

Variants of the Focusing NLS Equation: Derivation, Justification, and Open Problems Related to Filamentation

Éric Dumas, David Lannes, and Jérémie Szeftel

Abstract The focusing cubic NLS is a canonical model for the propagation of laser beams. In dimensions 2 and 3, it is known that a large class of initial data leads to finite-time blow-up. Now, physical experiments suggest that this blow-up does not always occur. This might be explained by the fact that some physical phenomena neglected by the standard NLS model become relevant at large intensities of the beam. Many ad hoc variants of the focusing NLS equation have been proposed to capture such effects. In this paper, we derive some of these variants from Maxwell's equations and propose some new ones. We also provide rigorous error estimates for all the models considered. Finally, we discuss some open problems related to these modified NLS equations.

1 Introduction

The cubic, focusing, nonlinear Schrödinger (NLS) equation in space dimension d is given by

$$\begin{cases} i\partial_\tau v + \Delta v + |v|^2 v = 0, & \tau > 0, x \in \mathbb{R}^d, \\ v(0, x) = v_0(x), & x \in \mathbb{R}^d. \end{cases} \quad (1)$$

It is a canonical model for the propagation of laser beams.

É. Dumas (✉)

Institut Fourier, Université Joseph Fourier, 100, rue des Mathématiques,
BP 74, 38402 Saint Martin d'Hères, France
e-mail: Eric.Dumas@ujf-grenoble.fr

D. Lannes

Institut de Mathématiques de Bordeaux, Université de Bordeaux & CNRS
UMR 5251, 351, cours de la Libération, France
e-mail: David.Lannes@math.u-bordeaux1.fr

J. Szeftel

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie,
4, place Jussieu, 75005 Paris, France
e-mail: Jeremie.Szeftel@upmc.fr

From a result of Ginibre and Velo [1], (1) is locally well-posed in $H^1 = H^1(\mathbb{R}^d)$ for $d = 1, 2, 3$, and thus, for $v_0 \in H^1$, there exist $0 < T \leq +\infty$ and a unique solution $v(\tau) \in \mathcal{C}([0, T), H^1)$ to (1) and either $T = +\infty$, we say the solution is global, or $T < +\infty$ and then $\lim_{\tau \uparrow T} \|\nabla v(\tau)\|_{L^2} = +\infty$, we say the solution blows up in finite time.

The NLS equation (1) also admits the following (formal) conservation laws:

$$L^2\text{-norm: } \|v(\tau)\|_{L^2}^2 = \|v_0\|_{L^2}^2;$$

$$\text{Energy: } E(v(\tau)) = \frac{1}{2} \int |\nabla v(\tau, x)|^2 dx - \frac{1}{4} \int |v(\tau, x)|^4 dx = E(v_0);$$

$$\text{Momentum: } \text{Im}(\int \nabla v(\tau, x) \overline{v(\tau, x)} dx) = \text{Im}(\int \nabla v_0(x) \overline{v_0(x)} dx).$$

It is also known that a large group of symmetries leaves the equation invariant: if $v(\tau, x)$ solves (1), then $\forall (\lambda_0, \tau_0, x_0, \beta_0, \gamma_0) \in \mathbb{R}_*^+ \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, so does

$$u(\tau, x) = \lambda_0 v(\lambda_0^2 \tau + \tau_0, \lambda_0 x + x_0 - \beta_0 t) e^{i \frac{\beta_0}{2} \cdot (x - \frac{\beta_0}{2} \tau)} e^{i \gamma_0}. \quad (2)$$

The scaling symmetry $u(\tau, x) = \lambda_0 v(\lambda_0^2 \tau, \lambda_0 x)$ leaves the homogeneous Sobolev space $\dot{H}^{s_c}(\mathbb{R}^d)$ invariant, where $s_c = d/2 - 1$.

Referring to conservation of the L^2 -norm by the flow, (1) is said to be L^2 -subcritical if $s_c < 0$, L^2 -critical if $s_c = 0$, and L^2 -supercritical if $s_c > 0$. Thus, (1) is L^2 -subcritical if $d = 1$, L^2 -critical if $d = 2$, and L^2 -supercritical if $d \geq 3$. In the subcritical case, global existence (in $\mathcal{C}([0, \infty), L^2)$) holds for arbitrarily large data in L^2 . It turns out that in this case, global existence (in $\mathcal{C}([0, \infty), H^1)$) also holds for arbitrarily large data in H^1 , due to the conservation of mass and energy. In the critical and supercritical cases however, there exists stable finite-time blow-up dynamics. This has been known since the 60s using global obstructive arguments based on the virial identity (see, e.g., [2]).¹

There is, however, a discrepancy between the blow-up results predicted by (1) and physical observations. Indeed, while the blow-up signifies a break-down of the solution v , physical observations show in many cases that lasers begin to focus according to the scenarios associated with (1) but depart from this behavior slightly before the focusing time. The reason advanced by physicists is that some physical phenomena that have been neglected to derive (1) become relevant at high intensities, and therefore near focusing. This phenomenon is called *filamentation*: defocusing physical phenomena are triggered at high intensities and halt the collapse of the beam. This interplay between diffraction, self-focusing, and defocusing mechanisms allows for the beam to propagate along several times the focusing distance (called Rayleigh length in optics) and the resulting structure is called *filament*.

Many variants of (1) have been derived in optics to take into account these additional physical phenomena and reproduce the filamentation mechanism. In many

¹Much more is known about the finite-time blow-up dynamics for the focusing NLS and we refer the interested reader to [3–10] and the references therein.

cases, it is a mathematical open problem to prove whether these additional terms prevent focusing or not, and a fortiori to understand the modification of the dynamics induced by them.

Rather than adding as usual ad hoc modifications to (1) in order to take new physical effects into account, we choose here to rigorously derive such modifications from Maxwell's equations. We then comment on some of the most physically relevant open mathematical problems that these modified equations raise and that are natural milestones towards the understanding of filamentation. These variants can roughly be classified into two groups, depending on whether they take ionization processes into account or not.

Notation 1. In the brief presentation below, we denote by z the direction of propagation of the laser, by $X_\perp = (x, y)$ the transverse variables, and by $\Delta_\perp = \partial_x^2 + \partial_y^2$ the transverse Laplace operator. In dimension $d = 2$, the variable y is omitted (and hence $\Delta_\perp = \partial_x^2$), while in dimension $d = 1$, functions depend only on z (so that $\Delta_\perp = 0$).

1.1 Models Without Ionization Processes

We give below a family of variants to (1) that incorporate many physical phenomena neglected by (1). It is of course possible to look at one or several of these additional effects simultaneously. We state the equations in their most general form, starting with a family of *scalar* NLS equations, and then give the corresponding *vectorial*—and more general—form of these equations. Let us therefore consider

$$iP_2(\varepsilon\nabla)\partial_\tau v + (\Delta_\perp + \alpha_1\partial_z^2)v + i\alpha_2v + (1 + i\varepsilon\alpha_3 \cdot \nabla)[(1 + f(\varepsilon^r|v|^2))|v|^2v] = 0, \quad (3)$$

where v is a complex-valued function. Here, $\varepsilon > 0$ is a (small) parameter; $P_2(\varepsilon\nabla)$ is a second (at most) order, self-adjoint, positive operator; $\alpha_1 = 0, \pm 1$; $\alpha_2 \geq 0$; $\alpha_3 \in \mathbb{R}^d$; $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a smooth mapping vanishing at the origin, and $r > 0$. The physical meaning of these terms is commented below:

1. **Nonlinearity.** The cubic nonlinearity in NLS corresponds to a first order approximation of the nonlinear optical phenomena. At high intensities, it is often worth including some next order terms captured here by the additional term $f(\varepsilon^r|v|^2)$. We consider here three situations:
 - (a) Cubic nonlinearity: $f = 0$.
 - (b) Cubic/quintic nonlinearity: $f(r) = -r$.
 - (c) Saturated nonlinearity: f is a smooth function on \mathbb{R}^+ vanishing at the origin and such that $(1 + f(r))r$ is bounded on \mathbb{R}^+ (e.g., $f(r) = -r/(1 + r)$).
2. **Group velocity dispersion (GVD).** The coefficient α_1 accounts for the dispersion of the group velocity and three different situations are possible:

