{0,I,\cdot}$ is found solving

$$\begin{cases} \pi_1\mathcal{U}_{0,I,\cdot} = \mathcal{U}_{0,I,\cdot}, \\ (\partial_T + v_1\partial_Z)\mathcal{U}_{0,I,\cdot} = 0, \\ \partial_\tau\mathcal{U}_{0,I,\cdot} + i\dfrac{v_1}{2k_l}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,I,\cdot} + \pi_1\Lambda(\mathcal{U}_{0,I,\cdot}) = 0 \end{cases} \tag{3.5}$$

in $D^s_{\tau^*}$.

If the nonlinearities are not studied in detail, the analysis of the components with a purely continuous spectrum is strictly the same as in the dispersive case. Provided that the continuous spectrum component $\mathcal{U}_{0,II}$ of $\mathcal{U}_0$ satisfies Assumption 2.2 (absence of low frequencies), we find as in section 2 that $\mathcal{U}_{0,II}$ must satisfy

$$\begin{cases} \pi(D_{t_0,z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}, \\ (\partial_T - \omega'(D_{z_0})\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau\mathcal{U}_{0,II} + i\dfrac{\omega'(D_{z_0})}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} + (\partial_T - \omega'(D_{z_0})\partial_Z)\pi(D_{t_0,z_0})\mathcal{U}_{1,II} \\ \qquad + \pi(D_{t_0,z_0})\psi^\delta(D_{z_0})[f(\mathcal{U}_0)]_{II} = 0, \end{cases} \tag{3.6}$$

where ψ^δ denotes the infrared cutoff introduced in Definition 2.2.

Yet, we can still simplify these equations in decomposing $\mathcal{U}_{0,II}$ in the form

$$\mathcal{U}_{0,II} = \mathcal{U}_{0,II,1} + \cdots + \mathcal{U}_{0,II,N}$$

such that the spectrum of $\mathcal{U}_{0,II,j}$ is included in $\mathcal{D}_j$ for all $j = 1, \ldots, N$.

Hence, (3.6) read

$$\begin{cases} \pi_j \mathcal{U}_{0,II,j} = \mathcal{U}_{0,II,j}, \quad j = 1, \dots, N, \\ (\partial_T + v_j \partial_Z)\mathcal{U}_{0,II,j} = 0, \quad j = 1, \dots, N, \\ \partial_\tau \mathcal{U}_{0,II,j} - i\dfrac{v_j}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II,j} + (\partial_T + v_j\partial_Z)\pi_j \mathcal{U}_{1,II,j} \\ \qquad + \pi_j \psi^\delta(D_{z_0})[f(\mathcal{U}_0)]_{II,j} = 0, \quad j = 1, \dots, N, \end{cases}$$

where we recall that $-v_j$ is the slope of line $\mathcal{D}_j$.

We now study the nonlinearity $\pi_j[f(\mathcal{U}_0)]_{II,j}$ which appears in the profile equations. With the same kind of argument as in the proof of Lemma 2.5, we can obtain the following lemma.

LEMMA 3.2. *Let $\mathcal{V}_{II,j} \in A^s_{\tau^*}$, $j = 1, \dots, N$, be N profiles with a purely continuous spectrum such that* Sp $\mathcal{V}_{II,j} \subset \mathcal{D}_j$. *Take also $a, b \in \mathbb{C}^n$. Then one has*

(i) $\pi_j F(\mathcal{V}_{II,k}, \mathcal{V}_{II,l}, \mathcal{V}_{II,m}) = 0$*, unless $j = k = l = m$;*

(ii) $\pi_j F(ae^{ik\theta}, \mathcal{V}_{II,l}, \mathcal{V}_{II,m}) = 0$ *for all $k \in Z$ unless $j = k = l = m = 1$;*

(iii) $\pi_j F(ae^{ik\theta}, be^{il\theta}, \mathcal{V}_{II,m}) = 0$ *for all $(k, l) \in Z^2$, unless $l + k = 0$ and $j = m$.*

Remark 3.1. The main difference between the dispersive and nondispersive cases is that we can have nonzero interactions between oscillations with a purely continuous spectrum in the nondispersive case. What the lemma says is that, in order to produce a nonzero interaction, these oscillations must have support on the same line as the characteristic variety. Therefore, the evolution equations of the modes $\mathcal{U}_{0,II,j}$ can be nonlinear.

