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High-frequency nonlinear optics: from the nonlinear Schrödinger approximation to ultrashort-pulses equations

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After a brief introduction and physical motivation, we show how the nonlinear Schrödinger (NLS) equation can be derived from a general class of nonlinear hyperbolic systems. Its purpose is to describe the behaviour of high-frequency oscillating wave packets over a large time-scale that requires us to take into account diffractive effects. We then show that the NLS approximation fails for short pulses and propose some alternative models, including a modified Schrödinger equation with improved frequency dispersion. It turns out that these models have better properties and are quite accurate for short pulses. For *ultrashort* pulses, however, they must also be abandoned for more complex approaches. We give the main steps for such an analysis and explain one striking fact about ultrashort pulses: their dynamics in dispersive media is *linear*.

1. Generalities

This paper is the result of the 'first session of crash-courses in analysis and nonlinear PDEs' held at Heriott-Watt University in March 2009. The aim of this course was to present various mathematical techniques used to describe different kinds of laser pulses, from standard beams to ultrashort pulses. The material for this course comes from the references quoted in the bulk of the text, in particular [1–4, 9, 12–14, 17].

1.1. Motivation

We are interested here in describing the behaviour of fast oscillating solutions to semilinear hyperbolic systems. More precisely, we are concerned with the following initial-value problem:

$$\left.\begin{aligned} \partial_t \boldsymbol{U} + A(\partial)\boldsymbol{U} + \frac{1}{\varepsilon} E\boldsymbol{U} = \varepsilon T(\boldsymbol{U},\boldsymbol{U},\boldsymbol{U}), \qquad & t \geqslant 0, \\ \boldsymbol{U}|_{t=0} = \mathcal{U}^0\left(x, \frac{\boldsymbol{k}\cdot x}{\varepsilon}\right), \qquad & x \in \mathbb{R}^d, \end{aligned}\right\} \tag{1.1}$$

where $\boldsymbol{U}$ is an $\mathbb{R}^n$-valued function ($n \geqslant 1$) depending on the time variable t and the space variable $x \in \mathbb{R}^d$ ($d \geqslant 1$),

$$\boldsymbol{U} : (t,x) \in \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^n;$$

the operator $A(\partial)$ is defined as

$$A(\partial) = \sum_{j=1}^{d} A_j \partial_j,$$

where ∂_j is the differentiation operator with respect to the jth spatial coordinate. The following assumption is made on the matrices A_j and E.

ASSUMPTION 1.1. The matrices A_j $(j = 1, \dots, d)$ are constant coefficient $n \times n$, real-valued *symmetric* matrices.

The matrix E is a constant coefficient $n \times n$, real-valued *skew symmetric* matrix.

We also assume that the mapping

$$T \colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$

is *trilinear*. We still denote by T its trilinear extension to $\mathbb{C}^{3n}$, while its symmetrization is denoted by T^{S},

$$\begin{aligned} 6T^{\mathrm{S}}(A,B,C) = {} & T(A,B,C) + T(A,C,B) + T(B,A,C) \\ & + T(B,C,A) + T(C,A,B) + T(C,B,A); \end{aligned}$$

for the sake of simplicity, we often write

$$T^{\mathrm{S}}(A) := T^{\mathrm{S}}(A, A, \bar{A}). \tag{1.2}$$

Finally, $\boldsymbol{k} \in \mathbb{R}^d$ is the (spatial) *wavenumber* of the oscillations and $\varepsilon \ll 1$ is a small parameter (the order of the wavelength of the oscillations).

EXAMPLE 1.2. An important class of initial conditions for (1.1) are 'wave packets'. The initial condition is then a fast oscillation modulated by an *envelope* u^0:

$$\mathcal{U}^0\left(x, \frac{\boldsymbol{k} \cdot x}{\varepsilon}\right) = u^0(x) \mathrm{e}^{\mathrm{i}(\boldsymbol{k} \cdot x)/\varepsilon} + \text{c.c.},$$

where here and later 'c.c.' denotes the complex conjugate.

REMARK 1.3. The factor 'ε' in front of the nonlinearity in (1.1) is due to the *size* of the initial conditions considered and to a rescaling. One could equivalently consider a nonlinearity of size $O(1)$ with initial conditions of size $O(\varepsilon^{1/2})$.

1.2. Two important examples

The rationale of this work is the description of physical phenomena arising in nonlinear optics. We give below two important examples that satisfy the above assumptions. It is important to bear them in mind for all that follows.

EXAMPLE 1.4. The Maxwell–Lorentz equations in a dispersive medium can be written

$$\begin{aligned}\partial_t B + \operatorname{curl} E &= 0,\\ \partial_t E - \operatorname{curl} B + \frac{1}{\varepsilon} Q &= 0,\\ \partial_t Q - \frac{1}{\varepsilon}(E - P) &= \varepsilon |P|^2 P,\\ \partial_t P - \frac{1}{\varepsilon} Q &= 0,\end{aligned}$$

where B and E are respectively the magnetic and electric fields, while P is the polarization of the medium.

EXAMPLE 1.5. A useful toy model for the study of Maxwell's equation is the Klein–Gordon system

$$\partial_t \boldsymbol{U} + \begin{pmatrix} 0 & \nabla \\ \nabla^{\mathrm{T}} & 0 \end{pmatrix} + \frac{1}{\varepsilon} \begin{pmatrix} 0 & -v \\ v^{\mathrm{T}} & 0 \end{pmatrix} = \varepsilon |\boldsymbol{U}|^2 \begin{pmatrix} 0 & -v \\ v^{\mathrm{T}} & 0 \end{pmatrix} \boldsymbol{U},$$

where $v \in \mathbb{R}^d \setminus \{0\}$ is a constant vector.

1.3. Diffractive optics

We are interested here in *diffractive* optics. For the sake of clarity, let us explain here roughly what this means (see § 2 for full details). A quick look at the linear part of (1.1) shows that one should have $Eu = 0$ in the limit $\varepsilon \to 0$. This is a matricial equation that gives us some information on the vectorial structure of the solution: it leads to the *polarization condition* (see § 2.2.1). At the next order, the dynamics should be described by the hyperbolic part of the equations; it is therefore no surprise that a precise analysis shows that the solution roughly travels along rays: this is *geometric optics* (see § 2.2.2).

Diffractive optics describe what happens when the solution is observed during larger time-scale than the natural scale of geometrics. The failure of the approximation of geometric optics is then described by the Schrödinger equation (see § 2.2.3). Note also that the scaling on the nonlinearity has been chosen in such a way that nonlinear and diffractive effects occur at the same time.

1.4. Generalizations

A good reference for the physical modelling leading to systems of the form (1.1) and variants in nonlinear optics is Donnat's thesis [11]. (See also [5, 19].)

The analysis presented here (in particular in § 2) can often be generalized to more general types of semilinearity (the well-known Maxwell–Bloch systems have a quadratic nonlinearity, for instance) and even to quasilinear systems [13, 14, 17]. The case of larger solutions (or, equivalently, stronger nonlinearities in (1.1)) has also been considered: see [7, 8] and in particular [15], where the important concept of *transparency* is investigated.

2. The nonlinear Schrödinger (NLS) approximation

We are interested here in the case where the initial condition to (1.1) is a wave packet,

$$\mathcal{U}^0\left(x, \frac{\boldsymbol{k}\cdot x}{\varepsilon}\right) = u^0(x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x)/\varepsilon} + \text{c.c.} \tag{2.1}$$

The derivation and justification of the NLS approximation presented below is inspired by [13, 14, 17]. Among other references for this problem are [16, 21].

2.1. Well-posedness of (1.1) over large times

The general goal of this section is to describe the behaviour of solutions to (1.1) over large time-scales of order $O(1/\varepsilon)$ when the initial condition is as in (2.1). The first thing to check is naturally that such solutions do exist! This is granted by theorem 2.1. This theorem gives additional important information on the solution, namely that it can be described in terms of a *profile* U,

$$\boldsymbol{U}(t,x) = U\left(t, x, \frac{\boldsymbol{k}\cdot x - \omega t}{\varepsilon}\right), \tag{2.2}$$

with $U(t,x,\theta)$ periodic with respect to θ and for any $\omega \in \mathbb{R}$, provided that U solves the *profile equation*

$$\left.\begin{aligned} \partial_t U + A(\partial)U + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega D_\theta, \boldsymbol{k}D_\theta)U = \varepsilon T(U,U,U), \\ U|_{t=0}(x,\theta) = u^0(x)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.} \end{aligned}\right\} \tag{2.3}$$

Here, we used the notation

$$\mathcal{L}(\omega D_\theta, \boldsymbol{k}D_\theta) = -\omega D_\theta + A(\boldsymbol{k})D_\theta + \frac{E}{\mathrm{i}}, \tag{2.4}$$

with $D_\theta = -\mathrm{i}\partial_\theta$ and $A(\boldsymbol{k}) = \sum_{j=1}^d A_j k_j$.

THEOREM 2.1. *Let assumption 1.1 be satisfied, and let $s > d/2$, $u^0 \in H^s(\mathbb{R}^d)^n$ and $\mathcal{U}^0$ be as in (2.1). Let also $\omega \in \mathbb{R}$.*

There exist $T > 0$ such that for all $\varepsilon \in (0,1)$ there is a unique solution $\boldsymbol{U} \in C([0,T/\varepsilon]; H^s(\mathbb{R}^d))^n$ to (1.1). Moreover, one can write $\boldsymbol{U}$ in the form

$$\boldsymbol{U}(t,x) = U\left(t, x, \frac{\boldsymbol{k}\cdot x - \omega t}{\varepsilon}\right),$$

where U solves the profile equation (2.3).

Proof. Quite obviously, a solution $\boldsymbol{U}$ to (1.1) is given by (2.2) if (2.3) admits a solution $U \in C([0,T/\varepsilon]; H^{s,k})^n$ $(k \geqslant 1)$, where

$$H^{s,k}(\mathbb{R}^d \times \mathbb{T}) = \left\{ f = \sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}n\theta},\ |f|_{H^{s,k}} < \infty \right\} \tag{2.5}$$

and with

$$|f|^2_{H^{s,k}} = \sum_{n\in\mathbb{Z}} (1+n^2)^k |f_n|^2_{H^s}.$$

That such a solution exists is proved in the following lemma.

LEMMA 2.2. *Let $U^0 \in H^{s,k}(\mathbb{R}^d \times \mathbb{T})^n$ ($s > d/2$, $k \geqslant 1$). Then there exist $T > 0$ and a unique solution $U \in C([0, T/\varepsilon]; H^{s,k}(\mathbb{R}^d \times \mathbb{T}))$ to the profile equation (2.3) with initial condition U^0.*

Proof. Let us define $S(t)$ as

$$S(t) = \exp\bigg(- tA(\partial) - \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega D_\theta, kD_\theta)\bigg);$$

from assumption 1.1, this operator is unitary on all $H^{s,k}(\mathbb{R}^d \times \mathbb{T})$. Since, moreover, $H^{s,k}(\mathbb{R}^d \times \mathbb{T})$ is a Banach algebra for $s > d/2$ and $k \geqslant 1$, one can construct a (unique) solution to (2.3) by a standard iterative scheme

$$U^{l+1}(t) = S(t)U^0 + \varepsilon \int_0^t S(t-t')T(U^k, U^k, U^k)(t')\,\mathrm{d}t';$$

owing to the ε in front of the integral, the sequence converges to a solution on $[0, T/\varepsilon]$ for some $T > 0$ *independent of* ε. □

To prove the uniqueness of the solution let $\boldsymbol{U}_1, \boldsymbol{U}_2 \in C([0,T]; H^s(\mathbb{R}^d)^n)$ solve (1.1) on $[0, T/\varepsilon]$. Then $\boldsymbol{V} := \boldsymbol{U}_1 - \boldsymbol{U}_2$ solves

$$\begin{aligned} \partial_t \boldsymbol{V} + A(\partial)\boldsymbol{V} + \frac{1}{\varepsilon}E\boldsymbol{V} &= R, \\ \boldsymbol{V}|_{t=0} &= 0, \end{aligned}$$

with

$$R = 3\varepsilon T^{\mathrm{S}}(\boldsymbol{U}_2, \boldsymbol{U}_2, \boldsymbol{V}) - 3\varepsilon T^{\mathrm{S}}(\boldsymbol{U}_1, \boldsymbol{U}_1, \boldsymbol{V}) - \varepsilon T(\boldsymbol{V}, \boldsymbol{V}, \boldsymbol{V}).$$

In particular, one has, for all $0 \leqslant t \leqslant T/\varepsilon$,

$$(R(t), \boldsymbol{V}(t)) \leqslant \varepsilon C(|\boldsymbol{U}_1|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)}, |\boldsymbol{U}_2|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)})|\boldsymbol{V}(t)|_2^2.$$

It is thus easy to deduce from an L^2-energy estimate on the equation and Gronwall's lemma that $\boldsymbol{V} = 0$. □

2.2. Formal derivation of the nonlinear Schrödinger approximation

We want to construct an approximation $\boldsymbol{U}_{\mathrm{app}}$ to the solution $\boldsymbol{U}$ of (1.1) as $\varepsilon \to 0$. According to theorem 2.1, it is natural to look for $\boldsymbol{U}_{\mathrm{app}}$ in the form

$$\boldsymbol{U}_{\mathrm{app}}(t,x) = U_{\mathrm{app}}\bigg(t, x, \frac{\boldsymbol{k}\cdot x - \omega t}{\varepsilon}\bigg), \tag{2.6}$$

where U_{app} is an approximate solution to the *profile equation* (2.3).

We look for U_{app} in the form

$$\begin{aligned} U_{\mathrm{app}}(t,x,\theta) = {} & (u_{0,1}(\varepsilon t, t, x)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}) \\ & + \varepsilon(u_{1,1}(\varepsilon t, t, x)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}) + \varepsilon^2 U_2(\varepsilon t, t, x, \theta). \end{aligned} \tag{2.7}$$

We use the BKW method to determine this approximate solution. More precisely, plugging (2.7) into the profile equation (2.3) yields

$$\partial_t U_{\rm app} + A(\partial)U_{\rm app} + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega D_\theta, kD_\theta)U_{\rm app} - \varepsilon T(U_{\rm app}, U_{\rm app}, U_{\rm app}) \\ = \varepsilon^{-1}R_{-1} + \varepsilon^0 R_0 + \varepsilon^1 R_2 + \varepsilon^2 R^\varepsilon; \quad (2.8)$$

the method consists in choosing $u_{0,1}$, $u_{1,1}$ and U_2 in order to cancel R_{-1}, R_0 and R_1.

Remark 2.3. Note the introduction of the *slow time variable* $\tau = \varepsilon t$ in $u_{0,1}$, $u_{1,1}$ and u_2. Its purpose is to capture the secular evolution of the solution on large time-scales of order $O(1/\varepsilon)$.