- (a) No GVD: $\alpha_1 = 0$.
 - (b) Anomalous GVD: $\alpha_1 = 1$.
 - (c) Normal GVD: $\alpha_1 = -1$.
3. Damping. The coefficient α_2 accounts for damping phenomena:
- (a) No damping: $\alpha_2 = 0$.
 - (b) Damping: $\alpha_2 > 0$.
4. Off-axis variations of the group velocity. The operator $P_2(\varepsilon\nabla)$ is here to account for the fact that self-focusing pulses become asymmetric due to the variation of the group velocity of off-axis rays.² The operator $P_2(\varepsilon\nabla)$ is a second order, self-adjoint, and positive operator in the sense that $(P_2(\varepsilon\nabla)u, u) \geq C|u|_*^2$, with $|u|_*^2 \geq |u|_{L^2}$. The norm $|\cdot|_*$ may also control derivatives of u ; we consider three cases:
- (a) No off-axis dependence: $P_2(\varepsilon\nabla) = 1$, and therefore $|\cdot|_* = |\cdot|_{L^2}$.
 - (b) Full off-axis dependence: the norm $|\cdot|_*$ controls all first order derivatives, $|u|_*^2 \sim |u|_{L^2}^2 + \varepsilon^2 |\nabla u|_{L^2}^2$.
 - (c) Partial off-axis dependence: the norm $|\cdot|_*$ controls some but not all first order derivatives. More precisely, there exist j ($j < d$) linearly independent vectors $\mathbf{v}_j \in \mathbb{R}^d$ such that $|u|_*^2 \sim |u|_{L^2}^2 + \varepsilon^2 \sum_{k=1}^j |\mathbf{v}_k \cdot \nabla u|_{L^2}^2$.
5. Self-steepening of the pulse. The operator $(1 + i\varepsilon\boldsymbol{\alpha}_3 \cdot \nabla)$ in front of the nonlinearity accounts for off-axis dependence of the nonlinearity, responsible for the possible formation of *optical shocks*. Various cases are considered here:
- (a) No self-steepening. This corresponds to $\boldsymbol{\alpha}_3 = 0$ and to the usual situation where the nonlinearity does not contain any derivative.
 - (b) Longitudinal self-steepening. When $\boldsymbol{\alpha}_3$ is colinear to \mathbf{e}_z , there is a derivative in the nonlinearity along the direction z of propagation of the laser.
 - (c) Transverse self-steepening. When $\boldsymbol{\alpha}_3 \neq 0$ and $\boldsymbol{\alpha}_3 \cdot \mathbf{e}_z = 0$, there is a derivative in the nonlinearity along a direction orthogonal to the direction of propagation.
 - (d) Oblique self-steepening. When $\boldsymbol{\alpha}_3$ is neither colinear nor orthogonal to \mathbf{e}_z .

Remark 1. The standard Schrödinger equation (1) is obtained with $P_2(\varepsilon\nabla) = 1$, $\alpha_2 = 0$, $\boldsymbol{\alpha}_3 = 0$, $f = 0$, and $\alpha_1 = 1$. Using the above terminology, it corresponds to a cubic nonlinearity, without damping terms, off-axis variation of the group velocity and self-steepening, and with anomalous GVD.

As previously said, (3) stems from a more general *vectorial* equation. For the sake of simplicity, we give here the equation corresponding to the cubic case (or

²This phenomenon is often referred to in optics as *space-time focusing* [11]; we do not use this terminology here because this would be misleading. Indeed, physicists usually take z as the evolution variable and treat t as a space variable. This amounts to permuting t and z in (3) and elsewhere.

$f = 0$ in (3)0):

$$iP_2(\varepsilon\nabla)\partial_\tau\mathbf{v} + (\Delta_\perp + \alpha_1\partial_z^2)\mathbf{v} + i\alpha_2\mathbf{v} + \frac{1}{3}(1 + i\varepsilon\alpha_3 \cdot \nabla)[(\mathbf{v} \cdot \mathbf{v})\bar{\mathbf{v}} + 2|\mathbf{v}|^2\mathbf{v}] = 0, \quad (3)_{\text{vect}}$$

where \mathbf{v} is now a \mathbb{C}^2 -valued function.

Remark 2. Equation (3) is in fact a particular case of (3)_{vect} corresponding to initial data living on a one-dimensional subspace of \mathbb{C}^2 . Indeed, if the initial condition to (3)_{vect} has the form $\mathbf{v}|_{\tau=0} = v^0(x)\mathbf{v}_0$ with $\mathbf{v}_0 \in \mathbb{R}^2$ and v^0 a scalar-valued function, then the solution to (3)_{vect} takes the form $\mathbf{v}(\tau, x) = v(\tau, x)\mathbf{v}_0$, where v solves (3) with initial condition v^0 .

1.2 Models with Ionization Processes

In addition to the physical phenomena taken into account in (3), it is necessary at high intensities to include ionization processes for a correct description of the laser pulse. The reason why this phenomenon is singled out here is because a system of two equations must be considered instead of the single equation (3). In the most simple case (i.e., $P_2 = 1, f = 0, \alpha_2 = 0, \alpha_3 = 0$; for a more general model, see (50)), this system is given by

$$\begin{cases} i(\partial_t + c_g\partial_z)u + \varepsilon(\Delta_\perp + \alpha_1\partial_z^2)u + \varepsilon(|u|^2 - \rho)u = -i\varepsilon c(\alpha_4|u|^{2K-2}u + \alpha_5\rho u), \\ \partial_t\rho = \varepsilon\alpha_4|u|^{2K} + \varepsilon\alpha_5\rho|u|^2, \end{cases} \quad (4)$$

with $\alpha_4, \alpha_5 \geq 0, c > 0$, and where ρ is the density of electrons created by ionization, while $\mathbf{c}_g = c_g\mathbf{e}_z$ is the group velocity associated with the laser pulse.

The system (4) does not directly compare to (3) and (3)_{vect}; indeed, (4) is written in the fixed frame of the laboratory, while (3) and (3)_{vect} are written in a frame moving at the group velocity $\mathbf{c}_g = c_g\mathbf{e}_z$ and with respect to a rescaled time $\tau = \varepsilon t$. Rather than (4), the NLS equation with ionization used in the physics community is its version written in the same variables as (3) and (3)_{vect}. More precisely, if we set

$$u(t, X_\perp, z) = v(\varepsilon t, X_\perp, z - c_g t), \quad \rho(t, X_\perp, z) = \tilde{\rho}(\varepsilon t, X_\perp, z - c_g t),$$

and $\tau = \varepsilon t$, Eq. (4) are *approximated*³ by the following ones,

³The approximation lies in the equation on ρ . In the new variables, the second equation of (4) is given by

$$\varepsilon\partial_\tau\tilde{\rho} - c_g\partial_z\tilde{\rho} = \varepsilon\alpha_4|v|^{2K} + \varepsilon\alpha_5\tilde{\rho}|v|^2.$$

In the physics literature, the term $\varepsilon\partial_\tau\tilde{\rho}$ is neglected, and this corresponds to (5).

$$\begin{cases} i\partial_\tau v + (\Delta_\perp + \alpha_1 \partial_z^2)v + (|v|^2 - \tilde{\rho})v = -ic(\alpha_4 |v|^{2K-2}v + \alpha_5 \tilde{\rho}v), \\ -c_g \partial_z \tilde{\rho} = \varepsilon \alpha_4 |v|^{2K} + \varepsilon \alpha_5 \tilde{\rho} |v|^2. \end{cases} \quad (5)$$

The ionization processes taken into account by the systems (4) and (5) are:

1. Photo-ionization. This corresponds to $\alpha_4 > 0$ and $K > 0$ (K is the number of photons necessary to liberate one electron).
2. Collisional ionization. When $\alpha_5 > 0$, a term corresponding to collisional ionization is added to the evolution equation on ρ .

Remark 3. 1. The coupling with ρ can of course be added to any equation of the family (3).

2. A vectorial variant of (4) and (5) can also be derived in the same lines as (3)_{vect}.
3. When $\alpha_4 = \alpha_5 = 0$ and $\rho|_{t=0} = 0$ (respectively $\lim_{z \rightarrow -\infty} \tilde{\rho} = 0$) one recovers (3) from (4) (respectively (5)).

The rest of the paper is as follows. In Sect. 2, we recall the Maxwell equations, we give an abstract formulation and we discuss the spaces of initial data used for the Cauchy problem. In Sect. 3, we prove our main result about the rigorous derivation of general abstract versions of the models (3), (3)_{vect}, and (4), (5). In Sect. 4, we analyze the role of the various parameters in (3) and (4) or (5). In particular, we consider whether they indeed prevent the breakdown in finite time or not. We also formulate a number of interesting open problems for these modified NLS equations. Finally, an appendix contains explicit computations for a physically relevant system of Maxwell's equations which show that the abstract models derived in Sect. 3 take indeed the form of (3), (3)_{vect}, and (4).

1.3 Notations

We denote by

- $\mathcal{F}u(\xi) = \hat{u}(\xi)$, the Fourier transform of u with respect to the space variables $x \in \mathbb{R}^d$.
- $\mathcal{F}_t u(\omega)$, the Fourier transform of u with respect to the time variable t .
- $f(D)$, the Fourier multiplier with symbol $f(\xi)$: $\widehat{f(D)u}(\xi) = f(\xi)\hat{u}(\xi)$.
- $f(D_t)$, Fourier multipliers with respect to time.
- $\Lambda = (1 - \Delta)^{1/2}$, the Fourier multiplier with symbol $(1 + |\xi|)^{1/2}$.