Thanks to Lemma 3.2, the nonlinearity $\pi_j[f(\mathcal{U}_0)]_{II,j}$ may be written in the form

$$\pi_1[f(\mathcal{U}_0)]_{II,1} = \sum_{k\in Z} \pi_1 F^S(\mathcal{U}_{0,I,2k+1}, \overline{\mathcal{U}_{0,I,2k+1}}, \mathcal{U}_{0,II,1}) \\ + \pi_1 f'(\mathcal{U}_{0,II,1})(\mathcal{U}_{0,I}) + \pi_1 f(\mathcal{U}_{0,II,1}),$$

and, when $j \geq 2$,

$$\pi_j[f(\mathcal{U}_0)]_{II,j} = \sum_{k\in Z} \pi_j F^S(\mathcal{U}_{0,I,2k+1}, \overline{\mathcal{U}_{0,I,2k+1}}, \mathcal{U}_{0,II,1}) + \pi_j f(\mathcal{U}_{0,II,j}).$$

3.2. Solving the profile equations. Inspired here again by [10] and [11], and using the expression of the nonlinearities given above, we decompose the equations on $\mathcal{U}_{0,II,j}$ as follows:

$$\begin{cases} \pi_1 \mathcal{U}_{0,II,1} = \mathcal{U}_{0,II,1}, \\ (\partial_T + v_1\partial_Z)\mathcal{U}_{0,II,1} = 0, \\ \partial_\tau \mathcal{U}_{0,II,1} - i\dfrac{v_1}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II,1} \\ \qquad + \psi^\delta(D_{z_0})\pi_1 \sum_{k\in Z} F^S(\mathcal{U}_{0,I,2k+1}, \overline{\mathcal{U}_{0,I,2k+1}}, \mathcal{U}_{0,II,1}) \\ \qquad + \psi^\delta(D_{z_0})\pi_1 f'(\mathcal{U}_{0,II,1})(\mathcal{U}_{0,I}) + \psi^\delta(D_{z_0})\pi_1 f(\mathcal{U}_{0,II,1}) = 0 \\ (\partial_T + v_1\partial_Z)\mathcal{U}_{1,II,1} = 0 \end{cases} \tag{3.7}$$

and for $j \geq 2$,

$$\left\{ \begin{aligned} &\pi_j \mathcal{U}_{0,II,j} = \mathcal{U}_{0,II,j}, \\ &(\partial_T + v_j \partial_Z)\mathcal{U}_{0,II,j} = 0, \\ &\partial_\tau \mathcal{U}_{0,II,j} - i\frac{v_j}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II,j} + \psi^\delta(D_{z_0})\pi_j f(\mathcal{U}_{0,II,j}) = 0 \\ &(\partial_T + v_j \partial_Z)\pi_j \mathcal{U}_{1,II,j} = -\psi^\delta(D_{z_0})\pi_j \sum_{k\in Z} F^S(\mathcal{U}_{0,I,2k+1}, \overline{\mathcal{U}_{0,I,2k+1}}, \mathcal{U}_{0,II,1}). \end{aligned} \right. \tag{3.8}$$

These profile equations can be solved. However, we will restrict ourselves to the case where $\mathcal{U}_{0,II} = \mathcal{U}_{0,II,1}$, i.e., where the spectrum of $\mathcal{U}_{0,II}$ is included in $\mathcal{D}_1$. The general case would be technically more difficult and is irrelevant for the physical examples considered in this paper.

PROPOSITION 3.3. *Let $\sigma \geq s$ and $\mathbb{R} > 0$ such that $\mathbf{U}^0 = \mathbf{U}_I^0 + \mathbf{U}_{II,1}^0 \in A_0^\sigma$ and $\|\mathbf{U}^0\|_{A_0^\sigma} \leq R$. Suppose, moreover, that $\mathbf{U}_I^0 = \sum_{j\in Z} \mathbf{U}_{I,2j+1}^0 e^{i(2j+1)\theta}$ with $\mathbf{U}_{I,\cdot}^0 \in D_0^\sigma$. Assume finally that*

$$\pi_1 \mathbf{U}_{I,1,.}^0 = \mathbf{U}_{I,1,.}^0, \quad \text{Sp } \mathbf{U}_{II,1}^0 \subset \mathcal{D}_1, \quad \textit{and} \quad \pi_1 \mathbf{U}_{II,1}^0 = \mathbf{U}_{II,1}^0.$$

Then there exists $\tau_2^* > 0$, which depends only on R, such that there exists:

- *a unique $\mathcal{U}_{0,I,\cdot} = \pi_1 \mathcal{U}_{0,I,\cdot} \in D_{\tau_2^*}^\sigma$ solution of*

$$\left\{ \begin{aligned} &(\partial_T + v_1 \partial_Z)\mathcal{U}_{0,I,\cdot} = 0, \\ &\partial_\tau \mathcal{U}_{0,I,\cdot} + i\frac{v_1}{2k_l}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,I,1} + \pi_1 \Lambda(\mathcal{U}_{0,I,\cdot}) = 0, \\ &\mathcal{U}_{0,I,\cdot}|_{\tau=T=0} = \mathbf{U}_{I,\cdot}^0, \end{aligned} \right. \tag{3.9}$$

where $\Lambda(\mathcal{U}_{0,I,.})$ is given by (3.4)*;*

- *a unique $\mathcal{U}_{0,II,1}^\delta = \pi_1 \mathcal{U}_{0,II,1}^\delta \in A_{\tau_2^*}^\sigma$ solution of*