2.2.1. *Cancelling the terms of order $O(\varepsilon^{-1})$: the dispersion relation and the polarization condition*

Quite obviously, the term R_{-1} that appears in (2.8) is given by

$$R_{-1} = \mathrm{i}\mathcal{L}(\omega, \boldsymbol{k})u_{0,1}\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.}$$

and the equation $R_{-1} = 0$ is thus equivalent to

$$\mathcal{L}(\omega, \boldsymbol{k})u_{0,1} = 0. \quad (2.9)$$

This equation has non-trivial solutions if and only if $\det \mathcal{L}(\omega, \boldsymbol{k}) = 0$. This leads us to define a very important object, the *characteristic variety*.

Definition 2.4. The *characteristic variety* associated to (1.1) is the real algebraic variety $\mathcal{C}_\mathcal{L} \subset \mathbb{R}^{d+1}$ defined as

$$\mathcal{C}_\mathcal{L} = \{(\omega, \boldsymbol{k}) \in \mathbb{R} \times \mathbb{R}^d, \ \det \mathcal{L}(\omega, \boldsymbol{k}) = 0\}.$$

If $(\omega, \boldsymbol{k}) \in \mathcal{C}_\mathcal{L}$, then we say that ω and $\boldsymbol{k}$ satisfy the *dispersion relation*.

Example 2.5. The characteristic variety associated to the Maxwell equations of example 1.4 is the union of seven sheets. Three of them are flat (namely, $\omega = 0$ and $\omega = \pm\sqrt{2}$) and four are curved sheets. Among these last four, two are smooth on $\mathbb{R}^d$ and do not contain the origin, and the last two admit a single singularity at the origin.

Example 2.6. For the Klein–Gordon equations of example 1.5, the characteristic variety is given by the two smooth hypersurfaces $\omega = \pm\sqrt{\boldsymbol{k}^2 + v^2}$. One therefore expects the same dispersive behaviour as for the two smooth curved sheets that do not contain the origin in the previous example.

We can then conclude that $u_{0,1}$ is a non-trivial (i.e. non-zero) solution of (2.9) if and only if

$$(\omega, \boldsymbol{k}) \in \mathcal{C}_\mathcal{L} \quad \text{and} \quad u_{0,1} \in \ker \mathcal{L}(\omega, \boldsymbol{k}) \setminus \{0\}. \quad (2.10)$$

The condition $u_{0,1} \in \ker \mathcal{L}(\omega, \boldsymbol{k}) \setminus \{0\}$ is called the *polarization condition*. Of course, if we work with *scalar* equations rather than *systems*, this condition disappears.

2.2.2. *Cancelling the terms of order $O(\varepsilon^0)$: transport at the group velocity*

One can compute that R_0 in (2.8) is given by

$$R_0 = (\mathrm{i}\mathcal{L}(\omega,\boldsymbol{k})u_{1,1} + (\partial_t + A(\partial))u_{0,1})\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.},$$

so that the equation $R_0 = 0$ is equivalent to

$$\mathcal{L}(\omega,\boldsymbol{k})u_{1,1} = \mathrm{i}(\partial_t + A(\partial))u_{0,1}. \tag{2.11}$$

Recall that it follows directly from assumption 1.1 that $\mathcal{L}(\omega,\boldsymbol{k})$ is a Hermitian matrix. In particular, the orthogonal complement of its kernel is its range, and the following notation makes sense.

NOTATION 2.7. For all $(\omega,\boldsymbol{k}) \in \mathbb{R}^{1+d}$, we write $\Pi(\omega,\boldsymbol{k})$, the orthogonal projection onto $\ker \mathcal{L}(\omega,\boldsymbol{k})$, and, with slight abuse of notation, $\mathcal{L}(\omega,\boldsymbol{k})^{-1}$, the partial inverse of $\mathcal{L}(\omega,\boldsymbol{k})$ defined and with values in the range of $\mathcal{L}(\omega,\boldsymbol{k})$. We consequently set $\mathcal{L}(\omega,\boldsymbol{k})^{-1}$ to be identically zero on $\ker \mathcal{L}(\omega,\boldsymbol{k})$.

We can now state the following straightforward but useful lemma.

LEMMA 2.8. *Let $(\omega,\boldsymbol{k}) \in \mathbb{R}^{1+d}$ and $a, b \in \mathbb{C}^n$. Then the following two assertions are equivalent:*

(i) *one has $\mathcal{L}(\omega,\boldsymbol{k})a = b$;*

(ii) *one has $\Pi(\omega,\boldsymbol{k})b = 0$ and $(\mathrm{Id} - \Pi(\omega,\boldsymbol{k}))a = \mathcal{L}(\omega,\boldsymbol{k})^{-1}b$.*

According to this lemma, (2.11) is equivalent to

$$\begin{aligned} \Pi(\omega,\boldsymbol{k})(\partial_t + A(\partial))u_{0,1} &= 0, \\ (\mathrm{Id} - \Pi(\omega,\boldsymbol{k}))u_{1,1} &= \mathrm{i}\mathcal{L}(\omega,\boldsymbol{k})^{-1}(\partial_t + A(\partial))u_{0,1}. \end{aligned}$$

Now, recalling that (2.10) yields $\Pi(\omega,\boldsymbol{k})u_{0,1} = u_{0,1}$, we can rewrite these two equations as

$$\left.\begin{aligned} \Pi(\omega,\boldsymbol{k})(\partial_t + A(\partial))\Pi(\omega,\boldsymbol{k})u_{0,1} &= 0, \\ (\mathrm{Id} - \Pi(\omega,\boldsymbol{k}))u_{1,1} &= \mathrm{i}\mathcal{L}(\omega,\boldsymbol{k})^{-1}A(\partial)\Pi(\omega,\boldsymbol{k})u_{0,1}, \end{aligned}\right\} \tag{2.12}$$

where we also used the fact that $\mathcal{L}(\omega,\boldsymbol{k})^{-1}\Pi(\omega,\boldsymbol{k}) = 0$.

The first equation of (2.12) is at first sight a *matricial* evolution equation. The fact that it is in fact *scalar* is a very important fact observed by Lax. In the present form, it can be found in [12].

LEMMA 2.9. *Let $(\omega,\boldsymbol{k})$ be a smooth point of $\mathcal{C}_{\mathcal{L}}$. Then one has*

$$\Pi(\omega,\boldsymbol{k})A(\partial)\Pi(\omega,\boldsymbol{k}) = \Pi(\omega,\boldsymbol{k})\nabla_{\boldsymbol{k}}\varpi \cdot \nabla,$$

where $\varpi(\cdot)$ is a local parametrization of $\mathcal{C}_{\mathcal{L}}$ in a neighbourhood of $(\omega,\boldsymbol{k})$.

Proof. By definition of ϖ, we have, for $\boldsymbol{k}'$ in a neighbourhood of $\boldsymbol{k}$,

$$\varpi(\boldsymbol{k}')\Pi(\varpi(\boldsymbol{k}'),\boldsymbol{k}') = \Pi(\varpi(\boldsymbol{k}'),\boldsymbol{k}')\left(A(\boldsymbol{k}') + \frac{E}{\mathrm{i}}\right).$$

Differentiating this identity with respect to $\boldsymbol{k}'_j$ ($j=1,\dots,d$), and writing Π instead of $\Pi(\underline{\omega}(\boldsymbol{k}'),\boldsymbol{k}')$, we get

$$\partial_j\underline{\omega}(\boldsymbol{k}')\Pi+\underline{\omega}(\boldsymbol{k}')\partial_j\Pi=(\partial_j\Pi)\left(A(\boldsymbol{k}')+\frac{E}{\mathrm{i}}\right)+\Pi A_j.$$

Multiplying this identity on the right and on the left by Π, and using the fact that $\Pi^2=\Pi$, we get

$$\partial_j\underline{\omega}(\boldsymbol{k}')\Pi=\Pi A_j\Pi,$$

and the result follows easily. □

REMARK 2.10. A geometric statement of lemma 2.9 is that the characteristic variety of the operator $\Pi(\omega,\boldsymbol{k})A(\partial)\Pi(\omega,\boldsymbol{k})$ is the tangent plane to $\mathcal{C}_{\mathcal{L}}$ at $(\omega,\boldsymbol{k})$. If $(\omega,\boldsymbol{k})$ is a singular point of $\mathcal{C}_{\mathcal{L}}$, then it can be shown [17] that the characteristic variety of $\Pi(\omega,\boldsymbol{k})A(\partial)\Pi(\omega,\boldsymbol{k})$ is the tangent cone to $\mathcal{C}_{\mathcal{L}}$ at $(\omega,\boldsymbol{k})$.

DEFINITION 2.11. The quantity $\nabla_{\boldsymbol{k}}\underline{\omega}\in\mathbb{R}^d$ is called the *group velocity* and will be denoted by $c_{\mathrm{g}}(\boldsymbol{k})$.

By lemma 2.9, we can rewrite (2.12) as

$$\left.\begin{aligned}(\partial_t+c_{\mathrm{g}}(\boldsymbol{k})\cdot\nabla)u_{0,1}&=0,\\ (\mathrm{Id}-\Pi(\omega,\boldsymbol{k}))u_{1,1}&=\mathrm{i}\mathcal{L}(\omega,\boldsymbol{k})^{-1}A(\partial)\Pi(\omega,\boldsymbol{k})u_{0,1}.\end{aligned}\right\}\tag{2.13}$$

The main information given by this step of the computation is that $u_{0,1}$ *is transported at the group velocity*. This is the core of *geometric optics*, which states that high-frequency waves propagate along rays. In *diffractive optics*, this behaviour is somehow altered, as we show now.

2.2.3. *Cancelling the terms of order $O(\varepsilon^1)$: diffractive and nonlinear effects*

After some computations, one can check that the R_2 term in (2.8) is given by

$$R_2=\mathrm{i}\mathcal{L}(\omega D_\theta,\boldsymbol{k}D_\theta)U_2+(\partial_t+A(\partial))U_1+\partial_\tau U_0-T(U_0,U_0,U_0),$$

where we wrote $U_j=u_{j,1}\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.}$ ($j=0,1$). One of the main differences from the previous orders of expansion is that the nonlinearity occurs at this level. In particular, the harmonics ±1 are not the only ones present in R_2 and one needs to include the harmonics ±3. Indeed, one has

$$T(U_0,U_0,U_0)=(T(u_{0,1},u_{0,1},u_{0,1})\mathrm{e}^{3\mathrm{i}\theta}+\text{c.c.})+(3T^{\mathrm{S}}(u_{0,1})\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.}),$$

where $T^{\mathrm{S}}(u_{0,1})$ is as defined in (1.2).

It follows that one can decompose R_2 into

$$R_2=(r_{2,1}\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.})+(r_{2,3}\mathrm{e}^{3\mathrm{i}\theta}+\text{c.c.}),$$

and the equation $R_2=0$ is equivalent to $r_{2,1}=0$ and $r_{2,3}=0$.

First harmonic. The equation $r_{2,1} = 0$ can be written

$$\mathcal{L}(\omega, \boldsymbol{k})u_{2,1} = \mathrm{i}(\partial_t + A(\partial))u_{1,1} + \mathrm{i}\partial_\tau u_{0,1} - 3\mathrm{i}T^{\mathrm{S}}(u_{0,1}).$$

Owing to lemma 2.8, this is equivalent to

$$\begin{aligned}\Pi(\omega, \boldsymbol{k})(\partial_t + A(\partial))u_{1,1} + \partial_\tau u_{0,1} &= 3\Pi(\omega, \boldsymbol{k})T^{\mathrm{S}}(u_{0,1}),\\ (\mathrm{Id} - \Pi(\omega, \boldsymbol{k}))u_{2,1} &= \mathrm{i}\mathcal{L}(\omega, \boldsymbol{k})^{-1}(\partial_t + A(\partial))u_{1,1} - 3\mathrm{i}\mathcal{L}(\omega, \boldsymbol{k})^{-1}T^{\mathrm{S}}(u_{0,1}),\end{aligned}$$

where we have used the fact that $\Pi(\omega, \boldsymbol{k})u_{0,1} = u_{0,1}$ and $\mathcal{L}(\omega, \boldsymbol{k})^{-1}\Pi(\omega, \boldsymbol{k}) = 0$. Note that at this point only the component of $u_{1,1}$ polarized along $\mathrm{Id} - \Pi(\omega, \boldsymbol{k})$ has been determined (by (2.13)). We are therefore free to set

$$\Pi(\omega, \boldsymbol{k})u_{1,1} = 0,$$

and the above equations yield

$$\partial_\tau u_{0,1} + \mathrm{i}\Pi(\omega, \boldsymbol{k})A(\partial)\mathcal{L}(\omega, \boldsymbol{k})^{-1}A(\partial)\Pi(\omega, \boldsymbol{k})u_{0,1} = 3\Pi(\omega, \boldsymbol{k})T^{\mathrm{S}}(u_{0,1})$$

and

$$\begin{aligned}(\mathrm{Id} - \Pi(\omega, \boldsymbol{k}))u_{2,1} = -\mathcal{L}(\omega, \boldsymbol{k})^{-1}(\partial_t + A(\partial))\mathcal{L}(\omega, \boldsymbol{k})^{-1}A(\partial)\Pi(\omega, \boldsymbol{k})u_{0,1}\\ - 3\mathrm{i}\mathcal{L}(\omega, \boldsymbol{k})^{-1}T^{\mathrm{S}}(u_{0,1}).\end{aligned}$$

The first of these equations gives the evolution of $u_{0,1}$ according to the slow time variable τ; recall that the evolution with respect to the time variable t is the transport at the group velocity found above. This new equation looks matricial, but, as for the transport equation, an important simplification holds.

LEMMA 2.12. *Let $(\omega, \boldsymbol{k})$ be a smooth point of $\mathcal{C}_{\mathcal{L}}$. Then one has*

$$\Pi(\omega, \boldsymbol{k})A(\partial)\mathcal{L}(\omega, \boldsymbol{k})^{-1}A(\partial)\Pi(\omega, \boldsymbol{k}) = -\tfrac{1}{2}\underline{\omega}''(\boldsymbol{k})(\partial, \partial),$$

where $\underline{\omega}(\cdot)$ is a local parametrization of $\mathcal{C}_{\mathcal{L}}$ in a neighbourhood of $(\omega, \boldsymbol{k})$.

Proof. The proof is in the same spirit as the proof of lemma 2.9. We refer the reader to [13]. □

The slow evolution of $u_{0,1}$ is thus described by the equation

$$\partial_\tau u_{0,1} - \tfrac{1}{2}\mathrm{i}\underline{\omega}''(\boldsymbol{k})(\partial, \partial)u_{0,1} = 3\Pi(\omega, \boldsymbol{k})T^{\mathrm{S}}(u_{0,1}). \tag{2.14}$$

Third harmonic. The equation $R_{2,3} = 0$ can be written

$$\mathcal{L}(3\omega, 3\boldsymbol{k})u_{2,3} = -\mathrm{i}T(u_{0,1}, u_{0,1}, u_{0,1}). \tag{2.15}$$

We now make the following assumption on $\mathcal{C}_{\mathcal{L}}$, which is generically satisfied in dispersive media.[1]

ASSUMPTION 2.13. The point $(3\omega, 3\boldsymbol{k})$ does not belong to $\mathcal{C}_{\mathcal{L}}$.