2 The Maxwell Equations and an Abstract Mathematical Formulation

2.1 The Maxwell Equations

The Maxwell equations in a non-magnetizable medium are a set of two equations coupling the evolution of the *magnetic field* B to the *electric induction* D ,

$$\begin{cases} \partial_t B + \operatorname{curl} E = 0, \\ \partial_t D - \frac{1}{\mu_0} \operatorname{curl} B = -J, \end{cases} \quad (6)$$

where D is given in terms of the *electric field* E and a *polarization* P —modeling the way the dipole moment per unit volume depends on the strength of the electric field—by the relation

$$D = \epsilon_0 E + P, \quad (7)$$

and where we used standard notations ϵ_0 and μ_0 for the *electric permittivity* and *magnetic permeability* in vacuum. The evolution equations (6) go along with two constitutive laws,

$$\nabla \cdot D = \rho, \quad \nabla \cdot B = 0, \quad (8)$$

where ρ is the electric charge density.

As a consequence of the relation $\nabla \cdot D = \rho$ and the second equation of (6) we get the continuity equation coupling ρ to the *current density* J ,

$$\partial_t \rho + \nabla \cdot J = 0. \quad (9)$$

Introducing the speed of light in vacuum

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}},$$

Eq. (6) can also be rewritten as a set of two evolution equations on the magnetic field B and the electric field E ,

$$\begin{cases} \partial_t B + \operatorname{curl} E = 0, \\ \partial_t E - c^2 \operatorname{curl} B = -\frac{1}{\epsilon_0} \partial_t P - \frac{1}{\epsilon_0} J. \end{cases} \quad (10)$$

In order to get a closed system of equations, we still need two physical informations:

1. A description of the polarization response to the electric field \mathbf{E} .
2. A description of the current density \mathcal{J} .

We first address the description of the polarization response in absence of current density and then proceed to describe the modification to be made when current density is included.

2.1.1 The Polarization Response to the Electric Field

Throughout this section, we assume that there is no charge nor current density ($\rho = 0$, $\mathcal{J} = 0$). The general case will be handled in Sect. 2.1.2 below.

There exist various ways to describe the polarization \mathbf{P} ; we use here a simple and natural model called “nonlinear anharmonic oscillator” according to which the polarization is found by solving the second order ODE

$$\partial_t^2 \mathbf{P} + \Omega_1 \partial_t \mathbf{P} + \Omega_0^2 \mathbf{P} - \nabla V_{\text{NL}}(\mathbf{P}) = \epsilon_0 b \mathbf{E}, \quad (11)$$

where $b \in \mathbb{R}$ is a coupling constant and $\Omega_0, \Omega_1 > 0$ are frequencies, and where V_{NL} accounts for nonlinear effects. When such effects are neglected, the description (11) goes back to Lorentz [12] and expresses the fact that electrons are bound to the nucleus by a spring force. Nonlinearities have been added to this description by Bloembergen [13] and Owyyoung [14] and the mathematical investigation of their influence was initiated by Donnat et al. [15, 16] (see also [17]).

Remark 4. In physics books, the polarization \mathbf{P} is often sought as an expansion

$$\mathbf{P} = \epsilon_0 [\chi^1[\mathbf{E}] + \chi^2[\mathbf{E}, \mathbf{E}] + \chi^3[\mathbf{E}, \mathbf{E}, \mathbf{E}] + \dots],$$

where the operator χ^1 is called the *linear susceptibility* of the material, while for $j > 1$, the operators χ^j are the *jth order nonlinear susceptibilities*. It is easy to check that the linear susceptibility corresponding to (11) is given by the nonlocal (in time) operator

$$\chi^1[\mathbf{E}] = \chi^1(D_t)\mathbf{E} \quad \text{with} \quad \chi^1(\omega) = \frac{b}{\Omega_0^2 - \omega^2 + i\Omega_1\omega},$$

where we used the Fourier multiplier notation,

$$\mathcal{F}_t[\chi^1(D_t)\mathbf{E}](\omega) = \chi^1(\omega)\mathcal{F}_t\mathbf{E}(\omega).$$

Example 1. Typical examples for $V_{\text{NL}}(\mathbb{P})$ are

1. Cubic nonlinearity:

$$V_{\text{NL}}(\mathbb{P}) = \frac{a_3}{4}|\mathbb{P}|^4 \quad \text{and therefore} \quad \nabla V_{\text{NL}}(\mathbb{P}) = a_3|\mathbb{P}|^2\mathbb{P}.$$

2. Cubic/quintic nonlinearity:

$$V_{\text{NL}}(\mathbb{P}) = \frac{a_3}{4}|\mathbb{P}|^4 - \frac{a_5}{6}|\mathbb{P}|^6 \quad \text{and therefore} \quad \nabla V_{\text{NL}}(\mathbb{P}) = a_3|\mathbb{P}|^2\mathbb{P} - a_5|\mathbb{P}|^4\mathbb{P}.$$

3. Saturated nonlinearity: there exists a function $v_{\text{sat}}: \mathbb{R}^+ \rightarrow \mathbb{R}$, with v'_{sat} and v''_{sat} bounded on \mathbb{R}^+ and such that

$$V_{\text{NL}}(\mathbb{P}) = \frac{1}{2}v_{\text{sat}}(|\mathbb{P}|^2) \quad \text{and therefore} \quad \nabla V_{\text{NL}}(\mathbb{P}) = v'_{\text{sat}}(|\mathbb{P}|^2)\mathbb{P};$$

for instance, one can take

$$v_{\text{sat}}(r) = \frac{a_3}{2} \frac{r^2}{1 + (2a_5/3a_3)r},$$

in which case $\nabla V_{\text{NL}}(\mathbb{P}) = a_3|\mathbb{P}|^2\mathbb{P} - a_5|\mathbb{P}|^4\mathbb{P} + \text{h.o.t.}$, and is therefore the same at the origin as in (2) above, up to higher-order terms (seventh-order terms here).

We show in section “The Case Without Charge nor Current Density” in Appendix 1 that Maxwell’s equations can be put under the following dimensionless form⁴ for all the nonlinearities considered in Example 1,

$$\begin{cases} \partial_t \mathbb{B} + \text{curl } \mathbb{E} = 0, \\ \partial_t \mathbb{E} - \text{curl } \mathbb{B} + \frac{1}{\varepsilon} \sqrt{\gamma} \mathbb{Q}^\sharp = 0, \\ \partial_t \mathbb{Q}^\sharp + \varepsilon^{1+p} \omega_1 \mathbb{Q}^\sharp - \frac{1}{\varepsilon} (\sqrt{\gamma} \mathbb{E} - \omega_0 \mathbb{P}^\sharp) = \varepsilon \frac{\gamma}{\omega_0^3} (1 + f(\varepsilon' |\mathbb{P}^\sharp|^2)) |\mathbb{P}^\sharp|^2 \mathbb{P}^\sharp, \\ \partial_t \mathbb{P}^\sharp - \frac{1}{\varepsilon} \omega_0 \mathbb{Q}^\sharp = 0, \end{cases} \quad (12)$$

where γ , ω_0 , ω_1 , r , and p are constants, $0 < \varepsilon \ll 1$ is a small parameter (the ratio of the duration of an optical cycle over the duration of the pulse, see section “The Case Without Charge nor Current Density” in Appendix 1), while f is a smooth function vanishing at the origin and

$$\mathbb{P}^\sharp = \frac{\omega_0}{\sqrt{\gamma}} \mathbb{P}, \quad \mathbb{Q}^\sharp = \frac{\varepsilon}{\omega_0} \partial_t \mathbb{P}^\sharp = \frac{\varepsilon}{\sqrt{\gamma}} \partial_t \mathbb{P}.$$

⁴The constitutive laws (8) are omitted because they are propagated by the equations if they are initially satisfied.

2.1.2 The Case with Charge and Current Density

The main mechanism at stake in laser filamentation is certainly the local ionization of the medium: once a powerful self-focusing laser beam reaches high enough intensities, it ionizes the medium around itself. It leaves behind a narrow channel of plasma, hereby causing local defocusing that prevents blowup.

Taking current density into account, we come back to the set of Eqs. (10)–(11), and a physical description of the current density \mathcal{J} is needed. This current density has the form

$$\mathcal{J} = \mathcal{J}_e + \mathcal{J}_i, \quad (13)$$

where \mathcal{J}_e and \mathcal{J}_i are, respectively, the free electron and ionization current densities.