$$\left\{ \begin{aligned} &(\partial_T + v_1 \partial_Z)\mathcal{U}_{0,II,1}^\delta = 0, \\ &\partial_\tau \mathcal{U}_{0,II,1}^\delta - i\frac{v_1}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II,1}^\delta \\ &\qquad + \psi^\delta(D_{z_0})\pi_1 \sum_{k\in Z} F^S(\mathcal{U}_{0,I,2k+1}, \overline{\mathcal{U}_{0,I,2k+1}}, \mathcal{U}_{0,II,1}^\delta) \\ &\qquad + \psi^\delta(D_{z_0})\pi_1 f'(\mathcal{U}_{0,II,1}^\delta)(\mathcal{U}_{0,I}) + \psi^\delta(D_{z_0})\pi_1 f(\mathcal{U}_{0,II,1}^\delta) = 0 \\ &\mathcal{U}_{0,II,1}^\delta|_{\tau=T=0} = \psi^\delta(D_{z_0})\mathbf{U}_{II,1}^0. \end{aligned} \right. \tag{3.10}$$

Proof. The first equation of (3.9) is automatically solved by looking for $\mathcal{U}_{0,I,\cdot}$ in the form $\mathcal{U}_{0,I,2j+1}(\tau, T, X, Y, Z) = \mathtt{U}_{0,I,2j+1}(\tau, Z - v_1 T, X, Y)$ for all $j \in Z$. System (3.9) then reduces to the Cauchy problem

$$\left\{ \begin{aligned} &\partial_\tau \mathtt{U}_{0,I,\cdot} + i\frac{v_1}{2k_l}(\partial_X^2 + \partial_Y^2)\mathtt{U}_{0,I,1} + \pi_1 \Lambda(\mathtt{U}_{0,I,\cdot}) = 0, \\ &\mathtt{U}_{0,I,\cdot}|_{\tau=0} = \mathbf{U}_{I,\cdot}^0, \end{aligned} \right.$$

which is easily solved by Picard iterates in $\mathcal{C}([0, \tau_2^*], D_0^\sigma)$ since its linear part defines a unitary group on D_0^σ, which is a Banach algebra.

For the continuous spectrum component, we cannot give an explicit expression of the solution as in Proposition 2.6, because we have to deal with the nonlinearities. However, the presence of nonlinearities is counterbalanced by the simple form of the transport equation. (We have here a common group velocity for all the frequencies.) In order for $\mathcal{U}^\delta_{0,II,1}$ to satisfy this transport equation, we look for it in the form

$$\mathcal{U}^\delta_{0,II,1}(\tau,T,X,Y,Z,t_0,z_0) = \mathtt{U}^\delta_{0,II,1}(\tau,Z-v_1T,X,Y,t_0,z_0),$$

so that (3.10) reduces to the Cauchy problem

$$\begin{cases} \partial_\tau \mathtt{U}^\delta_{0,II,1} - i\dfrac{v_1}{2D_{z_0}}(\partial_X^2+\partial_Y^2)\mathtt{U}^\delta_{0,II,1} \\ \qquad + \psi^\delta(D_{z_0})\pi_1 \displaystyle\sum_{k\in Z} F^S(\mathtt{U}_{0,I,2k+1},\overline{\mathtt{U}_{0,I,2k+1}},\mathtt{U}^\delta_{0,II,1}) \\ \qquad + \psi^\delta(D_{z_0})\pi_1 f'(\mathtt{U}^\delta_{0,II,1})(\mathtt{U}_{0,I}) + \psi^\delta(D_{z_0})\pi_1 f(\mathtt{U}^\delta_{0,II,1}) = 0 \\ \mathtt{U}^\delta_{0,II,1}|_{\tau=0} = \psi^\delta(D_{z_0})\mathbf{U}^0_{II,1}. \end{cases}$$

This Cauchy problem is solved in $B^\sigma_{\tau_2^*}$ by Picard iterates, using estimates similar to those of Lemma 1.5. ∎

Remark 3.2. (i) With $\mathcal{U}_0$ being given by Proposition 3.3, system (3.7) is then solved by taking $\pi_1\mathcal{U}_{1,II,1} = 0$.

(ii) The results of Proposition 3.3 also hold for $\delta = 0$, i.e., there exists a unique solution $\mathcal{U}_{0,II} = \mathcal{U}_{0,II,1} = \pi_1\mathcal{U}_{0,II,1}$ to

$$\begin{cases} (\partial_T + v_1\partial_Z)\mathcal{U}_{0,II,1} = 0, \\ \partial_\tau\partial_{z_0}\mathcal{U}_{0,II,1} + \dfrac{v_1}{2}(\partial_X^2+\partial_Y^2)\mathcal{U}_{0,II,1} \\ \qquad + \pi_1 \displaystyle\sum_{k\in Z}\partial_{z_0} F^S(\mathcal{U}_{0,I,2k+1},\overline{\mathcal{U}_{0,I,2k+1}},\mathcal{U}_{0,II,1}) \\ \qquad + \pi_1\partial_{z_0} f'(\mathcal{U}_{0,II,1})(\mathcal{U}_{0,I}) + \pi_1\partial_{z_0} f(\mathcal{U}_{0,II,1}) = 0, \\ \mathcal{U}_{0,II,1}|_{\tau=T=0} = \mathbf{U}^0_{II,1}, \end{cases} \tag{3.11}$$

and the convergence property of Proposition 2.13 can easily be extended to the present case.