Under this assumption, $u_{2,3}$ is found in terms of $u_{0,1}$ by inverting $\mathcal{L}(3\omega, 3\boldsymbol{k})$ in (2.15).

[1] The term 'dispersive' refers to the fact that the group velocity $c_{\mathrm{g}}(\boldsymbol{k})$ depends on $|\boldsymbol{k}|$.

2.2.4. *Summary*

We have shown in the previous section that if one can find $u_{0,1}(\tau,t,x)$ satisfying

$$\begin{aligned}\Pi(\omega,\boldsymbol{k})u_{0,1}&=u_{0,1},\\(\partial_t+c_{\rm g}(\boldsymbol{k})\cdot\nabla)u_{0,1}&=0,\\\partial_\tau u_{0,1}-\tfrac{1}{2}\mathrm{i}\underline{\omega}''(\boldsymbol{k})(\partial,\partial)&=3\Pi(\omega,\boldsymbol{k})T^{\rm S}(u_{0,1}),\\u_{0,1}|_{t=\tau=0}(x)&=u^0(x),\end{aligned}$$

then it is possible to find $u_{1,1}$ and U_2 in terms of $u_{0,1}$ in order to cancel R_{-1}, R_0 and R_1 in (2.8).

This systems looks overdeterminated; however, it is possible to reduce it by writing

$$u_{0,1}(\tau,t,x)=\underline{u}(\tau,x-c_{\rm g}(\boldsymbol{k})t);$$

provided that $\Pi(\omega,\boldsymbol{k})u^0=u^0$, the above equations indeed simplify to

$$\left.\begin{aligned}\partial_\tau\underline{u}-\tfrac{1}{2}\mathrm{i}\underline{\omega}''(\boldsymbol{k})(\partial_y,\partial_y)\underline{u}&=3\Pi(\omega,\boldsymbol{k})T^{\rm S}(\underline{u}),\\\underline{u}|_{\tau=0}(y)&=u^0(y),\end{aligned}\right\}\tag{2.16}$$

where $y=x-c_{\rm g}(\boldsymbol{k})t$.

Since (2.16) is locally well posed in $H^s(\mathbb{R}^d)$ ($s>d/2$), we can now define the *nonlinear Schrödinger approximation* $\boldsymbol{U}_{\rm NLS}$ to (1.1) by

$$\boldsymbol{U}_{\rm NLS}(t,x)=u_{\rm NLS}(t,x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x-\omega t)/\varepsilon}+\text{c.c.}\tag{2.17}$$

with $u_{\rm NLS}=\underline{u}(\varepsilon t,x-c_{\rm g}(\boldsymbol{k})t)$.

Remark 2.14. An equivalent way of defining $u_{\rm NLS}$ is that it is the solution of the initial-value problem

$$\begin{aligned}\partial_t u_{\rm NLS}+c_{\rm g}(\boldsymbol{k})\cdot\nabla u_{\rm NLS}-\tfrac{1}{2}\varepsilon\mathrm{i}\underline{\omega}''(\boldsymbol{k})(\partial,\partial)u_{\rm NLS}&=3\varepsilon\Pi(\omega,\boldsymbol{k})T^{\rm S}(u_{\rm NLS}),\\u_{\rm NLS}|_{t=0}(y)&=u^0.\end{aligned}$$

2.3. Rigorous justification of the NLS approximation

The formal derivation of the NLS approximation is justified by the following theorem.

Theorem 2.15. *Let assumptions 1.1 and 2.13 be satisfied. Let also $s>d/2$ and $u^0\in H^{s+3}(\mathbb{R}^d)^n$ be such that*

$$\exists\omega\in\mathbb{R},\quad(\omega,\boldsymbol{k})\in\mathcal{C}_{\mathcal{L}}\quad\text{and}\quad\Pi(\omega,\boldsymbol{k})u^0=u^0.$$

Let also $\underline{u}\in C([0,T];H^{s+3}(\mathbb{R}^d))$ (for some $T>0$) be the unique solution to (2.16).

Then there exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$, the unique solution $\boldsymbol{U}$ of (1.1) provided by theorem 2.1 exists on $[0,T/\varepsilon]$, and one has

$$|\boldsymbol{U}(t,x)-\boldsymbol{U}_{\rm NLS}(t,x)|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)}\leqslant\varepsilon C(T,|u^0|_{H^{s+3}}),$$

where $\boldsymbol{U}_{\rm NLS}$ is as in (2.17).

Proof. As explained in § 2.2.4, if one knows $\underline{u} \in C([0,T];H^{s+3}(\mathbb{R}^d))^n$, it is possible to construct $U_{\rm app} \in C([0,T/\varepsilon];H^{s,1})$, solving (2.8) with $R_{-1}=R_0=R_1=0$ and $(R^\varepsilon)_\varepsilon$ bounded in $C([0,T/\varepsilon];H^{s,1})$. The following lemma shows that the solution U of the profile equation (2.3) provided by lemma 2.2 exists on the same time interval $[0,T/\varepsilon]$ (if ε_0 is small enough) and remains close to $U_{\rm app}$.

LEMMA 2.16. *Let $U_{\rm app} \in C([0,T/\varepsilon];H^{s,1})$ solve the profile equation (2.3) up to a residual $\varepsilon^2 R^\varepsilon$ with $(R^\varepsilon)_\varepsilon$ bounded in $C([0,T/\varepsilon];H^{s,1})$, and assume, moreover, that*

$$\exists C_1>0, \quad |U_{\rm app}|_{t=0} - (u^0(x)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.})|_{H^{s,1}} \leqslant \varepsilon C_1.$$

Then there exists $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$ there exists a unique solution $U \in C([0,T/\varepsilon];H^{s,1}(\mathbb{R}^d\times\mathbb{T})^n)$ to (2.3). Moreover,

$$|U-U_{\rm app}|_{L^\infty([0,T/\varepsilon];H^{s,1})} \leqslant \varepsilon C(T,C_1,|U_{\rm app}|_{L^\infty([0,T/\varepsilon];H^{s,1})},|R^\varepsilon|_{L^\infty([0,T/\varepsilon];H^{s,1})}).$$

Proof. We already know from the proof of theorem 2.1 that (2.3) admits a solution in $C([0,T'/\varepsilon];H^{s,1}(\mathbb{R}^d\times\mathbb{T})^n)$ for some $T'>0$ possibly smaller than T. We prove at the same time that one can take $T'=T$ for ε small enough and that the estimate of the lemma holds.

It is sufficient to prove that V defined as

$$U = U_{\rm app} + \varepsilon V$$

exists and is bounded (with respect to ε) in $C([0,T/\varepsilon];H^{s,1}(\mathbb{R}^d\times\mathbb{R}))$. The equation satisfied by V is

$$\begin{aligned}\partial_t V + A(\partial)V + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega D_\theta, kD_\theta)V \\ = -\varepsilon R^\varepsilon + 3\varepsilon T^{\rm S}(U_{\rm app},U_{\rm app},V) + 3\varepsilon^2 T^{\rm S}(U_{\rm app},V,V) + \varepsilon^3 T^{\rm S}(V,V,V). \quad (2.18)\end{aligned}$$

Since $S(t)$ (as defined in the proof of lemma 2.2) is unitary on $H^{s,1}(\mathbb{R}^d\times\mathbb{T})$, we deduce the following estimate on V:

$$|V(t)|_{H^{s,1}} \leqslant \varepsilon(C_1+Mt) + \varepsilon C(M)\int_0^t (1+\varepsilon^2|V(t')|_{H^{s,1}}^2)|V(t')|_{H^{s,1}}\,\mathrm{d}t',$$

where M is such that

$$|R^\varepsilon|_{L^\infty([0,T/\varepsilon];H^{s,1})} \leqslant M \quad \text{and} \quad |U_{\rm app}|_{L^\infty([0,T/\varepsilon];H^{s,1})} \leqslant M.$$

Now, let $M' = (C_1+MT)\exp(2TC(M))$ and let us define T_0 as

$$T_0 = \sup\{T''\in\mathbb{R},\ 0<T''\leqslant T',\ \text{for all } 0\leqslant t\leqslant T''/\varepsilon,\ |V(t)|_{H^{s,1}}\leqslant M'\}.$$

We now show that $T''=T'$ if ε_0 is small enough. By definition of T_0 and the above estimate on $|V(t)|_{H^{s,1}}$ one has that, for all $0\leqslant t\leqslant T_0/\varepsilon$,

$$|V(t)|_{H^{s,1}} \leqslant (C_1+MT_0) + \varepsilon C(M)\int_0^t (1+\varepsilon^2 M'^2)|V(t')|_{H^{s,1}}\,\mathrm{d}t'.$$

By Gronwall's lemma, we thus get

$$|V(t)|_{H^{s,1}} \leqslant (C_1+MT_0)\exp(T_0C(M)(1+\varepsilon^2M'^2));$$

clearly, if ε is small enough, the right-hand side is smaller than M' and we deduce that $T_0 = T'$. It is also easy to deduce that one can take $T' = T$, since if T'/ε were the maximal existence time for V, one would have $\lim_{t\to T'/\varepsilon} |V(t)|_{H^{s,1}} = \infty$, which is excluded by the above bound. □

With $U_{\rm app}$ as constructed in the previous sections, the estimate of the lemma yields

$$|U - U_{\rm app}|_{L^\infty([0,T/\varepsilon];H^{s,1})} \leqslant \varepsilon C(T, |u^0|_{H^{s+3}}). \tag{2.19}$$

Moreover, one has

$$\begin{aligned}|\boldsymbol{U} - \boldsymbol{U}_{\rm NLS}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)}&\\ \leqslant |U(t,x,\theta) - (\underline{u}(\varepsilon t, x - c_{\rm g}(\boldsymbol{k})t)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.})|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d\times\mathbb{T})};&\end{aligned}$$

from the continuous embedding $L^\infty([0,T/\varepsilon];H^{s,1}) \subset L^\infty([0,T/\varepsilon]\times\mathbb{R}^d\times\mathbb{T})$, one deduces

$$\begin{aligned}|\boldsymbol{U} - \boldsymbol{U}_{\rm NLS}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leqslant |U - U_{\rm app}|_{L^\infty([0,T/\varepsilon];H^{s,1})}&\\ + |U_{\rm app} - (\underline{u}(\varepsilon t, x - c_{\rm g}(\boldsymbol{k})t)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.})|_{L^\infty([0,T/\varepsilon];H^{s,1})},&\end{aligned}$$

and the result is thus a consequence of (2.19) and the explicit expressions found for $u_{1,1}$ and U_2 in the previous sections. □

3. Short pulse approximations

We have seen in theorem 2.15 that the NLS approximation is accurate for initial data of the form (2.1). It is important to note that the wavelength of the *envelope* u^0 is $O(1)$ in (2.1). We are interested in this section in investigating what happens for *short pulses*, i.e. when the wavelength of the amplitude is of order $\beta \ll 1$. We follow here the approach of [9, 10].

3.1. Shortcomings of the NLS approximation

3.1.1. *Dispersive properties*

The information on the dispersive properties of (1.1) is given by $\mathcal{C}_{\mathcal{L}}$. More precisely, in order for a plane wave

$$\boldsymbol{A}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}'\cdot x - \omega' t)}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x - \omega t)/\varepsilon}, \quad \boldsymbol{A} \in \mathbb{C}^n,$$

to be a solution to the linear part of (1.1) it is necessary that

$$(\omega + \varepsilon\omega', \boldsymbol{k} + \varepsilon\boldsymbol{k}') \in \mathcal{C}_{\mathcal{L}}.$$

If ω_1 is a local parametrization of $\mathcal{C}_{\mathcal{L}}$ in a neighbourhood of $(\omega, \boldsymbol{k})$ (so that $\omega_1(\boldsymbol{k}) = \omega$), then this condition can be rewritten as

$$\omega' = \frac{\omega_1(\boldsymbol{k} + \varepsilon\boldsymbol{k}') - \omega}{\varepsilon}. \tag{3.1}$$

Let us now investigate the dispersive properties of the NLS approximation. Note that a plane wave of the above form corresponds to (the linear part of) the approximation (2.16), (2.17) if $u = \mathrm{e}^{\mathrm{i}(\boldsymbol{k}'\cdot x-\omega' t)}$ solves

$$\partial_t u + c_{\mathrm{g}}(\boldsymbol{k})\cdot\nabla u - \tfrac{1}{2}\mathrm{i}\varepsilon\omega_1''(\boldsymbol{k})(\partial,\partial)u = 0,$$

that is, if

$$\omega' = c_{\mathrm{g}}(\boldsymbol{k})\cdot\boldsymbol{k}' + \tfrac{1}{2}\varepsilon\omega_1''(\boldsymbol{k})(\boldsymbol{k}',\boldsymbol{k}'). \tag{3.2}$$

Clearly, (3.2) is the second-order approximation of (3.1) at $(\omega,\boldsymbol{k})$ and thus provides a very good approximation for plane waves localized (in frequency) near $(\omega,\boldsymbol{k})$. Since any solution can be seen as a sum of plane waves, this means that the dispersive properties of the Schrödinger approximation will be good for initial data of the form

$$\boldsymbol{U}|_{t=0}(x) = u^0(x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x)/\varepsilon} + \text{c.c.}, \qquad \Pi_1(\boldsymbol{k})u^0 = u^0$$

if the spectrum of u^0 (the support of its Fourier transform) is of size $O(1)$ (with respect to ε). More generally, if this spectrum is of size $O(1/\beta)$, then frequencies in a neighbourhood of size $O(\varepsilon/\beta)$ of $(\omega,\boldsymbol{k})$ in $\mathcal{C}_{\mathcal{L}}$ are involved and (3.2) may fail to be a good approximation to (3.1) if β is small (indeed, (3.2) is quadratic for large $\boldsymbol{k}'$, while (3.1) is at most linear). The conclusion of this discussion is the following.

(C1) The dispersive properties of the Schrödinger approximation are bad if the spectrum of the envelope is large.

3.1.2. *The slowly varying envelope approximation*

One of the implicit assumptions made in order to derive the NLS approximation is that the solution can be described as a fast oscillation modulated by an *envelope* $u_{\mathrm{env}}(t,x)$,

$$\boldsymbol{U}(t,x) \sim u_{\mathrm{env}}(t,x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x-\omega t)/\varepsilon};$$

this envelope is then approximated through the NLS equation (namely, $u_{\mathrm{env}}(t,x) \sim u_{\mathrm{NLS}}(t,x)$, with u_{NLS} as in (2.17)). This assumption is called *slowly varying envelope approximation (SVEA)* or *envelope approximation*.

Without entering into too much detail at this point of the discussion, it seems likely that the following holds.

(C2) The SVEA is valid, provided that the envelope does not vary too much at the scale of the wavelength (equal to ε here).