Free Electron Current Density

Partial ionization of the material medium by the laser generates free electrons, with charge q_e ($= -1.6 \times 10^{-19}$ C). This induces a free electron current density $\mathcal{J}_e = q_e \rho_e v_e$, where ρ_e is the electron density, and v_e is the electron velocity. A rough,⁵ but standard model in nonlinear optics is to take (see [19] and the references therein),

$$\mathbf{E}(t, X) \sim E_{01}(t, X)e^{i(\mathbf{k}_1 \cdot X - \omega_1 t)} + \text{c.c.}, \quad (14)$$

where \mathbf{k}_1 and ω_1 are the laser wave number and pulsation, respectively, with $|\partial_t E_{01}| \ll |\omega_1 E_{01}|$ (slowly varying envelope approximation (SVEA)); the polarization current is then taken under the form

$$\mathcal{J}_e \sim J_{01}(t, X)e^{i(\mathbf{k}_1 \cdot X - \omega_1 t)} + \text{c.c.} \quad \text{with} \quad J_{01} = i \frac{q_e^2}{\omega_1 m_e} \rho_e E_{01}, \quad (15)$$

where m_e is the electron mass. The drawback of this model is that it assumes that the electric field and the current density field can be written at leading order as wave packets (i.e., are given under the form (14)–(15)). In particular, it does not provide any relation between the current density \mathcal{J} and the electric field \mathbf{E} that could be used in Maxwell's equations (10). We therefore propose here such a relation, namely,

$$\mathcal{J}_e = \frac{q_e^2}{\omega_1 m_e} \mathcal{H} \left(\frac{\mathbf{k}_1}{k_1^2} \cdot D \right) (\rho_e \mathbf{E}),$$

⁵This approximation can be deduced formally by assuming that ions are at rest and that electron motion is described by the compressible Euler system (see, for instance, [18]). Neglecting electron collisions, such a model yields $\partial_t \mathcal{J}_e = (q_e^2/m_e) \rho_e \mathbf{E}$ which formally yields (15) assuming that \mathbf{E} is as in (14) and that ρ_e is not oscillating at leading order. It would of course be interesting to provide a rigorous justification to these approximations.

where $k_1 = |\mathbf{k}_1|$, and \mathcal{H} is the regularization of the Hilbert transform given by the Fourier multiplier

$$\mathcal{H}(D_z) = \frac{\sqrt{2} i D_z}{(1 + D_z^2)^{1/2}}. \quad (16)$$

Quite obviously, this is consistent with the usual model (15) since this latter is recovered at leading order when the electric field is a wave packet under the form (14).

Finally, the evolution of the electron density ρ_e is given by a source term S representing external plasma sources. Taking into account photo-ionization and collisional ionization, but neglecting electron recombination (see, for instance, [18] for richer models), we have

$$S = W(I)(\rho_{\text{nt}} - \rho_e) + \frac{\sigma}{U_i} \rho_e I,$$

where the intensity is $I = |\mathbf{E}|^2$ and ρ_{nt} is the constant density of neutral species. In the regime considered here,⁶ ρ_e is negligible compared to ρ_{nt} and the photo-ionization rate $W(I)$ takes the form

$$W(I) = \sigma_K I^{2K},$$

for some constant coefficient $\sigma_K > 0$ and with $K > 1$ the number of photons needed to liberate one electron. The collisional ionization cross-section σ depends on the laser frequency, and U_i is the ionization potential. Summing up, we get the following expression for the free electron current \mathcal{J}_e and $\rho = \rho_e$,

$$\begin{cases} \mathcal{J}_e = \frac{q_e^2}{\omega_l m_e} \mathcal{H}\left(\frac{\mathbf{k}_1}{k_1^2} \cdot D\right)(\rho \mathbf{E}), \\ \partial_t \rho = \sigma_K \rho_{\text{nt}} |\mathbf{E}|^{2K} + \frac{\sigma}{U_i} \rho |\mathbf{E}|^2. \end{cases} \quad (17)$$

2.1.3 Ionization Current Density

It is also necessary to take into account losses due to photo-ionization. We therefore introduce an ionization current density \mathcal{J}_i such that $\mathcal{J}_i \cdot \mathbf{E}$ represents the energy lost by the laser to extract electrons (per time and volume unit). More precisely, $\mathcal{J}_i \cdot \mathbf{E}$ is equal to the energy necessary to extract one electron (given by the ionization

⁶For higher intensities, electrons can tunnel out the Coulomb barrier of atoms, and $W(I)$ is modified.

potential U_i) multiplied by the number of electrons per time and volume unit (given by $\partial_t \rho$). Using the second equation of (17), this gives

$$\mathbb{J}_i \cdot \mathbf{E} = U_i \sigma_K \rho_{\text{nt}} |\mathbf{E}|^{2K} + \sigma \rho |\mathbf{E}|^2.$$

We therefore take

$$\mathbb{J}_i = (U_i \sigma_K \rho_{\text{nt}} |\mathbf{E}|^{2K-2} + \sigma \rho) \mathbf{E}. \quad (18)$$

We show in section “The Case with Charge and Current Density” in Appendix 1 that after nondimensionalization, the set of equations (10), (11), (13), (17), (18) (for the nonlinearities considered in Example 1) becomes

$$\left\{ \begin{array}{l} \partial_t \mathbf{B} + \text{curl } \mathbf{E} = 0, \\ \partial_t \mathbf{E} - \text{curl } \mathbf{B} + \frac{1}{\varepsilon} \sqrt{\gamma} \mathbf{Q}^\sharp \\ \quad = -\varepsilon \mathcal{H} \left(\varepsilon \frac{\mathbf{k}}{k^2} \cdot D_x \right) (\rho \mathbf{E}) - \varepsilon c_0 (c_1 |\mathbf{E}|^{2K-2} + c_2 \rho) \mathbf{E}, \\ \partial_t \mathbf{Q}^\sharp + \varepsilon^{1+p} \omega_1 \mathbf{Q}^\sharp - \frac{1}{\varepsilon} (\sqrt{\gamma} \mathbf{E} - \omega_0 \mathbf{P}^\sharp) = \varepsilon \frac{\gamma}{\omega_0^3} (1 + f(\varepsilon' |\mathbf{P}^\sharp|^2)) |\mathbf{P}^\sharp|^2 \mathbf{P}^\sharp, \\ \partial_t \mathbf{P}^\sharp - \frac{1}{\varepsilon} \omega_0 \mathbf{Q}^\sharp = 0, \\ \partial_t \rho = \varepsilon c_1 |\mathbf{E}|^{2K} + \varepsilon c_2 \rho |\mathbf{E}|^2, \end{array} \right. \quad (19)$$

with the same notations as in (12) for the constants $\gamma, \omega_0, \omega_1, r$, and p , the small parameter ε , and the function f . In addition, we have here constants $c_0, c_1, c_2 \geq 0$, and we also recall that the definition of the regularized Hilbert transform \mathcal{H} is given in (16).

2.2 Abstract Formulations

2.2.1 The Case Without Charge nor Current Density

We show in section “The Case Without Charge nor Current Density” in Appendix 1 that the Maxwell equations can be put under the dimensionless form (12), which itself has the form

$$\partial_t \mathbf{U} + A(\partial) \mathbf{U} + \frac{1}{\varepsilon} E \mathbf{U} + \varepsilon^{1+p} A_0 \mathbf{U} = \varepsilon F(\varepsilon, \mathbf{U}), \quad (20)$$

where \mathbf{U} is an \mathbb{R}^n ($n \geq 1$) valued function depending on the time variable t and the space variable $x \in \mathbb{R}^d$ ($d \geq 1$),

$$\mathbf{U}: (t, x) \in \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^n.$$

The operator $A(\partial)$ is defined as

$$A(\partial) = \sum_{j=1}^d A_j \partial_j,$$

where ∂_j is the differentiation operator with respect to the j th space coordinate. The matrix A_0 has size $n \times n$, p is a positive number, and the following assumption is made on the matrices A_j and E , and on the nonlinearity F .

- Assumption 1.** (i) *The matrices A_j ($j = 1, \dots, d$) are constant coefficient $n \times n$, real-valued, symmetric matrices.*
(ii) *The matrix E is a constant coefficient $n \times n$, real-valued, skew symmetric matrix.*
(iii) *There exists a smooth mapping $f: \mathbb{R}^+ \rightarrow \mathbb{R}$ vanishing at the origin, a real number $r > 0$, a quadratic form $Q: \mathbb{C}^n \mapsto \mathbb{R}^+$, and a trilinear symmetric mapping $T: (\mathbb{C}^n)^3 \rightarrow \mathbb{C}^n$ (with $T(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \subset \mathbb{R}^n$) such that*

$$\forall U \in \mathbb{C}^n, \quad F(\varepsilon, U) = (1 + f(\varepsilon^r Q(U))) T(U, \bar{U}, U).$$

Remark 5. There exist of course situations where the leading order of the nonlinearity is not cubic (it can be quadratic for non-centro-symmetric crystals for instance) or not of this form; since we are interested here in deriving variants of the standard cubic NLS equation, we restrict ourselves to this framework for the sake of simplicity.