3.3. Validity of the approximation. We show here that the approximate solution

$$u^\varepsilon(T,X,Y,Z) = \sqrt{\varepsilon}\,\mathcal{U}^\varepsilon\left(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon}\right),$$

with $\mathcal{U}^\varepsilon = \mathcal{U}_0 + \varepsilon\mathcal{U}_1 + \varepsilon^2\mathcal{U}_2$, is a good approximation of the exact solution of (1.1). As in section 2, the proof cannot be direct because the presence of low frequencies and $\mathcal{L}^{-1}$-regularity are not compatible. But we can mimic the reasoning of section 2 to obtain a stability result. Before stating the theorem, we recall that to any profile $\mathcal{V} \in A^\sigma_{\tau^*}$, we associate the profile $\underline{\mathcal{V}}^\varepsilon$ defined as

$$\underline{\mathcal{V}}^\varepsilon(\tau,X,Y,Z,t_0,z_0) := \mathcal{V}\left(\tau, \frac{\tau}{\varepsilon}, X, Y, Z, t_0, z_0\right).$$

THEOREM 3.4. *Suppose the characteristic variety $\mathcal{C}_\mathcal{L}$ is as in Assumption* 2.3.

Let $\mathbf{U}^0 = \mathbf{U}_I^0 + \mathbf{U}_{II,1}^0 \in A_0^{s+4}$ *such that* $\mathbf{U}_I^0 = \sum_{j\in Z} \mathbf{U}_{I,2j+1}^0 e^{i(2j+1)\theta}$, *with* $\mathbf{U}_{I,\cdot}^0 \in D_0^{s+4}$, *and suppose that*

$$\pi_1 \mathbf{U}_{I,\cdot}^0 = \mathbf{U}_{I,\cdot}^0, \quad \mathrm{Sp}\ \mathbf{U}_{II,1}^0 \subset \mathcal{D}_1, \quad \textit{and} \quad \pi_1 \mathbf{U}_{II,1}^0 = \mathbf{U}_{II,1}^0.$$

Then for $\tau_3^* = \min\{\tau_1^*, \tau_2^*\}$, *we have the following:*

(i) *The exact solution* $\mathbf{u}^\varepsilon$ *of* (1.1) *exists on* $[0, \tau_3^*/\varepsilon]$ *and can be written* $\mathbf{u}^\varepsilon(T, X, Y, Z) = \sqrt{\varepsilon}\mathbf{U}^\varepsilon(\varepsilon T, X, Y, Z, T/\varepsilon, Z/\varepsilon)$, *with* $\mathbf{U}^\varepsilon = \mathbf{U}_I^\varepsilon + \mathbf{U}_{II}^\varepsilon \in B_{\tau_3^*}^{s+4}$.

(ii) $\mathcal{U}_{0,I,\cdot}$ *is defined in* $D_{\tau_3^*}^{s+4}$ *as the unique solution of* (3.9), *and we define* $\mathcal{U}_{0,I}$ *as* $\mathcal{U}_{0,I} = \sum_{j\in Z} \mathcal{U}_{0,I,2j+1} e^{i(2j+1)\theta}$.

(iii) $\mathcal{U}_{0,II} = \mathcal{U}_{0,II,1}$ *is defined in* $A_{\tau_3^*}^{s+4}$ *as the unique solution of* (3.11).

(iv) *The profile* $\underline{\mathcal{U}}_0^\varepsilon \in B_{\tau_3^*}^s$ *associated to* $\mathcal{U}_0 = \mathcal{U}_{0,I} + \mathcal{U}_{0,II} \in A_{\tau_3^*}^s$ *approximates the singular equation* (1.5) *in the sense that*

$$\|\mathbf{U}_I^\varepsilon - \underline{\mathcal{U}}_{0,I}^\varepsilon\|_{B_{\tau_3^*}^s} = O(\varepsilon) \quad \textit{and} \quad \|\mathbf{U}_{II}^\varepsilon - \underline{\mathcal{U}}_{0,II}^\varepsilon\|_{B_{\tau_3^*}^s} = o(1).$$

(v) *We also have stability of the approximate solution* u_0^ε *defined with* $\mathcal{U}_0$,

$$\|\mathbf{u}_I^\varepsilon - u_{0,I}^\varepsilon\| = O(\varepsilon^{3/2}) \quad \textit{and} \quad \|\mathbf{u}_{II}^\varepsilon - u_{0,II}^\varepsilon\| = o(\sqrt{\varepsilon}),$$

where the norm can be taken either in $\mathcal{C}([0, \frac{\tau_3^*}{\varepsilon}] \times \mathbb{R}^3)^n$ *or in* $\mathcal{C}([0, \frac{\tau_3^*}{\varepsilon}], L^2(\mathbb{R}^3)^n)$.