3.1.3. *Short pulses*

We are interested here in one particular case where the two drawbacks of the NLS approximation described in §§ 3.1.1 and 3.1.2 can be met, namely, *short pulses*. More precisely, let us consider the case of initial data for (1.1) for the form

$$\boldsymbol{U}|_{t=0} = u^0_{(\beta)}(x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x)/\varepsilon} + \text{c.c.} \quad \text{with } u^0_{(\beta)}(x) = u^0\left(\frac{x}{\beta}\right), \tag{3.3}$$

for some $\beta \in (0,1]$. Clearly, when $\beta \to 0$, the typical width of the support of the envelope $u^0_{(\alpha)}$ gets smaller, and this is why we call the initial conditions, such as (3.3), 'short pulses' (see figure 1).

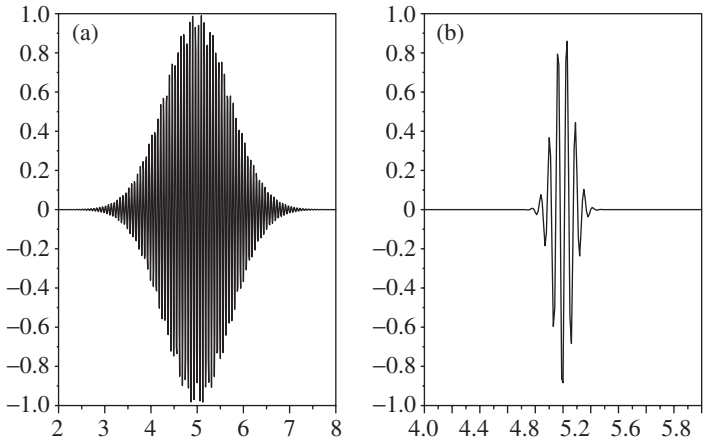

Figure 1. Initial condition $u^0_{(\beta)}(x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x)/\varepsilon} + \text{c.c.}$ with a Gaussian u^0, $\varepsilon = 0.01$, for (a) $\beta = 1$ and (b) $\beta = 0.1$.

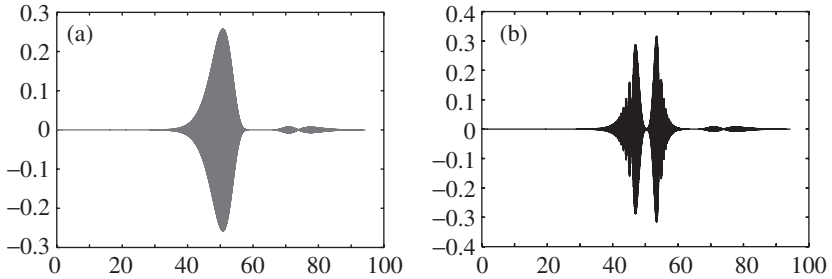

Figure 2. Short pulses: (a) the exact solution at time $t = 50$; (b) the difference between the exact solution and the NLS approximation with $\varepsilon = 0.01$, $\beta = 0.075$.

One may easily check that $u^0_{(\beta)}$ has spectrum of typical size $O(1/\beta)$ and that its variation at the scale of the wavelength ε is $O(\varepsilon/\beta)$. For small values of β, the conditions (C1) and (C2) are enforced and one cannot expect the NLS approximation to provide good results.

This intuition is confirmed by numerical computations; figure 2 shows an error of the order of 100% for the NLS approximation (the computations are made for the Klein–Gordon model of example 1.5).

3.2. Functional setting

Sobolev spaces are obviously not a good space to work with when studying short pulses. Indeed, if the initial condition for (1.1) is a short pulse with initial envelope $u^0_{(\beta)}$ as in (3.3), then

$$|u^{0,\beta}|_{H^s} \sim \frac{1}{\beta^{s-d/2}}$$

and thus grows to ∞ as $\beta \to 0$ when $s > d/2$. The constant $C(T, |U^0|_{H^{s+3}})$ in the estimate given by theorem 2.15 then becomes infinite, making it useless.

A better functional setting to study short pulses as in (3.3) is the Wiener algebra.

DEFINITION 3.1. The Wiener algebra $W(\mathbb{R}^d)$ is defined as

$$W(\mathbb{R}^d) = \{f \in \mathcal{S}'(\mathbb{R}^d),\ \hat{f} \in L^1(\mathbb{R}^d)\},$$

and is endowed with the norm $|f|_W = |\hat{f}|_{L^1}$.

More generally, $W^k(\mathbb{R}^d)$ ($k \in \mathbb{N}$) is the set of all $f \in W(\mathbb{R}^d)$ such that $\partial^\alpha f \in W(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}^d$, $|\alpha| \leqslant k$.

The following proposition gathers the properties of $W(\mathbb{R}^d)$ we shall need here.

PROPOSITION 3.2. *The space $(W(\mathbb{R}^d), |\cdot|_W)$ is a Banach algebra and is continuously embedded in $L^\infty(\mathbb{R}^d)$. Moreover, for all $\lambda > 0$ and $f \in W(\mathbb{R}^d)$, one has $|f(\cdot/\lambda)|_W = |f|_W$.*

REMARK 3.3. From the last statement of the proposition, one has $|u^0_{(\beta)}|_W = |u^0|_W$ if $u^0_{(\beta)}$ is as in (3.3). In particular, this quantity remains bounded as $\beta \to 0$.

REMARK 3.4. Since we replace the spaces $H^s(\mathbb{R}^d)$ ($s > d/2$) by $W(\mathbb{R}^d)$, it is natural to replace the spaces $H^{s,k}(\mathbb{R}^d \times \mathbb{T})$ ($k \geqslant 1$) defined in (2.5) by a Wiener algebra $W(\mathbb{R}^d \times \mathbb{T})$:

$$W(\mathbb{R}^d \times \mathbb{T}) = \left\{ f = \sum_{n\in\mathbb{Z}} f_n \mathrm{e}^{\mathrm{i}\theta},\ |f|_{W(\mathbb{R}^d\times\mathbb{T})} < \infty \right\},$$

with $|f|_{W(\mathbb{R}^d\times\mathbb{T})} = \sum_{n\in\mathbb{Z}} |f_n|_W$. Proposition 3.2 can easily be adapted to $W(\mathbb{R}^d \times \mathbb{T})$.

Using the properties of Wiener algebras given in proposition 3.2, it is possible to follow the same approach as in § 2, but replacing $H^{s+k}(\mathbb{R}^d)$ ($s > d/2$, $k \in \mathbb{N}$) Sobolev spaces by $W^k(\mathbb{R}^d)$. The estimate of theorem 2.15 would then be replaced by

$$|\boldsymbol{U}(t,x) - \boldsymbol{U}_{\mathrm{NLS}}(t,x)|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leqslant \varepsilon C(T, |U^0|_{W^3}).$$

For short pulses as in (3.3), one has $|U^0_{(\beta)}|_{W^3} = O(\beta^{-3})$ and this estimate is not more useful than the one given in theorem 2.15. This means that the whole approach developed in § 2 must be abandoned and a better adapted one developed. This is the goal of this section.

3.3. Validity of the SVEA

We recall that the SVEA consists in describing a fast oscillating wave packet by its envelope,

$$\boldsymbol{U}(t,x) \sim u_{\mathrm{env}}(t,x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x - \omega t)/\varepsilon} + \text{c.c.} \tag{3.4}$$

Since we know that the *exact* solution $\boldsymbol{U}$ to (1.1) can always be represented through (2.2) by a profile U solving the profile equation (2.3), the SVEA can be restated as

$$U(t,x,\theta) \sim u_{\mathrm{env}}(t,x)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.};$$

in other words, the SVEA only takes into account the harmonics ± 1 of the solution.

Physicists use a practical rule to assess the validity of this SVEA; it can be stated with our formalism as

$$|\nabla u^0|_\infty \ll \frac{1}{\varepsilon} \tag{3.5}$$

(which is of course a quantitative version of the condition (C2) stated in § 3.1.2).

In the next subsection, we investigate the validity of the SVEA (3.4), and compare it to the 'practical rule' (3.5).

3.3.1. *Formal derivation of the envelope equation*

We try here to find an equation that $u_{\rm env}$ must solve if the SVEA (3.4) is valid. As stated above, the SVEA is equivalent to

$$U(t,x,\theta) \sim u_{\rm env}(t,x)\mathrm{e}^{\mathrm{i}\theta} + \text{c.c.};$$

plugging this approximation into the profile equation (2.3) and keeping only the first harmonic in the Fourier expansion easily yields

$$\partial_t u_{\rm env} + A(\partial)u_{\rm env} + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega,\boldsymbol{k})u_{\rm env} = 3\varepsilon T^{\rm S}(u_{\rm env}).$$

Defining $D = -\mathrm{i}\partial$, we observe that

$$\begin{aligned} A(\partial) + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega,\boldsymbol{k}) &= A(\partial) + \frac{\mathrm{i}}{\varepsilon}(-\omega\,\mathrm{Id} + A(\boldsymbol{k})) \\ &= \frac{\mathrm{i}}{\varepsilon}(-\omega\,\mathrm{Id} + A(\boldsymbol{k} + \varepsilon D)) \\ &:= \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega,\boldsymbol{k}+\varepsilon D), \end{aligned}$$

where the notation in the latter is of course consistent with (2.4).

As a consequence of these computations, we say that $\boldsymbol{U}$ satisfies the SVEA if (3.4) holds, with $u_{\rm env}$ the solution of the *envelope equation*

$$\left.\begin{aligned} \partial_t u_{\rm env} + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega,\boldsymbol{k}+\varepsilon D)u_{\rm env} = 3\varepsilon T^{\rm S}(u_{\rm env}), \\ u_{\rm env}|_{t=0} = u^0. \end{aligned}\right\} \tag{3.6}$$

3.3.2. *Rigorous justification of the SVEA*

The main result of this section is theorem 3.8. It gives a rigorous justification of the SVEA (3.4), (3.5). The other approximations considered here (full dispersion model, improved Schrödinger, etc.) are quite easy consequences of the result.

Compared with § 2, some additional assumptions on $\mathcal{C}_{\mathcal{L}}$ are needed. This is made necessary by the fact that short pulses have a wider spectrum than standard pulses. Fortunately, these additional assumptions are satisfied for the applications.

Assumption 3.5. The characteristic variety $\mathcal{C}_{\mathcal{L}}$ and the frequency/wavenumber couple $(\omega,\boldsymbol{k})$ satisfy the following.

- There exist m functions $\omega_j \in C^\infty(\mathbb{R}^d\setminus\{0\})$ $(j=1,\dots,m)$ such that

$$\mathcal{C}_{\mathcal{L}}\setminus\{0\} = \bigcup_{j=1}^{m}\{(\omega_j(\boldsymbol{k}'),\boldsymbol{k}'),\ \boldsymbol{k}'\in\mathbb{R}^d\setminus\{0\}\};$$

  up to a renumbering, we assume that $(\omega,\boldsymbol{k}) = (\omega_1(\boldsymbol{k}),\boldsymbol{k})$.

- There exists a constant $c_0 > 0$ such that

$$\inf_{\boldsymbol{k}' \in \mathbb{R}^d} |\omega - \omega_j(\boldsymbol{k}')| \geqslant c_0, \quad j = 2, \dots, m.$$

EXAMPLE 3.6. This assumption is satisfied by the Maxwell and Klein–Gordon equations of examples 1.4 and 1.5 (see also examples 2.5 and 2.6).

NOTATION 3.7. We denote by $\Pi_j(\boldsymbol{k})$ $(j = 1, \dots, m)$ the eigenprojectors of the eigenvalues $\omega_j(\boldsymbol{k})$ of $A(\boldsymbol{k}) + E/\mathrm{i}$; in particular, we have

$$A(\boldsymbol{k}) + \frac{E}{\mathrm{i}} = \sum_{j=1}^{m} \omega_j(\boldsymbol{k}) \Pi_j(\boldsymbol{k}).$$

THEOREM 3.8. *Let assumptions 1.1, 2.13 and 3.5 be satisfied and let $u^0 \in W^1(\mathbb{R}^d)^n$ be such that $\Pi_1(\boldsymbol{k})u^0 = u^0$. Then we have the following.*

(i) *There exist $T > 0$ and a unique solution $u_{\mathrm{env}} \in C([0, T/\varepsilon]; W^1(\mathbb{R}^d)^n)$ to (3.6).*

(ii) *There exist $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution $\boldsymbol{U}$ to (1.1) provided by theorem 2.1 exists on $[0, T/\varepsilon]$ and*

$$|\boldsymbol{U} - \boldsymbol{U}_{\mathrm{SVEA}}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leqslant \varepsilon C(T, |U^0|_W)(1 + |\nabla U^0|_W),$$

where $\boldsymbol{U}_{\mathrm{SVEA}}(t, x) = u_{\mathrm{env}}(t, x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x - \omega t)/\varepsilon} + \mathrm{c.c.}$

REMARK 3.9. If we look at a family of initial conditions $(u^0_{(\beta)})_\beta$ bounded in $W(\mathbb{R}^d)$, which is of course the case for short pulses like (3.3), one deduces that

$$\text{the SVEA is valid if } |\nabla u^0_{(\beta)}|_W \ll \frac{1}{\varepsilon},$$

and theorem 3.8 thus provides a rigorous basis for the 'practical rule' (3.5). When working with short pulses with initial condition (3.3), it is easy to check that this condition reads simply $\varepsilon \ll \beta$.

REMARK 3.10. Working in the more classical framework of Sobolev spaces, one could establish an error estimate similar to that given by the theorem, but with H^s-norms $(s > d/2)$ instead of W-norms in the right-hand side of the estimates. For short pulses with initial data as in (3.3), the control would therefore be of the form

$$\varepsilon C\left(T, \frac{1}{\beta^{s-d/2}}\right) \frac{1}{\beta^{s-d/2+1}},$$

which is obviously useless when $\beta \to 0$.

Proof of theorem 3.8. (i) This is established by a fixed point argument as in theorem 2.1, thanks to the algebra properties of $W(\mathbb{R}^d)$ (see proposition 3.2). The proof also yields the bound

$$\sup_{0 \leqslant t \leqslant T/\varepsilon} |u_{\mathrm{env}}(t)|_W \leqslant C(T, |u^0|_W). \tag{3.7}$$

Differentiating the equation, one also gets

$$\sup_{0\leqslant t\leqslant T/\varepsilon} |\nabla u_{\rm env}(t)|_W \leqslant C(T,|u^0|_W)|\nabla u^0|_W. \tag{3.8}$$

(ii) Let us decompose $u_{\rm env}$ as

$$u_{\rm env} = u_1 + u_{\rm II}, \quad \text{with } u_{\rm II} = \sum_{j=2}^{m} u_j,$$

and where $u_j = \Pi_j(\boldsymbol{k}+\varepsilon D)u_{\rm env}$ (see notation 3.7).

The proof is divided into four steps.