Example 2. As previously said, the dimensionless version (12) of the Maxwell equations can be put under the form (20) and it satisfies Assumption 1 with $n = 12$, $\mathbf{U} = (\mathbf{B}, \mathbf{E}, \mathbf{Q}^\sharp, \mathbf{P}^\sharp)^T$. See Appendix 2 for more details.

2.2.2 The Case with Charge and Current Density

As shown in section “The Case with Charge and Current Density” in Appendix 1, the system (19) of Maxwell’s equations with partial ionization can be put under the general form

$$\begin{cases} \partial_t \mathbf{U} + A(\partial) \mathbf{U} + \frac{1}{\varepsilon} E \mathbf{U} + \varepsilon^{1+p} A_0 \mathbf{U} \\ = \varepsilon F(\varepsilon, \mathbf{U}) - \varepsilon \mathcal{H} \left(\varepsilon \frac{\mathbf{k}}{k^2} \cdot D \right) (\mathbf{W} C_1^T C_1 \mathbf{U}) - \varepsilon c C_1^T G(C_1 \mathbf{U}, \mathbf{W}), \\ \partial_t \mathbf{W} = \varepsilon G(C_1 \mathbf{U}, \mathbf{W}) \cdot C_1 \mathbf{U}, \end{cases} \quad (21)$$

where, as in Sect. 2.2.1, \mathbf{U} is an \mathbb{R}^n -valued function, whereas \mathbf{W} is an \mathbb{R} -valued function of $(t, x) \in \mathbb{R} \times \mathbb{R}^d$. The matrices A_j and E , as well as the nonlinearity F , satisfy Assumption 1. Concerning the other coefficients of the system, we assume the following.

- Assumption 2.** (i) *The real, constant matrix C_1 has size $m \times n$ (with $m \in \mathbb{N}$).*
(ii) *The constant c is positive.*
(iii) *There exist two real, positive constants c_1 and c_2 and an integer $K \geq 1$ such that*

$$\forall E \in \mathbb{C}^m, \forall w \in \mathbb{C}, \quad G(E, w) = c_1 |E|^{2K-2} E + c_2 w E.$$

Remark 6. As in Remark. 5 for the case without ionization, we can put Maxwell's equation with ionization terms (19) under the abstract form (21). Using the same notations as in Remark. 5, the matrix C_1 is the projection matrix such that $C_1 \mathbf{U} = \mathbb{E}$, and $\mathbf{w} = \rho$.

2.3 The Cauchy Problem

We are considering initial conditions that correspond to laser pulses. In the case without charge nor current density (20), they are fast oscillating *wave packets* slowly modulated by an envelope,

$$\mathbf{U}|_{t=0} = u^0(x) e^{i \frac{\mathbf{k} \cdot \mathbf{x}}{\varepsilon}} + \text{c.c.}, \quad (22)$$

where $\mathbf{k} \in \mathbb{R}^d$ is the (spatial) *wave-number* of the oscillations. Taking charge and current density into account (21), we need to provide initial conditions for \mathbf{W} ; since we are interested here in the situation where this quantity is created by the laser when it reaches high intensities near self-focusing, we take these initial conditions to be initially zero⁷ for the sake of clarity.

$$\mathbf{W}|_{t=0} = 0. \quad (23)$$

The evolution equation (20) (as well as (21)) being of semilinear nature, it is natural to work with Banach algebra in view of a resolution by Picard iterations. Throughout this article, we assume that $u^0 \in B$, with

$$B = H^{t_0}(\mathbb{R}^d)^n, \quad (t_0 > d/2)$$

⁷One could more generally and without supplementary difficulty consider non-oscillating initial conditions for the charge density \mathbf{W} .

or

$$B = W(\mathbb{R}^d)^n := \{f \in \mathcal{S}'(\mathbb{R}^d)^n, |f|_B := |\hat{f}|_{L^1} < \infty\}$$

(the so-called Wiener algebra, which is better adapted than $H^{t_0}(\mathbb{R}^d)^n$ to handle short pulses, see [20, 21]). In both cases, B is stable by translations in Fourier space (this ensures that if $u^0 \in B$ in (22) then $U_{|_{t=0}} \in B$) and is a Banach algebra in the sense that

$$\forall f, g \in B, \quad f \cdot g \in B \quad \text{and} \quad |f \cdot g|_B \lesssim |f|_B |g|_B.$$

For all $k \in \mathbb{N}$, we also define

$$B^{(k)} = \{f \in B, \forall \alpha \in \mathbb{N}^d, \forall |\alpha| \leq k, \partial^\alpha f \in B\},$$

endowed with its canonical norm.

We are interested in deriving asymptotics to the solution formed by (20)–(22), with initial envelope $u^0 \in B$, and more generally (21)–(23), if we want to be able to handle ionization processes in nonlinear optics. This requires a further assumption on the nonlinearity F , namely that F acts on B and is locally Lipschitz.

Assumption 3. *In addition to (iii) of Assumption 1, the mapping F satisfies, uniformly with respect to $\varepsilon \in [0, 1)$,*

(i) *For all $f \in B$, one has $F(\varepsilon, f) \in B$ and*

$$|F(\varepsilon, f)|_B \leq C(|f|_B)|f|_B.$$

(ii) *For all $f, g \in B$, one has*

$$|d_f F(\varepsilon, \cdot)g|_B \leq C(|f|_B)|g|_B.$$

Example 3. When $B = H^{t_0}(\mathbb{R}^d)^n$ ($t_0 > d/2$), Assumption 3 is always satisfied (by Moser's inequality); when $B = W(\mathbb{R}^d)^n$, the assumption holds for analytic nonlinearities.

3 Derivation of NLS-Type Equations

The Schrödinger approximation takes into account the *diffractive* effects that modify over large times the propagation along rays of standard geometrical optics. These diffractive effects are of linear nature and are known [15–17, 22, 23] to appear for time scales of order $O(1/\varepsilon)$ for the initial value problem formed by the linear part of (20) and (22). This is the reason why we are interested in proving the existence and describing the solutions to the (nonlinear) initial value problem (20)–(22) over such time scales.

For the sake of simplicity, the initial value problem (20)–(22) (no ionization) is first considered. Up to minor modifications, the results of Sects. 3.1–3.3 are known [20]; we reproduce them here because they are necessary steps to derive the family of NLS equations (3) and, for the sake of clarity, their proof is sketched in a few words. The general idea to derive the Schrödinger equations of Sect. 3.5 with improved dispersion was introduced in [20] but the computations are carried further here. We then derive in Sect. 3.6 a new class of models with derivative nonlinearity for which a local well-posedness result is proved. When applied to Maxwell’s equations, these derivative nonlinearities yield the so-called self-steepening operators; to our knowledge, this is the first rigorous explanation of these terms.

The asymptotic description of (21)–(22) (i.e., ionization is now included) is then addressed in Sect. 3.7.

3.1 The Profile Equation

We show here that under reasonable assumptions on F , solutions to the initial value problem (20)–(22) exist for times of order $O(1/\varepsilon)$ and that they can be written under a very convenient form using a *profile* U ,

$$\mathbf{U}(t, x) = U\left(t, x, \frac{\mathbf{k} \cdot x - \omega t}{\varepsilon}\right), \quad (24)$$

with $U(t, x, \theta)$ periodic with respect to θ and for any $\omega \in \mathbb{R}$, provided that U solves the *profile equation*

$$\begin{cases} \partial_t U + A(\partial)U + \frac{i}{\varepsilon} \mathcal{L}(\omega D_\theta, \mathbf{k} D_\theta)U + \varepsilon^{1+p} A_0 U = \varepsilon F(\varepsilon, U), \\ U|_{t=0}(x, \theta) = u^0(x) e^{i\theta} + \text{c.c.} \end{cases} \quad (25)$$

Here, we used the notation

$$\mathcal{L}(\omega D_\theta, \mathbf{k} D_\theta) = -\omega D_\theta + A(\mathbf{k}) D_\theta + \frac{E}{i}, \quad (26)$$

with $D_\theta = -i\partial_\theta$ and $A(\mathbf{k}) = \sum_{j=1}^d A_j k_j$.