4. Examples.

4.1. Lasers with large spectrums. As said in the introduction, we want to study the effects due to the fact that certain lasers have frequencies and wavenumbers which dribble around the theoretical value in a range greater than $O(\varepsilon)$ (typically $O(1)$). In order to make a model out of this phenomenon, we add to the theoretical (sinusoidal) oscillations a corrector with a purely continuous spectrum.

To describe the evolution of the electromagnetic field we use Maxwell equations coupled to a response of the medium by the polarization $\mathbf{p}$, which is described by the anharmonic oscillator model [14]. Once nondimensionalized [7], and omitting the divergence-free equations, the system reads

$$(M) \qquad \begin{cases} \partial_T \mathbf{e}^\varepsilon - \mathbf{curl}\ \mathbf{b}^\varepsilon + \dfrac{\sqrt{\gamma_a}}{\varepsilon}\mathbf{g}^\varepsilon & = 0, \\ \partial_T \mathbf{b}^\varepsilon + \mathbf{curl}\ \mathbf{e}^\varepsilon & = 0, \\ \partial_T \mathbf{p}^\varepsilon - \dfrac{\eta_a}{\varepsilon}\mathbf{q}^\varepsilon & = 0, \\ \partial_T \mathbf{q}^\varepsilon - \dfrac{1}{\varepsilon}(\sqrt{\gamma_a}\mathbf{e}^\varepsilon - \eta_a \mathbf{p}^\varepsilon) - \alpha\gamma_a^{3/2}|\mathbf{p}^\varepsilon|^2\mathbf{p}^\varepsilon & = 0. \end{cases}$$

This system (M) is of type (1.1),

$$L^\varepsilon(\partial)\mathbf{u}^\varepsilon + f(\mathbf{u}^\varepsilon) = 0,$$

with

$$\mathbf{u}^\varepsilon = (\mathbf{e}^\varepsilon, \mathbf{b}^\varepsilon, \mathbf{p}^\varepsilon, \mathbf{q}^\varepsilon)^T \in C^{12},$$

and the nonlinearity is of order 3 and reads

$$f(\mathbf{e}^\varepsilon, \mathbf{b}^\varepsilon, \mathbf{p}^\varepsilon, \mathbf{q}^\varepsilon) = (0, 0, 0, \alpha\gamma_a^{3/2}|\mathbf{p}^\varepsilon|^2\mathbf{p}^\varepsilon)^T.$$

Remark 4.1. The fact that the mapping F associated to f is not trilinear as in Assumption 1.2—since it is semilinear in one of its variables—is not important. Indeed, considering $\tilde{\mathbf{u}}^\varepsilon = (\mathbf{u}^\varepsilon, \overline{\mathbf{u}^\varepsilon}) \in C^{24}$ brings us back to this case.

The operator $L^\varepsilon(\partial)$ reads $L^\varepsilon(\partial) = \partial_T + A_1\partial_X + A_2\partial_Y + A_3\partial_Z + L_0/\varepsilon$, with

$$A_1\partial_X + A_2\partial_Y + A_3\partial_Z = \begin{pmatrix} 0 & -\mathbf{curl} & 0 & 0 \\ \mathbf{curl} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$L_0 = \begin{pmatrix} 0 & 0 & 0 & \sqrt{\gamma_a} Id \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta_a Id \\ -\sqrt{\gamma_a} Id & 0 & \eta_a Id & 0 \end{pmatrix}.$$

The characteristic variety $\mathcal{C}_\mathcal{L}$ associated to the symbol $\mathcal{L}(\omega, k) = \omega Id + kA_3 + L_0/i$ is defined by the algebraic equation

$$\omega^2(\omega^2 - 1 - \gamma_a)[(\omega^2 - \eta_a^2)(\omega^2 - k^2) - \gamma_a\omega^2]^2 = 0.$$

This characteristic variety has three singular points which are all of abscissa $k = 0$ and ordinate $0; \pm\sqrt{1+\gamma_a}$. If $(\omega_l, -k_l)$ is on a curved sheet of $\mathcal{C}_\mathcal{L}$, i.e., if $(\omega_l^2 - \eta_a^2)(\omega_l^2 - k_l^2) - \gamma_a\omega_l^2 = 0$, then the group velocity $\omega'(k_l)$ is given by

$$\omega'(k_l) = \frac{k_l}{\omega_l} \frac{\omega_l^2 - \eta_a^2}{(\omega_l^2 - k_l^2) + (\omega_l^2 - \eta_a^2) - \gamma_a}, \tag{4.1}$$

while the dispersive factor $\omega''(k_l)$ reads

$$\omega''(k_l) = \omega'(k_l)\frac{\omega_l - \omega'(k_l)}{\omega_l k_l} - 4\frac{\omega_l\omega'(k_l)^2}{k_l}\frac{\omega_l\omega'(k_l) - k_l}{\omega_l^2 - \eta_a^2}. \tag{4.2}$$

Notice that $\mathcal{C}_\mathcal{L}$ contains three plane sheets, so that Assumption 2.1 is not satisfied since these sheets are parallel. However, as mentioned earlier, the divergence-free conditions satisfied by the electromagnetic field allow us to consider that Assumption 2.1 is fulfilled.