Step 1. One has

$$\sup_{0\leqslant t\leqslant T/\varepsilon} |\partial_t u_1(t)|_W \leqslant C(T,|u^0|_W)(1+|\nabla u^0|_W).$$

In order to prove this inequality, let us apply the operator $\Pi_1(\boldsymbol{k}+\varepsilon D)$ to (3.6). One then gets

$$\partial_t u_1 + \frac{\mathrm{i}}{\varepsilon}(\omega_1(\boldsymbol{k}+\varepsilon D)-\omega)u_1 = 3\varepsilon\Pi_1(\boldsymbol{k}+\varepsilon D)T^{\rm S}(u_{\rm env}).$$

Since, by assumption, $\omega = \omega_1(\boldsymbol{k})$, one easily gets

$$|\omega_1(\boldsymbol{k}+\varepsilon D)-\omega|_W \leqslant \varepsilon|\nabla\omega_1|_{L^\infty}|\nabla u_1|_W$$

(note that one may infer $|\nabla\omega_1|_{L^\infty} < \infty$ from the observation that, for all $\boldsymbol{k}\neq 0$, $\partial_j\omega_1(\boldsymbol{k})\Pi_1(\boldsymbol{k}) = \Pi_1(\boldsymbol{k})A_j\Pi_1(\boldsymbol{k})$, as seen in the proof of lemma 2.9); (3.7) and (3.8) can then be used to give the desired bound on $\partial_t u_1$.

Step 2. We now want to prove that $u_{\rm II}$ remains of size $O(\varepsilon)$:

$$\sup_{t\in[0,T/\varepsilon]} |u_{\rm II}(t)|_W \leqslant \varepsilon C(T,|u^0|_W)(1+|\nabla u^0|_W).$$

Multiplying (3.6) by $\Pi_j(\boldsymbol{k}+\varepsilon D)$ ($j\geqslant 2$) gives

$$\begin{aligned}\partial_t u_j + \frac{\mathrm{i}}{\varepsilon}(\omega_j(\boldsymbol{k}+\varepsilon D)-\omega)u_j &\\ = 3\varepsilon\Pi_j(\boldsymbol{k}+\varepsilon D)T^{\rm S}(u) &\\ = 3\varepsilon\Pi_j(\boldsymbol{k}+\varepsilon D)T^{\rm S}(u_1) + 3\varepsilon\Pi_j(\boldsymbol{k}+\varepsilon D)(T^{\rm S}(u_{\rm env})-T^{\rm S}(u_1)).&\end{aligned}$$

With

$$S_j(t) = \exp\Big(-\mathrm{i}\frac{t}{\varepsilon}(\omega_j(\boldsymbol{k}+\varepsilon D)-\omega)\Big),$$

one therefore gets

$$\begin{aligned}u_j(t) = S_j(t)u_j^0 + 3\varepsilon\int_0^t S_j(t-t')\Pi_j(\boldsymbol{k}+\varepsilon D)T^{\rm S}(u_1)\,\mathrm{d}t' &\\ + 3\varepsilon\int_0^t S_j(t-t')\Pi_j(\boldsymbol{k}+\varepsilon D)(T^{\rm S}(u_{\rm env})-T^{\rm S}(u_1))\,\mathrm{d}t'.&\end{aligned} \tag{3.9}$$

We now bound the W-norm of the three terms on the right-hand side of (3.9).

- Estimate of $S_j(t)u_j^0$. Since $S_j(t)$ is unitary on $W(\mathbb{R}^d)^n$, one has

$$|S_j(t)u_j^0|_W = |u_j^0|_W = |\Pi_j(\boldsymbol{k}+\varepsilon D)u^0|_W.$$

Since, moreover, one can write

$$\Pi_j(\boldsymbol{k}+\varepsilon D)u^0 = (\Pi_j(\boldsymbol{k}+\varepsilon D) - \Pi_j(\boldsymbol{k}))u^0 + \Pi_j(\boldsymbol{k})u^0,$$

it follows from the orthogonality of the projectors Π_j $(j=1,\dots,m)$ that

$$\Pi_j(\boldsymbol{k}+\varepsilon D)u^0 = (\Pi_j(\boldsymbol{k}+\varepsilon D) - \Pi_j(\boldsymbol{k}))u^0.$$

Since the derivatives of $\Pi_j(\cdot)$ are in general not bounded near the origin, we cannot control the right-hand side by a Taylor expansion and we thus write

$$\begin{aligned}\Pi_j(\boldsymbol{k}+\varepsilon D)u^0 = (\Pi_j(\boldsymbol{k}+\varepsilon D) - \Pi_j(\boldsymbol{k}))1_{\{\varepsilon|D|\leqslant|\boldsymbol{k}|/2\}}u^0 \\ + (\Pi_j(\boldsymbol{k}+\varepsilon D) - \Pi_j(\boldsymbol{k}))1_{\{\varepsilon|D|\geqslant|\boldsymbol{k}|/2\}}u^0,\end{aligned}$$

where $1_{\{\varepsilon|\xi|\leqslant|\boldsymbol{k}|/2\}} = 1$ if $\varepsilon|\xi| \leqslant |\boldsymbol{k}|/2$ and 0 otherwise.

Using the fact that $\Pi_j(\cdot)$ is C^∞ on the ball of centre $\boldsymbol{k}$ and radius $|\boldsymbol{k}|/2$, we can bound the first term of the right-hand side in $W(\mathbb{R}^d)^n$-norm by $\varepsilon\text{const.}|\nabla u^0|_W$; one can also check that a similar estimate holds for the second term of the left-hand side since one has $1 \leqslant 2|\xi|\varepsilon/|\boldsymbol{k}|$ for all $\varepsilon|\xi| \geqslant |\boldsymbol{k}|/2$. We can thus conclude that

$$|S_j(t)u_j^0|_W \leqslant \varepsilon\text{const.}|\nabla u^0|_W. \tag{3.10}$$

- Estimate of

$$A := \varepsilon\int_0^t S_j(t-t')\Pi_j(\boldsymbol{k}+\varepsilon D)T^{\mathrm{S}}(u_1)\,\mathrm{d}t'.$$

Taking the Fourier transform of this term and integrating by parts yields

$$\begin{aligned}&\varepsilon\int_0^t \exp\left(-\mathrm{i}\frac{t-t'}{\varepsilon}(\omega_j(\boldsymbol{k}+\varepsilon\xi)-\omega)\right)\Pi_j(\boldsymbol{k}+\varepsilon\xi)\widehat{T^{\mathrm{S}}(u_1)}\,\mathrm{d}t' \\ &\quad= -\mathrm{i}\varepsilon\int_0^t \frac{1}{\omega_j(\boldsymbol{k}+\varepsilon\xi)-\omega} \\ &\qquad\times\varepsilon\exp\left(-\mathrm{i}\frac{t-t'}{\varepsilon}(\omega_j(\boldsymbol{k}+\varepsilon\xi)-\omega)\right)\Pi_j(\boldsymbol{k}+\varepsilon\xi)\widehat{\partial_t T^{\mathrm{S}}(u_1)}\,\mathrm{d}t' \\ &\qquad+\mathrm{i}\varepsilon\left[\frac{1}{\omega_j(\boldsymbol{k}+\varepsilon\xi)-\omega}\right. \\ &\qquad\qquad\left.\times\varepsilon\exp\left(-\mathrm{i}\frac{t-t'}{\varepsilon}(\omega_j(\boldsymbol{k}+\varepsilon\xi)-\omega)\right)\Pi_j(\boldsymbol{k}+\varepsilon\xi)\widehat{T^{\mathrm{S}}(u_1)}\right]_0^t.\end{aligned}$$

One deduces therefore, using assumption 3.5, that

$$\begin{aligned}&\sup_{t\in[0,T/\varepsilon]}|A(t)|_W \\ &\qquad\leqslant \text{const.}\frac{\varepsilon T}{c_0}\sup_{[0,T/\varepsilon]}|u_1|_W^2\sup_{[0,T/\varepsilon]}|\partial_t u_1|_W + \text{const.}\frac{\varepsilon^2}{c_0}\sup_{[0,T/\varepsilon]}|u_1|_W^3,\end{aligned}$$

so that, owing to (3.7), (3.8) and step 1,

$$\sup_{t\in[0,T/\varepsilon]} |A(t)|_W \leqslant \varepsilon C(T,|u^0|_W)(1+|\nabla u^0|_W). \tag{3.11}$$

- Estimate of

$$B := \varepsilon \int_0^t S_j(t-t')\Pi_j(\boldsymbol{k}+\varepsilon D)(T^{\mathrm{S}}(u_{\mathrm{env}})-T^{\mathrm{S}}(u_1))\,\mathrm{d}t'.$$

First recall that, owing to the trilinearity of T, one has, for all $t\in[0,T/\varepsilon]$,

$$|T^{\mathrm{S}}(u_{\mathrm{env}})(t)-T^{\mathrm{S}}(u_1)(t)|_W \leqslant \text{const.} \sup_{[0,T/\varepsilon]} |u_{\mathrm{env}}|_W^2 |u_{\mathrm{II}}(t)|_W;$$

using (3.7) and (3.8), we therefore obtain

$$\sup_{t\in[0,T/\varepsilon]} |B(t)|_W \leqslant \varepsilon C(T,|u^0|_W)\int_0^t |u_{\mathrm{II}}(t')|_W\,\mathrm{d}t'. \tag{3.12}$$

It is now a direct consequence of $(3.9)_j$ $(j=2,\dots,m)$ and (3.10)–(3.12) that, for all $t\in[0,T/\varepsilon]$,

$$|u_{\mathrm{II}}(t)|_W \leqslant \varepsilon C(T,|u^0|_W)(1+|\nabla u^0|_W) + \varepsilon C(T,|u^0|_W)\int_0^t |u_{\mathrm{II}}(t')|_W\,\mathrm{d}t',$$

and the result therefore follows from Gronwall's lemma.

Step 3. We construct here a solution $U\in W(\mathbb{R}^d\times\mathbb{T})$ to (2.3) as a perturbation of $U_{\mathrm{app}} = u_{\mathrm{env}}\mathrm{e}^{\mathrm{i}\theta}+\text{c.c.}$:

$$U(t,x,\theta) = U_{\mathrm{app}}(t,x,\theta)+\varepsilon V(t,x,\theta).$$

We want to prove that it is possible to find such a V on the time interval $[0,T/\varepsilon]$ and that V remains bounded (with respect to ε) in $C([0,T/\varepsilon];W(\mathbb{R}^d\times\mathbb{T})^n)$. The equation that V must solve is

$$\begin{aligned}\partial_t V + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega D_\theta, \boldsymbol{k}D_\theta+\varepsilon D)V \\ = T(u_{\mathrm{env}},u_{\mathrm{env}},u_{\mathrm{env}})\mathrm{e}^{3\mathrm{i}\theta}+\text{c.c.} \\ + (T(U_{\mathrm{app}}+\varepsilon V,U_{\mathrm{app}}+\varepsilon V,U_{\mathrm{app}}+\varepsilon V)-T(U_{\mathrm{app}},U_{\mathrm{app}},U_{\mathrm{app}})).\end{aligned}$$

Owing to assumption 2.13, we can look for V in the form

$$V(t,x,\theta) = V_0(t,x,\theta)+\varepsilon v_1(t,x)\mathrm{e}^{3\mathrm{i}\theta}+\text{c.c.},$$

with $v_1 = -\mathrm{i}\mathcal{L}(3\omega,3\boldsymbol{k})^{-1}T(u_1,u_1,u_1)$; the resulting equation on V_0 is

$$\partial_t V_0 + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(\omega D_\theta,\boldsymbol{k}D_\theta+\varepsilon D)V_0 = I_1+I_2+I_3, \tag{3.13}$$

with

$$\begin{aligned} I_1 &= (T(u_{\mathrm{env}},u_{\mathrm{env}},u_{\mathrm{env}})-T(u_1,u_1,u_1))\mathrm{e}^{3\mathrm{i}\theta}+\text{c.c.},\\ I_2 &= -\varepsilon(\partial_t+A(\partial))v_1\mathrm{e}^{3\mathrm{i}\theta}+\text{c.c.},\\ I_3 &= (T(U_{\mathrm{app}}+\varepsilon V,U_{\mathrm{app}}+\varepsilon V,U_{\mathrm{app}}+\varepsilon V)-T(U_{\mathrm{app}},U_{\mathrm{app}},U_{\mathrm{app}})).\end{aligned}$$

Let us now bound I_j $(j = 1, 2, 3)$ in $W(\mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$ and for all $t \in [0, T/\varepsilon]$.

- From (3.7), (3.8) and step 2, one gets
$$|I_1(t)|_{W(\mathbb{R}^d\times\mathbb{T})} \leqslant \varepsilon C(T, |u^0|_W)(1 + |\nabla u^0|_W). \tag{3.14}$$
- From the definition of v_1, (3.7), (3.8) and step 1, one has directly
$$|I_2(t)|_{W(\mathbb{R}^d\times\mathbb{T})} \leqslant \varepsilon C(T, |u^0|_W)(1 + |\nabla u^0|_W). \tag{3.15}$$
- From the trilinearity of T, one similarly obtains
$$|I_3(t)|_{W(\mathbb{R}^d\times\mathbb{T})} \leqslant \varepsilon C(T, |u^0|_W)(1 + |V_0(t)|_W + \varepsilon^2 |V_0(t)|_W^3). \tag{3.16}$$

Since the semigroup
$$S(t) = \exp\left(-\mathrm{i}\frac{t}{\varepsilon}\mathcal{L}(\omega D_\theta, \boldsymbol{k} D_\theta + \varepsilon D)\right)$$
is unitary on $W(\mathbb{R}^d \times \mathbb{T})$, the estimates (3.14)–(3.16) allow one to conclude the existence of a solution $V_0 \in C([0, T/\varepsilon]; W(\mathbb{R}^d \times \mathbb{T})^n)$ to (3.13) using a fixed-point formulation (the fact that the existence time can be taken equal to T for ε_0 small enough is obtained as in lemma 2.16). After a Gronwall argument, one also gets
$$\sup_{0\leqslant t\leqslant T/\varepsilon} |V_0(t)|_W \leqslant C(T, |u^0|_W)(1 + |\nabla u^0|_W). \tag{3.17}$$

Step 4 (completion of the proof). Since $U(t) - U_{\mathrm{app}}(t) = \varepsilon V(t)$, it follows from the above that
$$\sup_{t\in[0,T/\varepsilon]} |U(t) - U_{\mathrm{app}}(t)|_{W(\mathbb{R}^d\times\mathbb{T})} \leqslant \varepsilon C(T, |u^0|_W)(1 + |\nabla u^0|_W),$$
and the theorem follows therefore from the observation that
$$|\boldsymbol{U} - \boldsymbol{U}_{\mathrm{SVEA}}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leqslant \sup_{t\in[0,T/\varepsilon]} |U(t) - U_{\mathrm{app}}(t)|_{W(\mathbb{R}^d\times\mathbb{T})}.$$

□

3.4. The full dispersion model

The full dispersion model consists in approximating the exact solution to (1.1) by $\boldsymbol{U}_{\mathrm{FD}}$ defined as
$$\boldsymbol{U}_{\mathrm{FD}}(t, x) = u_{\mathrm{FD}}(t, x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x - \omega t)/\varepsilon} + \text{c.c.}, \tag{3.18}$$
where the (ε-dependent) profile u_{FD} solves the *full dispersion* scalar equation
$$\left.\begin{aligned} \partial_t u_{\mathrm{FD}} + \frac{\mathrm{i}}{\varepsilon}(\omega_1(\boldsymbol{k} + \varepsilon D) - \omega)u_{\mathrm{FD}} = 3\varepsilon\pi_1(\boldsymbol{k})T^{\mathrm{S}}(u_{\mathrm{FD}}),\\ u_{\mathrm{FD}}|_{t=0}(x) = u^0(x) \end{aligned}\right\} \tag{3.19}$$
with $\omega_1(\cdot)$ as in assumption 3.5.