Theorem 1. *Let $B = H^{t_0}(\mathbb{R}^d)^n$ or $B = W(\mathbb{R}^d)^n$ and $u^0 \in B$. Under Assumptions 1 and 3, there exists $T > 0$ such that for all $0 < \varepsilon \leq 1$ there exists a unique solution $\mathbf{U} \in C([0, T/\varepsilon]; B)$ to (20)–(22). Moreover, one can write \mathbf{U} under the form*

$$\mathbf{U}(t, x) = U\left(t, x, \frac{\mathbf{k} \cdot x - \omega t}{\varepsilon}\right),$$

where U solves the profile equation (25).

Proof. The proof is a slight adaptation of the one given in [20] in the trilinear case; consequently, we just give the main steps of the proof. Quite obviously, a solution U to (20)–(22) is given by (24) if (25) admits a solution $U \in C([0, T/\varepsilon]; H^k(\mathbb{T}; B))$ ($k \geq 1$) where

$$H^k(\mathbb{T}; B) = \left\{ f = \sum_{n \in \mathbb{Z}} f_n e^{in\theta}, |f|_{H^k(\mathbb{T}, B)} < \infty \right\} \quad (27)$$

and with $|f|_{H^k(\mathbb{T}, B)}^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^k |f_n|_B^2$. For $k \geq 1$, $H^k(\mathbb{T}, B)$ is a Banach algebra; moreover, the evolution operator $S(t)$ associated with the linear part of (25),

$$S(t) = \exp\left(-tA(\partial) - \frac{i}{\varepsilon} t \mathcal{L}(\omega D_\theta, k D_\theta)\right)$$

is unitary on $H^k(\mathbb{T}; B)$ (thanks to points i and ii of Assumption 1). One can therefore construct a (unique) solution to (25) by a standard iterative scheme

$$U^{l+1}(t) = S(t)U^0 + \varepsilon \int_0^t S(t-t') [F(\varepsilon, U^l)(t') - \varepsilon^p A_0 U^l] dt',$$

with $U^0 = U|_{t=0}$. Indeed, one has thanks to Assumption 3,

$$|U^{l+1}(t)|_{H^k(\mathbb{T}; B)} \leq |U^0|_{H^k(\mathbb{T}; B)} + \varepsilon \int_0^t [\varepsilon^p |A_0 U^l|_{H^k(\mathbb{T}; B)} + C(|U^l|_{H^k(\mathbb{T}; B)}) |U^l|_{H^k(\mathbb{T}; B)}];$$

thanks to the ε in front of the integral. An estimate of the same kind is valid for a difference of iterates, by point (ii) of Assumption 3. By a fixed point argument, this ensures that the sequence converges to a solution on $[0, T/\varepsilon]$ for some $T > 0$ independent of ε . Uniqueness then follows classically from an energy estimate on the difference of two solutions. \square

3.2 The Slowly Varying Envelope Approximation

The SVEA consists in writing the profile U under the form

$$U(t, x, \theta) \sim u_{\text{env}}(t, x) e^{i\theta} + \text{c.c.}; \quad (28)$$

plugging this approximation into the profile equation (25) and keeping only the first harmonic in the Fourier expansion yields easily (writing $u = u_{\text{env}}$),

$$\partial_t u + A(\partial)u + \frac{i}{\varepsilon} \mathcal{L}(\omega, \mathbf{k})u + \varepsilon^{1+p} A_0 u = \varepsilon F^{\text{env}}(\varepsilon, u),$$

where

$$F^{\text{env}}(\varepsilon, u) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} F(\varepsilon, ue^{i\theta} + \text{c.c.}) d\theta. \quad (29)$$

Example 4. With $F(u) = |u|^2 u$, one gets $F^{\text{env}}(u) = (u \cdot u)\bar{u} + 2|u|^2 u$.

Denoting $D = -i\nabla$, we observe that

$$\begin{aligned} A(\partial) + \frac{i}{\varepsilon} \mathcal{L}(\omega, \mathbf{k}) &= A(\partial) + \frac{i}{\varepsilon} (-\omega \text{Id} + A(\mathbf{k})) \\ &= \frac{i}{\varepsilon} (-\omega \text{Id} + A(\mathbf{k} + \varepsilon D)) := \frac{i}{\varepsilon} \mathcal{L}(\omega, \mathbf{k} + \varepsilon D), \end{aligned}$$

where the last notation is of course consistent with (26).

As a consequence of these computations, we see that in order for (28) to hold, it is necessary that $u = u_{\text{env}}$ satisfies the *envelope equation*

$$\begin{cases} \partial_t u + \frac{i}{\varepsilon} \mathcal{L}(\omega, \mathbf{k} + \varepsilon D)u + \varepsilon^{1+p} A_0 u = \varepsilon F^{\text{env}}(\varepsilon, u), \\ u|_{t=0} = u^0. \end{cases} \quad (30)$$

As implicitly assumed by omitting the fast oscillating scale in the argument of the envelope function $u_{\text{env}}(t, x)$, the envelope must not contain any fast oscillation. However,

- The singular part of the linear term in (30) creates fast oscillations with frequencies $\omega - \omega_j(\mathbf{k})$, where the $\omega_j(\mathbf{k})$ stand for the eigenvalues of $\mathcal{L}(0, \mathbf{k})$.
- The nonlinearity creates other oscillations that may resonate with the linear propagator.

There is one way to avoid the catastrophic effects of these two scenarios. Choosing $\omega = \omega_j(\mathbf{k})$ for some j and assuming that, up to $O(\varepsilon)$ terms, the initial envelope u^0 is contained in the corresponding eigenspace prevents the creation of oscillations by the linear propagator. This is the *polarization condition*. The nonlinearity will, however, create harmonics of the main oscillation $\mathbf{k} \cdot x - \omega_j(\mathbf{k})t$ and it is necessary to make a non-resonance assumption. What is called *characteristic variety* in the assumption below is the set $\mathcal{C}_{\mathcal{L}} \subset \mathbb{R}^{d+1}$ defined as

$$\mathcal{C}_{\mathcal{L}} = \{(\omega', \mathbf{k}') \in \mathbb{R}^{1+d}, \det \mathcal{L}(\omega', \mathbf{k}') = 0\}.$$

Let us also recall that we assumed that the nonlinearity is under the form

$$F(\varepsilon, U) = \left(1 + f(\varepsilon^r Q(U))\right) T(U, \bar{U}, U),$$

with $f(0) = 0$, Q a quadratic form and T a trilinear symmetric mapping. If U is a monochromatic oscillation, the nonlinearity $\varepsilon F(\varepsilon, U)$ creates third harmonic with size $O(\varepsilon)$, a fifth harmonic (if $f'(0) \neq 0$) with size $O(\varepsilon^{1+r})$, etc. The non-resonance condition stated below holds for the $(2p + 3)$ th harmonics, for all $p \geq 0$ such that $pr < 1$ (the contribution of higher harmonics is small enough to be controlled even if it is resonant).

Assumption 4. *The characteristic variety $\mathcal{C}_{\mathcal{L}}$ and the frequency/wave number couple (ω, \mathbf{k}) satisfy:*

1. *There exist m functions $\omega_j \in C^\infty(\mathbb{R}^d \setminus \{0\})$ ($j = 1, \dots, m$) such that*

$$\mathcal{C}_{\mathcal{L}} \setminus \{0\} = \bigcup_{j=1}^m \{(\omega_j(\mathbf{k}'), \mathbf{k}'), \mathbf{k}' \in \mathbb{R}^d \setminus \{0\}\};$$

up to a renumbering, we assume that $(\omega, \mathbf{k}) = (\omega_1(\mathbf{k}), \mathbf{k})$.

2. *There exists a constant $c_0 > 0$ such that*

$$\inf_{\mathbf{k}' \in \mathbb{R}^d} |\omega - \omega_j(\mathbf{k}')| \geq c_0, \quad j = 2, \dots, m.$$

3. *(Non-resonance assumption) One has $\pm(2p + 3)(\omega, \mathbf{k}) \notin \mathcal{C}_{\mathcal{L}}$, for all $p \geq 0$ such that $pr < 1$.*

Notation 2. We denote by $\pi_j(\mathbf{k})$ ($j = 1, \dots, m$) the eigenprojectors of the eigenvalues $\omega_j(\mathbf{k})$ of $A(\mathbf{k}) + (1/i)E$; in particular, we have

$$\mathcal{L}(0, \mathbf{k}) = A(\mathbf{k}) + \frac{1}{i}E = \sum_{j=1}^m \omega_j(\mathbf{k})\pi_j(\mathbf{k}).$$

Example 5. For Maxwell's equations, it is shown in Appendix 2 that Assumption 4 is satisfied with $m = 7$ in dimension $d = 3$, for $\omega \neq 0$. Explicit expressions for the eigenprojectors $\pi_j(\mathbf{k})$ are also provided in Appendix 2.