We consider initial conditions of the form

$$\mathbf{u}_\varepsilon^0 = \varepsilon^{1/2}\mathbf{U}^0(X, Y, Z, 0, Z/\varepsilon) = \varepsilon^{1/2}(\mathbf{E}^0, \mathbf{B}^0, \mathbf{P}^0, \mathbf{Q}^0)(X, Y, Z, 0, Z/\varepsilon),$$

where $\mathbf{U}^0$ is written $\mathbf{U}^0(X, Y, Z, t_0, z_0) = \mathbf{U}_{I,1}^0(X, Y, Z)e^{i\theta} + \text{c.c.} + \mathbf{U}_{II}^0(X, Y, Z, t_0, z_0)$, with the component $\mathbf{U}_{II}^0$ having a purely continuous spectrum. As mentioned above, $\mathbf{U}_{I,1}^0$ corresponds to the usual (small-spectrum) laser, i.e., the laser with time-space wavenumber equal to $(\omega_l, -k_l)$, while $\mathbf{U}_{II}^0$ corresponds to the large dribbling.

Moreover, the initial conditions are polarized,

$$\pi(\omega_l, -k_l)\mathbf{U}_{I,1}^0 = \mathbf{U}_{I,1}^0 \quad \text{and} \quad \pi(D_{t_0,z_0})\mathbf{U}_{II}^0 = \mathbf{U}_{II}^0.$$

The solution of diffractive optics reads

$$u^\varepsilon(T, X, Y, Z) = \varepsilon^{1/2}(\mathcal{U}_{0,I,1}(\varepsilon T, T, X, Y, Z)e^{i(\omega_l \frac{T}{\varepsilon} - k_l \frac{Z}{\varepsilon})} + \text{c.c.} \\ + \mathcal{U}_{0,II}\left(\varepsilon T, T, X, Y, Z, \frac{T}{\varepsilon}, \frac{Z}{\varepsilon}\right),$$

and the results of section 2 state that the profile $\mathcal{U}_{0,I,1}$ is given by

$$\partial_\tau \mathcal{U}_{0,I,1} + i\frac{\omega'(k_l)}{2k_l}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,I,1} + i\frac{\omega''(k_l)}{2}\partial_Z^2\mathcal{U}_{0,I,1} + \pi(\omega_l, -k_l)f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}}) = 0,$$

with here

$$f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}}) = \left(0,0,0,\alpha\gamma_a^{3/2}\left(2|\overrightarrow{\mathcal{P}}_{0,I,1}|^2\overrightarrow{\mathcal{P}}_{0,I,1} + (\overrightarrow{\mathcal{P}}_{0,I,1}.\overrightarrow{\mathcal{P}}_{0,I,1})\overline{\overrightarrow{\mathcal{P}}_{0,I,1}}\right)\right)^T$$

and ω' and ω'' given by (4.1)–(4.2).

As the waves which we consider here propagate along (OZ) with a wavenumber $\overrightarrow{k_l} = (0,0,k_l)$, the electric field is polarized on the plane (OXY). We can assume that it is polarized along (OX), i.e., $\overrightarrow{\mathcal{E}}_{0,I,1} = (\mathcal{E}_{0,I,1},0,0)^T$. Since $\pi(\omega_l,-k_l)\mathcal{U}_{0,I,1} = \mathcal{U}_{0,I,1}$, one therefore has $\overrightarrow{\mathcal{P}}_{0,I,1} = \eta_a\chi(\omega_l)\overrightarrow{\mathcal{E}}_{0,I,1}$, where the dielectric susceptibility $\chi(\omega_l)$ is given by

$$\chi(\omega_l) = \frac{\sqrt{\gamma_a}}{\eta_a^2 - \omega_l^2}.$$

The nonlinearity therefore reads

$$f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}}) = (0,0,0,3\alpha\gamma_a^{3/2}\eta_a^3\chi(\omega_l)^3|\mathcal{E}_{0,I,1}|^2\mathcal{E}_{0,I,1},0,0)^T.$$

In order to obtain the evolution equation on $\mathcal{E}_{0,I,1}$, one needs to compute the nonlinearity $\pi(\omega_l,-k_l)f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}})$ (in fact, computing its first component is enough). For all vectors $a = (a_1,0,0)^T \in C^3$, one has

$$\pi(\omega_l,-k_l)\begin{pmatrix}0\\0\\0\\a\end{pmatrix} = -\frac{i\omega_l\sqrt{\gamma_a}}{\eta_a^2-\omega_l^2}\begin{pmatrix}\frac{a_1}{N^2}\\ \cdot\\ \cdot\\ \cdot\end{pmatrix}$$

with

$$\begin{aligned} N^2 &= 1 + \frac{k_l^2}{\omega_l^2} + \eta_a^2\chi^2(\omega_l) + \omega_l^2\chi^2(\omega_l)\\ &= \sqrt{\gamma_a}\left(\frac{k_l^2+\omega_l^2}{k_l^2-\omega_l^2} + \frac{\eta_a^2+\omega_l^2}{\eta_a^2-\omega_l^2}\right)\chi(\omega_l)\\ &:= \sqrt{\gamma_a}\beta(\omega_l,k_l)\chi(\omega_l). \end{aligned}$$