The following corollary shows that the full dispersion scalar equation yields an approximation of the same precision as the envelope equation for times $t \in [0, T/\varepsilon]$.

COROLLARY 3.11 (full dispersion model). *Under the assumptions of theorem 3.8, there exists a unique solution* $u_{\rm FD} \in C([0,T/\varepsilon];W(\mathbb{R}^d)^n)$ *to (3.19) for some* $T>0$ *and all* $0<\varepsilon<\varepsilon_0$.

If ε_0 *is small enough, then the solution* $\boldsymbol{U}$ *to (1.1) exists on* $[0,T/\varepsilon]$ *and*

$$|\boldsymbol{U}-\boldsymbol{U}_{\rm FD}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leqslant \varepsilon C(T,|u^0|_W)(1+|\nabla u^0|_W),$$

where $\boldsymbol{U}_{\rm FD}$ *is as defined in (3.18).*

REMARK 3.12. The quantity $u_{\rm FD}$ remains $\mathbb{C}^n$-valued, but we call (3.19) a *scalar* approximation because the operator $\mathrm{i}(\omega_1(\boldsymbol{k}+\varepsilon D)-\omega)/\varepsilon$ is scalar, which is not the case for $\mathrm{i}\mathcal{L}(\omega,\boldsymbol{k}+\varepsilon D)/\varepsilon$ in the envelope equation (3.6). The interest of the FD model is that $\mathrm{i}(\omega_1(\boldsymbol{k}+\varepsilon D)-\omega)u/\varepsilon$ remains bounded for spectrally localized functions u, while $\mathrm{i}\mathcal{L}(\omega,\boldsymbol{k}+\varepsilon D)u/\varepsilon$ is of order $O(1/\varepsilon)$. The fast oscillations of the non-polarized modes must therefore be taken into account with the envelope approximation, and the discretization step must therefore be much smaller in numerical computations than for the FD model.

REMARK 3.13. Performing the same analysis as in remark 3.9, one can check that the 'practical rule' also applies for the FD model.

Proof. We omit the existence/uniqueness part of the corollary, since it is obtained with the same tools as for theorem 3.8 (in particular, taking a smaller ε_0 if necessary), the existence time of the envelope equation is larger than the existence time for (3.19) and we thus focus on the error estimate.

Denoting, as in the proof of theorem 3.8, $u_1=\pi_1(\boldsymbol{k}+\varepsilon D)u_{\rm env}$, where $u_{\rm env}$ is the solution of the envelope equation, from step 2 of the proof of theorem 3.8 one obtains

$$\sup_{t\in[0,T/\varepsilon]}|u_{\rm env}(t)-u_1(t)|_W \leqslant \varepsilon C(T,|u^0|_W)(1+|\nabla u^0|_W),$$

so that it suffices to control $|u_1(t)-u_{\rm FD}(t)|_W$ to prove corollary 3.11. Applying $\Pi_1(\boldsymbol{k}+\varepsilon D)$ to (3.6), one gets

$$\partial_t u_1+\frac{\mathrm{i}}{\varepsilon}(\omega_1(\boldsymbol{k}+\varepsilon D)-\omega)u_1 = 3\varepsilon\Pi_1(\boldsymbol{k}+\varepsilon D)T^{\rm S}(u_{\rm env}),$$

so that the difference $v=u_1-u_{\rm FD}$ solves

$$\left.\begin{aligned}\partial_t v+\frac{\mathrm{i}}{\varepsilon}(\omega_1(\boldsymbol{k}+\varepsilon D)-\omega)v = 3\varepsilon\Pi_1(\boldsymbol{k}+\varepsilon D)T^{\rm S}(u_{\rm env})-\varepsilon\Pi_1(\boldsymbol{k})T^{\rm S}(u_{\rm FD}),\\ v|_{t=0}(x)=\Pi_1(\boldsymbol{k}+\varepsilon D)u^0-u^0.\end{aligned}\right\} \quad (3.20)$$

Recall now that

$$\begin{aligned}&\Pi_1(\boldsymbol{k}+\varepsilon D)T^{\rm S}(u_{\rm env})-\Pi_1(\boldsymbol{k})T^{\rm S}(u_{\rm FD})\\ &\quad=(\Pi_1(\boldsymbol{k}+\varepsilon D)-\Pi_1(\boldsymbol{k}))T^{\rm S}(u_{\rm env})\\ &\qquad+\Pi_1(\boldsymbol{k})(T^{\rm S}(u_{\rm env})-T^{\rm S}(u_1))+\Pi_1(\boldsymbol{k})(T^{\rm S}(u_1)-T^{\rm S}(u_{\rm FD})).\end{aligned} \quad (3.21)$$

Since $|(\Pi_1(\boldsymbol{k}+\varepsilon D)-\Pi_1(\boldsymbol{k}))T^{\rm S}(u_{\rm env})|_W \leqslant \varepsilon\mathrm{const.}|\nabla T^{\rm S}(u_{\rm env})|_W$ (see the proof of (3.10)), one can use (3.7) and (3.8) to bound the first component of the right-hand side of (3.21) from above by $\varepsilon C(T,|u^0|_W)|\nabla u^0|_W$. The second component

of (3.21) can be estimated exactly like the term I_1 in (3.14), while the last one is bounded from above in $W(\mathbb{R}^d)$ by $C(|u_1|_W, |u_{\rm FD}|_W)|v|_W$. Since, moreover, $|u_1|_W$ is controlled by (3.7), (3.8) and a similar estimate also obviously holds for $|u_{\rm FD}|_W$, one may deduce that, for all $0 \leqslant t \leqslant T/\varepsilon$,

$$|T^{\rm S}(u_{\rm env}(t)) - T^{\rm S}(u_{\rm FD}(t))|_W \leqslant \varepsilon C(T, |u^0|_W)(1 + |\nabla u^0|_W) + C(T, |u^0|_W)|v(t)|_W.$$

This inequality, together with an energy estimate for (3.20) and a Gronwall argument, shows that

$$\sup_{t\in[0,T/\varepsilon]} |v(t)|_W \leqslant \varepsilon C(T, |u^0|_W)(1 + |\nabla u^0|_W),$$

where we also used the estimate $|\Pi_1(\boldsymbol{k} + \varepsilon D)u^0 - u^0|_W \leqslant \varepsilon\text{const.}(1 + |\nabla u^0|_W)$ (which is proved with the same arguments as (3.10)). □

3.5. The nonlinear Schrödinger equation

As seen in § 2 (see remark 2.14), the exact solution to (1.1) is approximated by $\boldsymbol{U}_{\rm NLS}$ defined as

$$\boldsymbol{U}_{\rm NLS}(t,x) = u_{\rm NLS}(t,x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x - \omega t)/\varepsilon} + \text{c.c.}, \tag{3.22}$$

where the (ε-dependent) profile $U_{(2)}$ solves the *nonlinear Schrödinger* equation

$$\left.\begin{aligned} \partial_t u_{\rm NLS} + c_{\rm g}(\boldsymbol{k})\cdot\nabla u_{\rm NLS} - \tfrac{1}{2}\varepsilon\mathrm{i}\omega''(\boldsymbol{k})(\partial,\partial)u_{\rm NLS} \\ = 3\varepsilon\Pi_1(\boldsymbol{k})T^{\rm S}(u_{\rm NLS}, u_{\rm NLS}, \bar{u}_{\rm NLS}), \\ u_{\rm NLS}|_{t=0}(y) = u^0. \end{aligned}\right\} \tag{3.23}$$

One can deduce from theorem 3.8 the following refinement of theorem 2.15.

Corollary 3.14 (Schrödinger approximation). *Under the assumptions of theorem 3.8, there exists a unique solution $u_{\rm NLS} \in C([0,T/\varepsilon); W(\mathbb{R}^d)^n)$ to (3.23), for some $T > 0$ and all $0 < \varepsilon < \varepsilon_0$.*

If, moreover, $u^0 \in W^3(\mathbb{R}^d)^n$ and ε_0 is small enough, then the solution $\boldsymbol{U}$ to (1.1) exists on $[0,T/\varepsilon]$ and

$$|\boldsymbol{U} - \boldsymbol{U}_{\rm NLS}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leqslant \varepsilon C(T, |u^0|_W)(1 + |\nabla u^0|_W + |\mathfrak{c}_{\rm NLS}|_\infty |u^0|_{W^3}),$$

where $\boldsymbol{U}_{\rm NLS}$ is as defined in (3.22) and

$$\mathfrak{c}_{\rm NLS}(\xi) := \frac{\omega_1(\boldsymbol{k}+\varepsilon\xi) - (\omega + \varepsilon c_{\rm g}(\boldsymbol{k})\cdot\xi + \varepsilon^2\frac{1}{2}\omega_1''(\xi,\xi))}{\varepsilon^3(1+|\xi|^3)}.$$

Remark 3.15.

(i) A third-order Taylor expansion of $\omega_1(\boldsymbol{k}+\varepsilon\xi)$ at $\xi = 0$ shows that $|\mathfrak{c}_{\rm NLS}|_\infty$ is finite and can be bounded from above independently from ε.

(ii) The component $|\mathfrak{c}_{\rm NLS}|_\infty|u^0|_{W^3}$ of the error estimate does not appear for the full dispersion model. It is due to the approximation of the non-local operator $\mathrm{i}(\omega_1(\boldsymbol{k}+\varepsilon D) - \omega)/\varepsilon$ (left-hand side of (3.19)) by the differential operator $c_{\rm g}(\boldsymbol{k})\cdot\nabla - \frac{1}{2}\varepsilon\mathrm{i}\omega_1''(\boldsymbol{k})(\partial,\partial)$ (left-hand side of (3.23)). This error term is thus a *linear* effect.

(iii) This additional term is responsible for the bad behaviour of the Schrödinger equation when modelling short pulses. For instance, for initial data like (3.3), the precision of the Schrödinger approximation is of order

$$O\left(\varepsilon\left(\frac{1}{\beta}+\frac{|\mathfrak{c}_{\rm NLS}|_\infty}{\beta^3}\right)\right) \quad \text{when } \beta \ll 1.$$

In order for the Schrödinger approximation to keep the same order of precision as the full dispersion model, one needs therefore to have

$$\frac{|\mathfrak{c}_{\rm NLS}|_\infty}{\beta^3} \lesssim \frac{1}{\beta},$$

which requires that $\beta^2 \geqslant |\mathfrak{c}_{\rm NLS}|_\infty$. This condition is far much restrictive than the practical rule $\beta \gg \varepsilon$.

(iv) The advantage of the formulation (2.16) of the NLS approximation over (3.23) is that it is *independent of* ε. However, (3.23) is interesting because it admits useful generalizations for short pulses, as shown in the next section.

Proof. As in the proof of corollary 3.11, we focus on the error estimate and omit the existence/uniqueness part of the proof.

The difference $v = u_{\rm FD} - u_{\rm NLS}$ of the solution of the full dispersion and Schrödinger equations solves the initial-value problem

$$\left.\begin{aligned} \partial_t v + \frac{\mathrm{i}}{\varepsilon}(\omega_1(\boldsymbol{k}+\varepsilon D)-\omega)v = 3\varepsilon \Pi_1(\boldsymbol{k})(T^{\rm S}(u_{\rm FD}) - T^s(u_{\rm NLS})) - \varepsilon^2 \mathcal{R}_2(D)u_{\rm NLS}, \\ v|_{t=0}(x) = 0, \end{aligned}\right\} \tag{3.24}$$

where, for all $\xi \in \mathbb{R}^d$,

$$\mathcal{R}_2(\xi) = \frac{1}{\varepsilon^3}(\mathrm{i}\omega_1(\boldsymbol{k}+\varepsilon\xi) - \mathrm{i}\omega - \mathrm{i}\varepsilon c_{\rm g}(\boldsymbol{k})\cdot\xi - \varepsilon^2 \tfrac{1}{2}\mathrm{i}\omega_1''(\xi,\xi)).$$

Note now that one has, for all $0 \leqslant t \leqslant \tau/\varepsilon$,

$$|\mathcal{R}_2(D)u_{\rm NLS}(t)|_W \leqslant |\mathfrak{c}_{\rm NLS}|_\infty |u_{\rm NLS}(t)|_{W^3},$$

with $\mathfrak{c}_{\rm NLS}(\cdot)$ as in the statement of the corollary; differentiating the Schrödinger equation (3.23) and estimating the W-norm of the solution, one also easily obtains

$$\sup_{t\in[0,T/\varepsilon]} |u_{\rm NLS}(t)|_{W^3} \leqslant C(T,|u^0|_W)(1+|u^0|_{W^3}).$$

Since the first term of the right-hand side of (3.24) can be bounded as in (3.14), one obtains from Gronwall's lemma applied to (3.24) that

$$\sup_{t\in[0,T/\varepsilon]} |v(t)|_W \leqslant C(T,|u^0|_W)(1+|\mathfrak{c}_{\rm NLS}|_\infty |u^0|_{W^3}),$$

which, together with corollary 3.11, yields the result. □

3.6. The nonlinear Schrödinger equation with improved dispersion relation

We derive here new approximations based on a family of modified Schrödinger equations, whose dispersive properties are closer to the exact model. Such an approximation $\boldsymbol{U}_{\rm imp}$ is defined as

$$\boldsymbol{U}_{\rm imp}(t,x) = u_{\rm imp}(t,x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x-\omega t)/\varepsilon} + \text{c.c.}, \tag{3.25}$$

where $u_{\rm imp}$ solves the *nonlinear Schrödinger equation with improved dispersion relation*:

$$\left.\begin{aligned}(1 - \mathrm{i}\varepsilon\boldsymbol{b}\cdot\nabla - \varepsilon^2\nabla\cdot B\nabla)\partial_t u_{\rm imp} + c_{\rm g}(\boldsymbol{k})\cdot\nabla u_{\rm imp} \\ - \tfrac{1}{2}\varepsilon\mathrm{i}(\omega_1''(\boldsymbol{k})(\partial,\partial) + 2\nabla\cdot(c_{\rm g}(\boldsymbol{k})\otimes\boldsymbol{b})\nabla)u_{\rm imp} + \varepsilon^2\boldsymbol{C}(\nabla)u_{\rm imp} \\ = 3\varepsilon\Pi_1(\boldsymbol{k})T^{\rm S}(u_{\rm imp}), \\ u_{\rm imp}|_{t=0}(x) = u^0(x),\end{aligned}\right\} \tag{3.26}$$

where $\boldsymbol{b}\in\mathbb{C}^d$, $B\in\mathcal{M}_{d\times d}(\mathbb{R})$ and $\boldsymbol{C}\colon \mathbb{C}^d\times\mathbb{C}^d\times\mathbb{C}^d\to\mathbb{C}$ is a trilinear mapping. We assume, moreover, that

$$B \text{ is symmetric positive,} \quad \boldsymbol{b}\in\operatorname{range}(B) \quad \text{and} \quad 4 - \boldsymbol{b}\cdot(B^{-1}\boldsymbol{b}) > 0 \tag{3.27}$$

(note that even though $B^{-1}\boldsymbol{b}$ is not unique when B is not definite, the scalar $\boldsymbol{b}\cdot(B^{-1}\boldsymbol{b})$ is uniquely defined). One then has the following result.