Theorem 2. *Let $B = H^0(\mathbb{R}^d)^n$ or $B = W(\mathbb{R}^d)^n$ and $u^0 \in B^{(1)}$, $r \in B$. Let Assumptions 1, 3, and 4 be satisfied and assume moreover that $u^0 = \pi_1(\mathbf{k})u^0 + \varepsilon r$. Then*

- (i) *There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $u \in C([0, T/\varepsilon]; B^{(1)})$ to (30) with initial condition u^0 .*
- (ii) *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution \mathbf{U} to (20) provided by Theorem 1 exists on $[0, T/\varepsilon]$ and*

$$\|\mathbf{U} - \mathbf{U}_{\text{SVEA}}\|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_B)(1 + |\nabla u^0|_B + |r|_B),$$

where $\mathbf{U}_{\text{SVEA}}(t, x) = u(t, x)e^{i(\mathbf{k}x - \omega t)/\varepsilon} + \text{c.c.}$

Proof. Here again, the proof is a small variation of the one given in [20] for the trilinear case and $B = W(\mathbb{R}^d)^n$. We just indicate the main steps of the proof:

Step 1. Existence and bounds of the solution u to (30) is established by a fixed point argument as in Theorem 1.

Step 2. We decompose u as

$$u = u_1 + u_{\Pi}, \quad \text{with } u_{\Pi} = \sum_{j=2}^m u_j,$$

and where $u_j = \pi_j(\mathbf{k} + \varepsilon D)u$ (see Notation 2).

Step 3. Thanks to the assumption that $\omega = \omega_1(\mathbf{k})$ one gets from the equation obtained by applying $\pi_1(\mathbf{k} + \varepsilon D)$ to (30) that $|\partial_t u_1(t)|_B$ is uniformly bounded on $[0, T/\varepsilon]$.

Step 4. Using a non-stationary phase argument (on the semigroup formulation) relying on point (ii) of Assumption 4 and the bound on $\partial_t u_1$ established in Step 3, and taking advantage of Assumption 3, we get that $(1/\varepsilon)|u_{\Pi}(t)|_B$ remains uniformly bounded on $[0, T/\varepsilon]$.

Step 5. Using the non-resonance condition iii of Assumption 4, one can show that the third and higher harmonics created by the nonlinearity remain of order $O(\varepsilon)$. More precisely, the solution $U \in H^1(\mathbb{T}; B)$ to (25) provided by Theorem 1 can be written as

$$U(t, x, \theta) = U_{\text{app}}(t, x, \theta) + \varepsilon V(t, x, \theta),$$

where $U_{\text{app}}(t, x, \theta) = u(t, x)e^{i\theta} + \text{c.c.}$, and V remains bounded (with respect to ε) in $C([0, T/\varepsilon]; H^1(\mathbb{T}; B)^n)$.

Step 6. Since $U(t) - U_{\text{app}}(t) = \varepsilon V(t)$, it follows from the above that

$$\sup_{t \in [0, T/\varepsilon]} |U(t) - U_{\text{app}}(t)|_{H^1(\mathbb{T}; B)} \leq \varepsilon C(T, |u^0|_B)(1 + |\nabla u^0|_B + |r|_B),$$

and the theorem follows therefore from the observation that

$$|\mathbf{U} - \mathbf{U}_{\text{SVEA}}|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \sup_{t \in [0, T/\varepsilon]} |U(t) - U_{\text{app}}(t)|_{H^1(\mathbb{T}; B)}. \quad \square$$

3.3 The Full Dispersion Model

The idea is to diagonalize (30) in order to work with a scalar equation. We project therefore (30) onto the eigenspace corresponding to the oscillating term. This naturally leads to introduce

$$u_{\text{FD}} = \pi_1(\mathbf{k} + \varepsilon D)u_{\text{env}}.$$

which leads to the *full dispersion* scalar⁸ equation (writing $u = u_{\text{FD}}$)

$$\begin{cases} \partial_t u + \frac{i}{\varepsilon}(\omega_1(\mathbf{k} + \varepsilon D) - \omega)u + \varepsilon^{1+p}\pi_1(\mathbf{k} + \varepsilon D)A_0 u \\ \quad = \varepsilon\pi_1(\mathbf{k} + \varepsilon D)F^{\text{env}}(\varepsilon, u) \\ u|_{t=0}(x) = u^0(x) \end{cases} \quad (31)$$

and with $\omega_1(\cdot)$ as in Assumption 4.

The following corollary shows that the full dispersion scalar equation yields an approximation of the same precision as the envelope equation for times $t \in [0, T/\varepsilon]$.

Corollary 1 (Full Dispersion Model). *Under the assumptions of Theorem 2,*

- (i) *There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $u \in C([0, T/\varepsilon]; B^{(1)})$ to (31) with initial condition u^0 .*
- (ii) *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution \mathbf{U} to (20) provided by Theorem 1 exists on $[0, T/\varepsilon]$ and*

$$\|\mathbf{U} - \mathbf{U}_{\text{FD}}\|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_B)(1 + |\nabla u^0|_B + |r|_B),$$

where $\mathbf{U}_{\text{FD}}(t, x) = u(t, x)e^{i(\mathbf{k} \cdot x - \omega t)/\varepsilon} + \text{c.c.}$.

Remark 7. Equation (31) does not correspond exactly to the “full dispersion” model of [20, 21], where the right-hand side is $\varepsilon\pi_1(\mathbf{k})F^{\text{env}}(\varepsilon, u)$ rather than $\varepsilon\pi_1(\mathbf{k} + \varepsilon D)F^{\text{env}}(\varepsilon, u)$ (but it can be found as an “intermediate model” in [24]). This change does not affect the estimate given in the corollary, but it is important to keep track of the frequency dependence of the polarization of the nonlinear term to introduce the “self-steepening” operators in Sect. 3.6. Note also that the “full dispersion” model is related to the so-called unidirectional pulse propagation equation used in nonlinear optics [19, 25].

Proof. This is actually a by-product of the proof of Theorem 2 since u_{FD} coincides with u_1 in Step 2 of the proof of Theorem 2. \square

3.4 The NLS Equation

The NLS equation is easily derived from (31) using Taylor expansions

$$\begin{aligned} \frac{i}{\varepsilon}(\omega_1(\mathbf{k} + \varepsilon D) - \omega) &= \mathbf{c}_g \cdot \nabla - \frac{i}{2} \nabla \cdot H_{\mathbf{k}} \nabla + O(\varepsilon^2), \\ \pi_1(\mathbf{k} + \varepsilon D) &= \pi_1(\mathbf{k}) + O(\varepsilon), \end{aligned}$$

⁸The linear propagator is a scalar operator but the equation remains vectorial because of the nonlinear term. Indeed, $\pi_1(\mathbf{k})$ is in general not of rank 1 (i.e., $\omega_1(\mathbf{k})$ is in general not of multiplicity one).

where $\mathbf{c}_g = \nabla \omega_1(\mathbf{k})$ and $H_{\mathbf{k}}$ stands for the Hessian of ω_1 at \mathbf{k} . Neglecting the $O(\varepsilon^2)$ terms in (31) we define the NLS approximation $u = u_{\text{NLS}}$ as the solution to

$$\begin{cases} \partial_t u + \mathbf{c}_g \cdot \nabla u - \varepsilon \frac{i}{2} \nabla \cdot H_{\mathbf{k}} \nabla u + \varepsilon^{1+p} \pi_1(\mathbf{k}) A_0 u = \varepsilon \pi_1(\mathbf{k}) F^{\text{env}}(\varepsilon, u); \\ u|_{t=0}(x) = u^0(x). \end{cases} \quad (32)$$

We then get easily (see [20, 21]) the following justification of the NLS approximation.

Corollary 2 (Schrödinger Approximation). *Under the assumptions of Theorem 2, one has for all $u^0 \in B^{(3)}$ such that $u_0 = \pi_1(\mathbf{k})u_0$,*

- (i) *There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $u \in C([0, T/\varepsilon]; B^{(3)})$ to (32) with initial condition u^0 .*
- (ii) *There exists $\varepsilon_0 > 0$ and $\mathfrak{c}_{\text{NLS}} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution \mathbf{U} to (20) provided by Theorem 1 exists on $[0, T/\varepsilon]$ and*

$$\|\mathbf{U} - \mathbf{U}_{\text{NLS}}\|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_B) (1 + |\nabla u^0|_B + \mathfrak{c}_{\text{NLS}} |u^0|_{B^{(3)}}),$$

where $\mathbf{U}_{\text{NLS}}(t, x) = u(t, x) e^{i(\mathbf{k} \cdot x - \omega t)/\varepsilon} + \text{c.c.}$

Remark 8. The component $\mathfrak{c}_{\text{NLS}} |u^0|_{B^{(3)}}$ in the error estimate of the corollary is due to the bad frequency behavior of the Schrödinger equation when the envelope of the oscillations ceases to be well localized in frequency. This is, for instance, the case for short pulses, chirped pulses [20, 21, 24], and near the focusing point. To describe such extreme situations, the standard NLS approximation does a poor job, and this is why various variants have been derived in physics (e.g., [18]).