We thus find

$$\pi(\omega_l,-k_l)f'(\mathcal{U}_{0,I,1})(\overline{\mathcal{U}_{0,I,1}}) = -3i\alpha\gamma_a\eta_a^3\frac{\omega_l}{\beta(\omega_l,k_l)}\chi(\omega_l)^3(|\mathcal{E}_{0,I,1}|^2\mathcal{E}_{0,I,1},\ldots),$$

and the evolution equation on $\mathcal{E}_{0,I,1}$ is therefore

$$\begin{aligned} &\partial_\tau\mathcal{E}_{0,I,1} + i\frac{\omega'(k_l)}{2k_l}(\partial_X^2+\partial_Y^2)\mathcal{E}_{0,I,1} + i\frac{\omega''(k_l)}{2}\partial_Z^2\mathcal{E}_{0,I,1}\\ &= -3i\alpha\gamma_a\eta_a^3\frac{\omega_l}{\beta(\omega_l,k_l)}\chi(\omega_l)^3|\mathcal{E}_{0,I,1}|^2\mathcal{E}_{0,I,1}. \end{aligned}$$

The purely continuous spectrum component of $\mathcal{U}_0$ is found solving

$$\partial_\tau \partial_{z_0} \mathcal{U}_{0,II} - \frac{\omega'(D_{z_0})}{2}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} - \frac{D_{z_0}\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{U}_{0,II} = 0.$$

Supposing as above that $\mathcal{E}_{0,II}$ is polarized along (OX), i.e., $\overrightarrow{\mathcal{E}}_{0,II} = (\mathcal{E}_{0,II}, 0, 0)^T$, we obtain

$$\partial_\tau \partial_{z_0} \mathcal{E}_{0,II} - \frac{\omega'(D_{z_0})}{2}(\partial_X^2 + \partial_Y^2)\mathcal{E}_{0,II} - \frac{D_{z_0}\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{E}_{0,II} = 0.$$

Remark 4.2. A usual direct computation of this equation without the help of the general results proved above would have led to a nonlinear equation. It was not obvious a priori that all the nonlinear terms would be negligible.

4.2. Short pulses. The second application we study concerns short-pulse lasers. Normally, the length of the pulse of a laser is long enough to contain many oscillations so that the phenomena considered can be described well enough by knowing the evolution of the envelope of these oscillations. For ultrashort pulses, this is no longer true (see Figure 1) since there may even be less than an oscillation. The profile we use to make a model of this phenomenon therefore has only a purely continuous spectrum component: the sinusoidal one (discrete spectrum) does not have time to appear. More precisely, we consider an initial condition for (M) of the form

$$\mathbf{u}_\varepsilon^0 = \varepsilon^{1/2}\mathbf{U}^0(X, Y, Z, 0, Z/\varepsilon) = \varepsilon^{1/2}(\mathbf{E}^0, \mathbf{B}^0, \mathbf{P}^0, \mathbf{Q}^0)(X, Y, Z, 0, Z/\varepsilon),$$

where $\mathbf{U}^0 \in A_0^s$ has a purely continuous spectrum. Moreover the initial condition is polarized as follows:

$$\pi(D_{t_0,z_0})\mathbf{U}^0 = \mathbf{U}^0.$$

In accordance with the results of section 2, the profile $\mathcal{U}_0 = \mathcal{U}_{0,II}$ of the approximate solution is found solving

$$\partial_\tau \partial_{z_0} \mathcal{U}_0 - \frac{\omega'(D_{z_0})}{2}(\partial_X^2 + \partial_Y^2)\mathcal{U}_0 - \frac{D_{z_0}\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{U}_0 = 0.$$

Considering the same model (M) as in the previous section and using the same notation thus yield the following equation for the nonzero component $\mathcal{E}_0$ of the electric field:

$$\partial_\tau \partial_{z_0} \mathcal{E}_0 - \frac{\omega'(D_{z_0})}{2}(\partial_X^2 + \partial_Y^2)\mathcal{E}_0 - \frac{D_{z_0}\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{E}_0 = 0,$$

where ω' and ω'' are given by (4.1)–(4.2).

Remark 4.3. Here again the fact that we would obtain a linear equation is not obvious. We also point out the fact that our framework allows us to find this equation in the physical *dispersive* case.