Corollary 3.16 (improved Schrödinger approximation). *Under the assumptions of theorem 3.8, there exists a unique solution $u_{\rm imp}\in C([0,T/\varepsilon);W(\mathbb{R}^d)^n)$ to (3.26), for some $T>0$ and all $0<\varepsilon<\varepsilon_0$.*

If, moreover, $u^0\in W^3(\mathbb{R}^d)^n$ and ε_0 is small enough, then the solution $\boldsymbol{U}$ to (1.1) exists on $[0,T/\varepsilon]$ and

$$|\boldsymbol{U} - \boldsymbol{U}_{\rm imp}|_{L^\infty([0,T/\varepsilon]\times\mathbb{R}^d)} \leqslant \varepsilon C(T,|u^0|_W)(1 + |\nabla u^0|_W + |\mathfrak{c}_{\rm imp}|_\infty|u^0|_{W^3}),$$

where $\boldsymbol{U}_{\rm imp}$ is as defined in (3.25) and

$$\begin{aligned}&\mathfrak{c}_{\rm imp}(\xi)\\ &:= \left(\omega_1(\boldsymbol{k}+\varepsilon\xi) - \left(\omega + \varepsilon\frac{c_{\rm g}(\boldsymbol{k})\cdot\xi + \frac{1}{2}\varepsilon(\omega_1''(\xi,\xi) + 2(c_{\rm g}(\boldsymbol{k})\cdot\xi)(\boldsymbol{b}\cdot\xi)) - \varepsilon^2\boldsymbol{C}(\xi)}{1+\varepsilon\boldsymbol{b}\cdot\xi+\varepsilon^2\xi\cdot B\xi}\right)\right)\\ &\quad\times\frac{1}{\varepsilon^3(1+|\xi|^3)}.\end{aligned}$$

Remark 3.17.

(i) As for the Schrödinger equation, one can check by a simple Taylor expansion that $|\mathfrak{c}_{\rm imp}|_\infty$ is finite and uniformly bounded with respect to ε.

(ii) Taking $\boldsymbol{b}=0$, $B=0$ and $\boldsymbol{C}=0$ (this choice satisfies (3.27)), one recovers the usual Schrödinger equation (3.23).

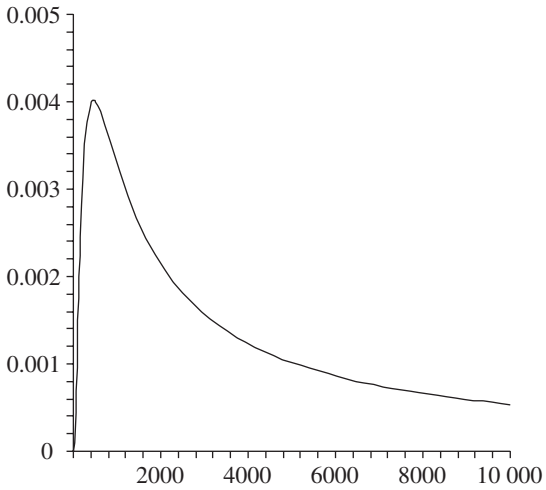

Figure 3. $\mathfrak{c}_{\rm imp}(\xi)/\mathfrak{c}_{\rm NLS}(\xi)$ for $\varepsilon = 0.01$ with the coefficients (3.28).

(iii) The interest of (3.26) with respect to (3.23) is that one can choose $\boldsymbol{b}$, B and $\boldsymbol{C}$ such that $\mathfrak{c}_{\rm imp} \ll \mathfrak{c}_{\rm NLS}$, thus improving considerably the accuracy of the approximation. In the one-dimensional case $d = 1$, it is possible to choose $\boldsymbol{b}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ in such a way that the dispersion relation for (3.26) is the $[3, 2]$-Padé expansion of the dispersion relation of (3.19). For the case of the Klein–Gordon system of example 1.5, this leads to

$$\boldsymbol{b} = \frac{2\boldsymbol{k}}{\boldsymbol{v}^2 + \boldsymbol{k}^2}, \qquad \boldsymbol{B} = \frac{\boldsymbol{v}^2 + 4\boldsymbol{k}^2}{4(\boldsymbol{v}^2 + \boldsymbol{k}^2)^2}, \qquad \boldsymbol{C} = \frac{\boldsymbol{k}(3\boldsymbol{v}^2 + 4\boldsymbol{k}^2)}{4(\boldsymbol{v}^2 + \boldsymbol{k}^2)^{5/2}}; \tag{3.28}$$

we illustrate in figure 3 how much one gains by working with (3.26) instead of (3.23) for the Klein–Gordon system of example 1.5 with $\boldsymbol{v} = \boldsymbol{k} = 1$.

(iv) The same analysis as in remark 3.15(iii) shows that the approximation provided by (3.26) is of the same order as the envelope approximation if $\beta^2 \geqslant |\mathfrak{c}_{\rm imp}|_\infty$. Since $|\mathfrak{c}_{\rm imp}|_\infty \ll |\mathfrak{c}_{\rm NLS}|_\infty$, this condition is much weaker than the corresponding one for the usual Schrödinger model. In some particular cases, this condition can even be weaker than the 'practical rule' $\varepsilon \ll \beta$ and the range of validity of the model will be determined by the latter.

Proof. Choosing $\xi_0 \in -\frac{1}{2}B^{-1}\boldsymbol{b}$, one can check that

$$1 + \boldsymbol{b}\cdot\xi + \xi\cdot B\xi = 1 - \tfrac{1}{4}\boldsymbol{b}\cdot(B^{-1}\boldsymbol{b}) + (\xi - \xi_0)\cdot B(\xi - \xi_0),$$

so that it follows from assumption (3.27) that $1 + \boldsymbol{b}\cdot\xi + \xi\cdot B\xi > 0$ (uniformly with respect to $\xi \in \mathbb{R}^d$). The operator $1 - \varepsilon \mathrm{i}\boldsymbol{b} - \varepsilon^2\nabla\cdot B\nabla$ is therefore invertible, and its inverse is the Fourier multiplier $(1 + \varepsilon\boldsymbol{b}\cdot D + \varepsilon^2 D\cdot BD)^{-1}$. Equation (3.26) can therefore be rewritten as

$$\partial_t u_{\rm imp} + \mathrm{i}\frac{c_{\rm g}(\boldsymbol{k})\cdot D + \frac{1}{2}\varepsilon(\omega_1''(D, D) + (c_{\rm g}(\boldsymbol{k})\cdot D)(D\cdot\boldsymbol{b})) - \varepsilon^2\boldsymbol{C}(D)}{(1 + \varepsilon\boldsymbol{b}\cdot D + \varepsilon^2 D\cdot BD)}u_{\rm imp} = 3(1 + \varepsilon\boldsymbol{b}\cdot D + \varepsilon^2 D\cdot BD)^{-1}\pi_1(\boldsymbol{k})T^{\rm S}(u_{\rm imp}).$$

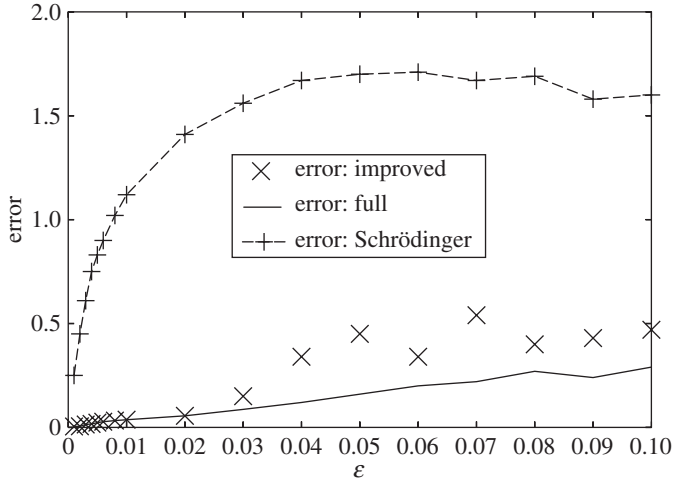

Figure 4. Short pulses: the errors $E_{(j)}(\varepsilon, \beta)$ for $\beta = 0.1$ and $\varepsilon \in [0.001, 0.1]$; $j = 1$ corresponds to FD, $j = 2$ to Schrödinger and $j = 3$ to the improved Schrödinger.

Since $(1 + \varepsilon \boldsymbol{b} \cdot D + \varepsilon^2 D \cdot BD)^{-1}$ is regularizing (of order -2) and acts on $W(\mathbb{R}^d)$ uniformly with respect to $\varepsilon > 0$, the proof of the result follows exactly the same lines as the proof of corollary 3.16 and we thus omit it. □

3.7. Numerical validation

We have already seen in § 3.1.3 (see figure 2) that the NLS approximation is completely inaccurate for short pulses. It is interesting to check numerically the validity of the models introduced in this section (we refer the reader to [9] for more detailed examples).

Figure 4 shows that for short pulses like (3.3) the full dispersion provides, as expected, a good approximation up to a ratio $\varepsilon/\beta \sim 0.25$. It also shows that, up the same ratio, the improved Schrödinger model provides an approximation of similar accuracy.

At this point of the discussion, we can draw the following conclusions:

- the NLS approximation is not accurate for short pulses like (3.3) when β is small.

- the full dispersion and improved Schrödinger models furnish good approximations for short pulses, provided that the ratio ε/β is small enough.

The FD and improved NLS models are better than NLS for short pulses because they successfully solve the first shortcoming of the NLS approximation (bad dispersive behaviour; see § 3.1.1). However, they do *not* address the second shortcoming, namely, the validity of the SVEA (see § 3.1.2) and this is why they fail when the ratio ε/β is not small enough (in accordance with the 'practical rule' (3.5)).

Pulses for which ε/β is not small are called *ultrashort pulses*; they require a completely different analysis, as sketched in the next section.

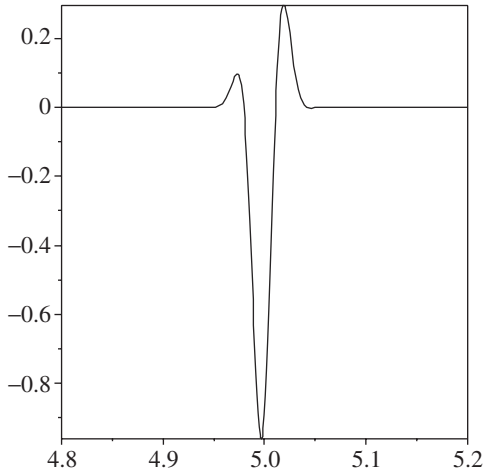

Figure 5. Initial condition $u^0_{(\beta)}(x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x)/\varepsilon} + \text{c.c.}$ with a Gaussian u^0, $\varepsilon = 0.01$, and for $\beta = 0.02$.

4. Ultrashort pulses

As explained in § 3.7, ultrashort pulses are so short that the SVEA must be abandoned. For short pulses of the form (3.3), this means that $\beta \sim \varepsilon$ (see figure 5).

The goal of this section is to propose an approach to describe the behaviour of such pulses in dispersive media. We refer the reader to [4] for more details and focus here on their qualitative behaviour. In particular, we show that their dynamics becomes *linear*, which is a striking difference with respect to the standard pulses studied in § 2.

Notation 4.1. For the sake of simplicity, we assume throughout this section that $\mathcal{C}_{\mathcal{L}}$ is symmetric around the axis (0ω). It follows that the group velocity $c_{\mathrm{g}}(\boldsymbol{k})$ is always collinear to $\boldsymbol{k}$. By abuse of notation, we also write $\mathcal{C}_{\mathcal{L}}$ for the intersection of $\mathcal{C}_{\mathcal{L}}$ with the plane containing both (0ω) and $\boldsymbol{k}$.

4.1. Modelling ultrashort pulses: different approaches

Alterman and Rauch [1–3] modelled ultrashort pulses in *non-dispersive media* ($E = 0$ in (1.1)) by replacing the fast oscillating term in the initial condition by a fast decaying one; more precisely, they considered the initial condition for (1.1),

$$\boldsymbol{U}|_{t=0} = \mathcal{U}^0\left(x, \frac{\boldsymbol{k}\cdot x}{\varepsilon}\right), \tag{4.1}$$

such that $\mathcal{U}^0(x,\tilde{z}) \to 0$ as $|\tilde{z}| \to \infty$, and consequently replaced the representation formula (2.2) by

$$u(t,x) \sim U\left(t, x, \frac{\boldsymbol{k}\cdot x - \omega(\boldsymbol{k})t}{\varepsilon}\right), \tag{4.2}$$

with $U(t,x,\tilde{z}) \to 0$ as $\tilde{z} \to \infty$. The Schrödinger equation (3.23) is then replaced by

$$\partial_t\partial_{\tilde{z}}U + (c_{\mathrm{g}}(\boldsymbol{k})\cdot\nabla)\partial_{\tilde{z}}U + \tfrac{1}{2}\varepsilon\mathcal{R}(\partial,\partial)U = \varepsilon\partial_{\tilde{z}}\tilde{F}(U), \quad U|_{t=0} = \mathcal{U}^0; \tag{4.3}$$

this approximation (rigorously justified) uses the fact that the group velocity $c_{\mathrm{g}}(\boldsymbol{k})$ does not depend on $|\boldsymbol{k}|$ and is therefore only valid in *non-dispersive media* ($E = 0$ in (1.1)). Alterman and Rauch's approach has been generalized in [6, 20] by taking into account the particularities of the optical susceptibility of some cubic nonlinear and weakly dispersive media such as silica, and finally obtaining a *quasilinear* variant of (4.3), which is rigorously justified in the linear case.

In order to model the propagation of ultrashort pulses in *dispersive* media, we use here the approach developed in [4] and based on the functional tools introduced in [18]. Another approach was developed in [22] by Texier; in studying the propagation of short waves (which can be seen as one-dimensional ultrashort pulses with transverse perturbations) by dispersive quasilinear hyperbolic systems, he derived an equation which echoes the Alterman–Rauch equation.

4.2. Functional setting

We have quite often used the representation (2.2) for the exact solution $\boldsymbol{U}$ to (1.1),

$$\boldsymbol{U}(t,x) = U\biggl(t, x, \frac{\boldsymbol{k}\cdot x - \omega t}{\varepsilon}\biggr),$$

where U solves the profile equation (2.3). It is worth noting that a more general representation formula is

$$\boldsymbol{U}(t,x) = \mathcal{U}\biggl(t, x, -\frac{t}{\varepsilon}, \frac{z}{\varepsilon}\biggr),$$

where $(0z)$ stands for the direction of $\boldsymbol{k}$.

If $\boldsymbol{V}(t,x)$ is a wave packet

$$\boldsymbol{V}(t,x) = v(t,x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x - \omega t)/\varepsilon} + \text{c.c.},$$

with $v \in C(\mathbb{R}; H^s(\mathbb{R}^d))$, then, using such a representation, one has

$$\boldsymbol{V}(t,x) = \mathcal{V}\biggl(t, x, -\frac{t}{\varepsilon}, \frac{z}{\varepsilon}\biggr),$$

with $\mathcal{V}(t,x,\tilde{t},\tilde{z}) = v(t,x)\mathrm{e}^{\mathrm{i}k\tilde{z}+\omega\tilde{t}} + \text{c.c.}$,where $k = |\boldsymbol{k}|$. In particular, taking the Fourier transform with respect to $(\tilde{t},\tilde{z})$ yields

$$\mathcal{F}_{\tilde{t},\tilde{z}}\mathcal{V}(t,x,\cdot,\cdot) = v(t,x)\delta_{(\omega,k)} + \overline{v(t,x)}\delta_{-(\omega,k)},$$

where $\delta_{\pm(\omega,k)}$ is the Dirac measure in $\mathbb{R}^{1+1}$ located at $\pm(\omega,k)$. In particular, $\mathcal{F}_{\tilde{t},\tilde{z}}\mathcal{V}$ is continuous with respect to t, and with values in the space of bounded variation and $H^s(\mathbb{R}^d)$-valued Borel measures on $\mathbb{R}^{1+1}$.

This motivates the following definition.

Definition 4.2.

(i) For all $s \in \mathbb{R}$, we define E^s as

$$E^s = \mathcal{F}^{-1}_{\tilde{\tau},\tilde{\zeta}}\bigl[\,\mathrm{BV}(\mathbb{R}^{1+1}_{\tilde{\tau},\tilde{\zeta}}; H^s(\mathbb{R}^d))\bigr],$$

where $\mathrm{BV}(\mathbb{R}^{1+1}; H^s(\mathbb{R}^d))$ denotes the set of bounded variation $H^s(\mathbb{R}^d)$-valued Borel measures on $\mathbb{R}^{1+1}$.

(ii) If $V \in E^s$, we call *spectrum* of V and denote by $\operatorname{Sp} V$ the support of the measure $\mathcal{F}_{\tilde{t},\tilde{z}} V$.

For all $T > 0$, we write $E^s_T = C([0,T]; E^s)$, endowed with its canonical norm.

EXAMPLE 4.3. The above discussion shows that the initial condition considered in (2.1) can be represented as

$$u^0(x)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot x)/\varepsilon} + \text{c.c.} = \mathcal{U}^0\left(x, 0, \frac{z}{\varepsilon}\right),$$

where $\boldsymbol{k} = k\boldsymbol{e}_z$ and $\mathcal{U}^0(x,\tilde{t},\tilde{z}) = u^0(x)\delta_{(\omega,k)} + \overline{u^0(x)}\delta_{-(\omega,k)}$, which belongs to E^s.

EXAMPLE 4.4. The new functional setting presented here allows one to work with oscillations having a *continuous* (rather than discrete) spectrum. More precisely, if $\mathcal{M} \subset \mathbb{R}^{d+1}$ is a submanifold with associated Lebesgue measure σ, then for all $\alpha \in L^1(\mathcal{M}; H^s(\mathbb{R}^d))$ one can define

$$V(x,\tilde{t},\tilde{z}) = \int_{\mathcal{M}} \mathrm{e}^{\mathrm{i}(\tilde{t},\tilde{z})\cdot(\omega,\boldsymbol{k})}\alpha(\omega,\boldsymbol{k})(x)\sigma(\mathrm{d}\omega,\mathrm{d}\boldsymbol{k}).$$

Then one has $V \in E^s$ and $\operatorname{Sp} V \subset \mathcal{M}$.

EXAMPLE 4.5. One can also check that the ansatz (4.2) used by Alterman and Rauch to model ultrashort pulses is a particular case of oscillation with continuous spectrum, as considered in example 4.4.

4.3. Generalizing (1.1) and (2.3) for ultrashort pulses

In order to model ultrashort pulses, we generalize the class of initial conditions considered in the previous sections as follows:

$$\boldsymbol{U}|_{t=0} = \mathcal{U}^0\left(x, 0, \frac{z}{\varepsilon}\right) \quad \text{with } \mathcal{U}^0 \in E^s. \tag{4.4}$$

As shown in example 4.3, this contains the class of initial conditions considered so far. The representation formulae (2.2), (2.3) can also be generalized into

$$\boldsymbol{U}(t,x) = \mathcal{U}\left(t, x, -\frac{t}{\varepsilon}, \frac{z}{\varepsilon}\right), \tag{4.5}$$

with

$$\left.\begin{aligned} \partial_t\mathcal{U} + A(\partial)\mathcal{U} + \frac{\mathrm{i}}{\varepsilon}\mathcal{L}(D_{\tilde{t}}, D_{\tilde{z}})U = \varepsilon T(U,U,U), \\ U|_{t=0}(x,\theta) = \mathcal{U}^0\left(x, 0, \frac{z}{\varepsilon}\right); \end{aligned}\right\} \tag{4.6}$$

this is a consequence of the following theorem.

THEOREM 4.6. *Let assumption 1.1 be satisfied, and let $s > d/2$, $\mathcal{U}^0 \in E^s$.*

There exists $T > 0$ and a unique solution $\boldsymbol{U} \in C([0,T/\varepsilon]; E^s)$ to (1.1) with initial condition (4.4). Moreover, one can write $\boldsymbol{U}$ in the form

$$\boldsymbol{U}(t,x) = \mathcal{U}\left(t, x, -\frac{t}{\varepsilon}, \frac{z}{\varepsilon}\right),$$

where $\mathcal{U}$ solves (4.6).

Proof. The proof is essentially an adaptation of the proof of theorem 2.1 to the present functional setting [4, 18]. □

4.4. Diffractive optics for ultrashort pulses

We follow the same strategy as for the derivation of the standard NLS approximation for wave packets (see § 2.2). We thus look for an approximation $\boldsymbol{U}_{\rm app}$ to the solution $\boldsymbol{U}$ of (1.1) in the form

$$\boldsymbol{U}_{\rm app}(t,x) = \mathcal{U}_{\rm app}\left(t, x, -\frac{t}{\varepsilon}, \frac{z}{\varepsilon}\right), \tag{4.7}$$

where $\mathcal{U}_{\rm app}$ is an approximate solution to the *profile equation* (4.6).

We look for $\mathcal{U}_{\rm app}$ in the form

$$\mathcal{U}_{\rm app}(t,x,\tilde{t},\tilde{z}) = \mathcal{U}_0(\varepsilon t,t,x,\tilde{t},\tilde{z}) + \varepsilon\mathcal{U}_1(\varepsilon t,t,x,\tilde{t},\tilde{z}) + \varepsilon^2\mathcal{U}_2(\varepsilon t,t,x,\tilde{t},\tilde{z}). \tag{4.8}$$

The same BKW method is then used to determine the $\mathcal{U}_j$ $(j = 0, 1, 2)$ as in § 2.2. The three qualitative steps corresponding to the cancellation of the three leading terms also appear in the present case. Referring the reader to [4] for full details, we choose to insist on the new phenomena observed here due to the peculiar characteristics of ultrashort pulses.

4.4.1. *Cancelling the terms of order $O(\varepsilon^{-1})$: the dispersion relation and the polarization condition*

When standard wave packets were under investigation in § 2.2, this step reduced to the matricial equation (2.9) on the coefficient of the first harmonic. Since there is no such thing as harmonics for ultrashort pulses, this condition becomes frequency dependent (or pseudodifferential), namely,

$$\mathcal{L}(D_{\tilde{t}}, D_{\tilde{z}})\mathcal{U}_0 = 0. \tag{4.9}$$

Consequently, the dispersion relation and polarization condition (2.10) become

$$\mathrm{Sp}\,\mathcal{U}_0 \subset \mathcal{C}_{\mathcal{L}} \quad \text{and} \quad \Pi(D_{\tilde{t}}, D_{\tilde{z}})\mathcal{U}_0 = \mathcal{U}_0 \tag{4.10}$$

(that is, the spectrum of $\mathcal{U}_0$ is included in $\mathcal{C}_{\mathcal{L}}$ and the polarization condition becomes frequency dependent).

Remark 4.7. As shown in example 4.3, the functional framework used here *contains* the standard framework of § 2. Consequently, one may easily check that (4.10) degenerates in (2.10) if $\mathcal{U}_0$ is a standard oscillation.

4.4.2. *Cancelling the terms of order $O(\varepsilon^0)$: transport at the group velocity*

Forgetting about the technical details, let us focus on the qualitative information found at this step: $\mathcal{U}_0$ *must satisfy a frequency dependent transport equation in the* (t,x) *variables*. For each given frequency (with respect to the $(\tilde{t},\tilde{z})$ variables), the analysis is the same as in § 2.2.2. Since the group velocity depends on the frequency and that the spectrum of ultrashort pulses is not concentrated around $(\omega,\boldsymbol{k})$ it is not a surprise that the transport equation now becomes non-local with respect to

$(\tilde{t},\tilde{z})$. If the spectrum of $\mathcal{U}_0$ is supported on a smooth (part of a) characteristic sheet of $\mathcal{C}_{\mathcal{L}}$, this equation reads

$$(\partial_t + c_{\rm g}(D_{\tilde{z}})\partial_z)\mathcal{U}^0 = 0. \tag{4.11}$$

REMARK 4.8. In the non-dispersive case, $c_{\rm g}(D_{\tilde{z}})$ is a constant and (4.11) is a standard transport equation.

4.4.3. *Cancelling the terms of order $O(\varepsilon^1)$: diffractive and nonlinear effects*

For the same reasons as for the transport equation, the Schrödinger equation found at this step is non-local with respect to the $(\tilde{t},\tilde{z})$ variables. If the spectrum of $\mathcal{U}_0$ is supported on a smooth (part of a) characteristic sheet of $\mathcal{C}_{\mathcal{L}}$, this equation reads

$$\partial_\tau \mathcal{U}_0 - \tfrac{1}{2}\mathrm{i}\underline{\omega}''(D_{\tilde{z}})(\partial,\partial)\mathcal{U}^0 = \Pi(D_{\tilde{t}},D_{\tilde{z}})T^s(\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0). \tag{4.12}$$

4.5. Why the nonlinearities disappear for ultrashort pulses in dispersive media

The most striking fact regarding the behaviour of ultrashort pulses in dispersive media is that their dynamics is linear. Indeed, the right-hand-side of (4.12) vanishes!

PROPOSITION 4.9. *Let $s > d/2$ and $\mathcal{U}^0 \in E^s$ be such that $\mathcal{F}_{\tilde{t},\tilde{z}}\mathcal{U}^0$ has no atom located on $\mathcal{C}_{\mathcal{L}}$. Assume, moreover, that $\mathcal{C}_{\mathcal{L}}$ is nowhere flat. Then*

$$\Pi(D_{\tilde{t}},D_{\tilde{z}})T^s(\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0) = 0.$$

Proof. Let us prove that

$$\Pi(D_{\tilde{t}},D_{\tilde{z}})T(\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0, \overline{\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0}, \Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0) = 0.$$

For the other components of $T^s(\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0)$, the proof is exactly the same.

Let us denote $\mu = \mathcal{F}_{\tilde{t},\tilde{z}}(\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0)$ and by $v(\mu)$ its total variation. From the Radon–Nykodým property, one has, for all Borel sets $E \subset \mathbb{R}^2$,

$$\mu(E) = \int_E r_\mu(\xi)v(\mu)(\mathrm{d}\xi),$$

where r_μ is an H^s-valued integrable function such that $|r_\mu(\xi)|_{H^s} = 1$ for $v(\mu)$ almost every ξ. Let us now denote by ν the measure

$$\mathcal{F}_{\tilde{t},\tilde{z}}(\Pi(D_{\tilde{t}},D_{\tilde{z}})T(\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0, \overline{\Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0}, \Pi(D_{\tilde{t}},D_{\tilde{z}})\mathcal{U}^0));$$

the result will be proved if we can show that $v(\nu)(\mathbb{R}^2) = 0$.

By definition of the total variation, one has

$$v(\nu)(\mathbb{R}^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\Pi(\xi_1+\xi_2+\xi_3)F(r_\mu(\xi_1),r_\mu(\xi_2),r_\mu(\xi_3))|_{H^s} \times v(\mu)(\mathrm{d}\xi_1)v(\mu)(\mathrm{d}\xi_2)v(\mu)(\mathrm{d}\xi_3).$$

Since $\operatorname{Sp} \Pi \mathcal{U}_0 \subset \mathcal{C}_\mathcal{L}$, we deduce that $r_\mu(\xi) = 0$ $v(\mu)$-almost surely if $\xi \notin \mathcal{C}_\mathcal{L}$. We thus have

$$v(\nu)(\mathbb{R}^2) = \int_{\mathcal{C}_\mathcal{L}} \int_{\mathcal{C}_\mathcal{L}} \left[\int_{\mathcal{C}_\mathcal{L}} |\Pi(\xi_1+\xi_2+\xi_3) F(r_\mu(\xi_1), r_\mu(\xi_2), r_\mu(\xi_3))|_{H^s} v(\mu)(\mathrm{d}\xi_3) \right] \times v(\mu)(\mathrm{d}\xi_1) v(\mu)(\mathrm{d}\xi_2).$$

Now, if $\xi_1 + \xi_2 \neq 0$, then $\{\xi \in \mathcal{C}_\mathcal{L} : \xi_1 + \xi_2 + \xi_3 \in \mathcal{C}_\mathcal{L}\}$ is discrete (this is where the assumption that $\mathcal{C}_\mathcal{L}$ is nowhere flat is important). It is therefore of $v(\mu)$ measure equal to 0 since we assumed that $v(\mu)$ does not have atoms located on $\mathcal{C}_\mathcal{L}$. It is then easy to deduce that $v(\nu)(\mathbb{R}^2) = 0$, which concludes the proof. □

Remark 4.10. In the non-dispersive case, $\mathcal{C}_\mathcal{L}$ (understood in the sense of notation 4.1) is flat and the above proposition cannot apply. This is why the model equation (4.3) derived by Alterman and Rauch [1–3] for ultrashort pulses in non-dispersive media is *nonlinear*. One could of course re-derive it with the approach presented here. Note also that, for some weakly dispersive models that are somehow intermediate between the non-dispersive framework of Alterman and Rauch and the dispersive models considered here, nonlinearities must be taken into account [4, § 5].

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