In a nondispersive framework, the nonlinearities would not have vanished. Assuming that the spectrum of $\mathcal{U}_0 = \mathcal{U}_{0,II}$ is located on the line $\mathcal{D}_1$ of $\mathcal{C}_\mathcal{L}$ to which $(\omega_l, -k_l)$ belongs, we would find

$$\partial_\tau \partial_{z_0} \mathcal{U}_0 + \frac{v_1}{2}(\partial_X^2 + \partial_Y^2)\mathcal{U}_0 + \pi_1 \partial_{z_0} f(\mathcal{U}_0) = 0,$$

where $-v_1$ is the slope of D_1. This is Alterman and Rauch's equation [1], [2], [3].

5. Weakly dispersive case. We have seen in the previous sections that there is a radical difference between the behavior of the approximate solution of the dispersive case and that of the nondispersive case. In the former, the evolution of the continuous spectrum mode is uncoupled with the discrete spectrum mode and is linear; in the latter, it is coupled and nonlinear.

This behavioral gap means that the estimate $o(1)$ of Theorem 2.14 is somewhat critical for weakly dispersive systems.We propose in this section a few hints to improve this result.

In terms of BKW strategy, solving (2.23) or the following equations is equivalent:

$$(5.1)\quad \begin{cases} \pi(D_{t_0}, D_{z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}, \\ (\partial_T + \omega'(k_l)\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau \mathcal{U}_{0,II} + i\dfrac{\omega'(D_{z_0})}{2D_{z_0}}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} + \dfrac{-\omega'(k_l) - \omega'(D_{z_0})}{\varepsilon}\partial_Z \mathcal{U}_{0,II} \\ \qquad + i\dfrac{\omega''(D_{z_0})}{2}\partial_Z^2 \mathcal{U}_{0,II} = 6\pi(D_{t_0}, D_{z_0})F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}). \end{cases}$$

Note that the nonlinearity has been simplified according to the results of Lemmas 2.5 and 3.2, and that here $\pi_1(D_{t_0}, D_{z_0})\mathcal{U}_{1,II} = 0$.

In general, the Fourier multiplier $\frac{-\omega'(k_l)-\omega'(D_{z_0})}{\varepsilon}$ is of size $O(1/\varepsilon)$ and solving (2.23) is much more relevant than solving (5.1) since this former system does not involve ε. However, if the model considered is weakly dispersive in the sense that one can choose ω_0' in such a way that $\frac{-\omega'(k_l)-\omega'(D_{z_0})}{\varepsilon} = O(1)$, system (5.1) becomes more interesting because the transport equation is the same for all frequencies, as in the nondispersive case. Since it is also reasonable to suppose that such weakly dispersive systems also satisfy $\omega''(D_{z_0}) = O(\varepsilon)$, system (5.1) is equivalent in terms of BKW strategy to

$$(5.2)\quad \begin{cases} \pi(D_{t_0}, D_{z_0})\mathcal{U}_{0,II} = \mathcal{U}_{0,II}, \\ (\partial_T + \omega'(k_l)\partial_Z)\mathcal{U}_{0,II} = 0, \\ \partial_\tau\partial_{z_0} \mathcal{U}_{0,II} + \dfrac{\omega'(k_l)}{2}(\partial_X^2 + \partial_Y^2)\mathcal{U}_{0,II} + \dfrac{-\omega'(k_l) - \omega'(D_{z_0})}{\varepsilon}\partial_Z\partial_{z_0} \mathcal{U}_{0,II} \\ \qquad = 6\pi(D_{t_0}, D_{z_0})\partial_{z_0}F^S(\mathcal{U}_{0,I,1}, \overline{\mathcal{U}_{0,I,1}}, \mathcal{U}_{0,II}), \end{cases}$$

where we also have differentiated the last equation with respect to z_0.

Therefore, in the weakly dispersive case, one can obtain profile equations for $\mathcal{U}_{0,II}$ which are linear but coupled with the evolution equation of the discrete spectrum mode $\mathcal{U}_{0,I,1}$. System (5.2) thus provides an intermediate model between the dispersive case of section 2 and the nondispersive case of section 3, and hence partially fills the behavioral gap between these two situations.

Example. With the same notation as in section 4.1, the equation for the electric field $\mathcal{E}_{0,II}$ associated to a large-spectrum laser reads

$$\partial_\tau\partial_{z_0}\mathcal{E}_{0,II} + \frac{\omega'(k_l)}{2}(\partial_X^2 + \partial_Y^2)\mathcal{E}_{0,II} + \frac{-\omega'(k_l) - \omega'(D_{z_0})}{\varepsilon}\partial_Z\partial_{z_0}\mathcal{E}_{0,II}$$
$$= -6i\alpha\gamma_a\eta_a^3\chi_{\omega_l}^2\frac{D_{t_0}\chi(D_{z_0})}{\beta(D_{t_0}, D_{z_0})}|\mathcal{E}_{0,I,1}|^2\mathcal{E}_{0,II},$$

which is still linear but coupled with $\mathcal{E}_{0,I,1}$. We recall that by weakly dispersive we mean that $\frac{-\omega'(k_l)-\omega'(D_{z_0})}{\varepsilon} = O(1)$, and that we must simultaneously solve $(\partial_T + \omega'(k_l)\partial_Z)\mathcal{E}_{0,II} = 0$.

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