Remark 9. We assumed here that the polarization of the initial condition is exact (i.e., $r = 0$ in Theorem 2) for the sake of simplicity; indeed, the solution remains polarized along $\pi_1(\mathbf{k})$ for all times and computations on real physical models are much easier.

Example 6. In the frequent case where $\omega_1(\cdot)$ has a radial symmetry, and writing with a slight abuse of notation $\omega_1(\mathbf{k}) = \omega_1(k)$, with $k = |\mathbf{k}|$, we can write

$$\mathbf{c}_g = \omega_1'(k) \mathbf{e}_z, \quad H_{\mathbf{k}} = \frac{\omega_1'(k)}{|\mathbf{k}|} (I - \mathbf{e}_z \otimes \mathbf{e}_z) + \mathbf{e}_z \otimes \mathbf{e}_z \omega_1''(k),$$

where we assumed without loss of generality that $(0z)$ is the direction of the wave number \mathbf{k} , $\mathbf{k} = k \mathbf{e}_z$. In particular, (32) reads

$$\partial_t u + \omega_1'(k) \partial_z u - \varepsilon \frac{i}{2} \frac{\omega_1'(k)}{k} \Delta_{\perp} u - i \frac{\varepsilon}{2} \omega_1''(k) \partial_z^2 u + \varepsilon^{1+p} \pi_1(\mathbf{k}) A_0 u = \varepsilon \pi_1(\mathbf{k}) F^{\text{env}}(\varepsilon, u),$$

where $\Delta_{\perp} = \partial_x^2 + \partial_y^2$ is the Laplace operator in transverse variables. If we write $v(t, x, z) = u(\varepsilon t, x, z - \omega_1'(k)t)$, we get

$$\partial_t v - \frac{i}{2} \frac{\omega_1'(k)}{k} \Delta_{\perp} v - i \frac{1}{2} \omega_1''(k) \partial_z^2 v + \varepsilon^p \pi_1(\mathbf{k}) A_0 v = \pi_1(\mathbf{k}) F^{\text{env}}(\varepsilon, v).$$

3.5 The NLS Equation with Improved Dispersion Relation

We propose here to investigate further the Schrödinger equation with improved dispersion relation derived in [20]. As said in Remark 8, the NLS approximation has too bad dispersive properties to capture correctly the envelope of oscillating solutions to Maxwell's equations in extreme situations, where high frequencies are released. Indeed, the dispersion relation $\omega_1(\cdot)$ of (20) is approximated by the second order polynomial

$$\omega_{\text{NLS}}(\mathbf{k}') = \omega_1(\mathbf{k}) + \mathbf{c}_g \cdot (\mathbf{k}' - \mathbf{k}) + \frac{1}{2} (\mathbf{k}' - \mathbf{k}) \cdot H_{\mathbf{k}}(\mathbf{k}' - \mathbf{k}). \quad (33)$$

For Maxwell's equations and in dimension $d = 1$, Fig. 1 shows that this dispersion relation is very poor when \mathbf{k}' is not close to \mathbf{k} . The idea introduced in [20] is to replace the *linear* part of the Schrödinger approximation by a linear operator that differs from the linear part of the Schrödinger approximation by $O(\varepsilon^2)$ terms only, but whose dispersion relation is far better.

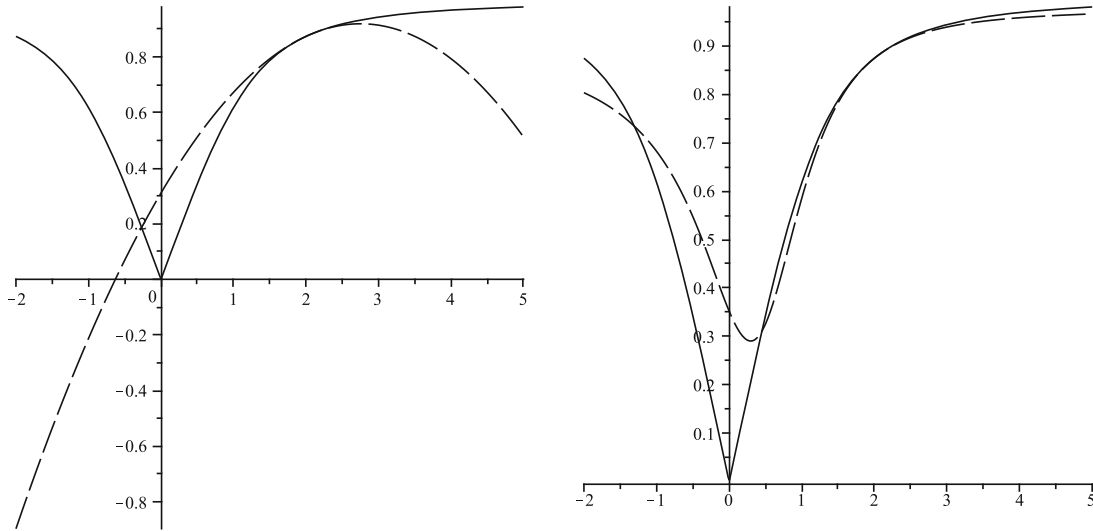


Fig. 1 The $\omega_{+,-}$ component (see Appendix 2) of the characteristic variety of Maxwell's equations (*solid*) and the dispersion relation corresponding to the Schrödinger approximation (*left*) and improved Schrödinger (*right*). Here $\mathbf{k} = 2$

More precisely, we consider an approximation under the form

$$\mathbf{U}_{\text{imp}}(t, x) = u_{\text{imp}}(t, x)e^{i(\mathbf{k}\cdot x - \omega t)/\varepsilon} + \text{c.c.}, \quad (34)$$

where $u = u_{\text{imp}}$ solves the *NLS equation with improved dispersion relation*

$$\begin{cases} (1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \partial_t u + \mathbf{c}_g \cdot \nabla u - \varepsilon \frac{i}{2} \nabla \cdot (H_{\mathbf{k}} + 2\mathbf{c}_g \otimes \mathbf{b}) \nabla u \\ + \varepsilon^2 C_3(\nabla)u + \varepsilon^{1+p} \pi_1(\mathbf{k}) A_0 u = \varepsilon \pi_1(\mathbf{k}) F^{\text{env}}(\varepsilon, u) \\ u|_{t=0}(x) = u^0(x), \end{cases} \quad (35)$$

where $\mathbf{b} \in \mathbb{C}^d$, $B \in \mathcal{M}_{d \times d}(\mathbb{R})$ and $C_3(\nabla)$ is a third-order homogeneous differential operator. We assume moreover that

$$B \text{ is symmetric positive, } \mathbf{b} \in \text{Range}(B), \text{ and } 4 - \mathbf{b} \cdot (B^{-1} \mathbf{b}) > 0 \quad (36)$$

(note that even though $B^{-1} \mathbf{b}$ is not unique when B is not definite, the scalar $\mathbf{b} \cdot (B^{-1} \mathbf{b})$ is uniquely defined). These assumptions ensure that the operator $(1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla)$ is invertible. This new model allows one to replace (33) by

$$\begin{aligned} \omega_{\text{imp}}(\mathbf{k}') &= \omega_1(\mathbf{k}) \\ &+ \frac{\mathbf{c}_g \cdot (\mathbf{k}' - \mathbf{k}) + \frac{1}{2}(\mathbf{k} - \mathbf{k}') \cdot (H_{\mathbf{k}} + 2\mathbf{c}_g \otimes \mathbf{b})(\mathbf{k}' - \mathbf{k}) - C_3(\mathbf{k}' - \mathbf{k})}{1 + \mathbf{b} \cdot (\mathbf{k}' - \mathbf{k}) + (\mathbf{k}' - \mathbf{k}) \cdot B(\mathbf{k}' - \mathbf{k})}. \end{aligned}$$

A good choice of \mathbf{b} , B , and C_3 allows a much better approximation of $\omega_1(\cdot)$, as shown in Fig. 1 for Maxwell's equation in dimension $d = 1$.

Exactly as for Corollary 2 we get the following result, where the only difference in the error estimate with respect to Corollary 2 is the constant \mathbf{c}_{imp} (which is much smaller than \mathbf{c}_{NLS} for good choices of \mathbf{b} , B , and C). We refer to [20] for the proof and numerical validations of this model for the approximation of short pulses and chirped pulses.

Corollary 3 (Schrödinger Approximation with Improved Dispersion). *Under the assumptions of Theorem 2, one has, for all $u^0 \in B^{(3)}$ such that $\pi_1(\mathbf{k})u_0 = u_0$,*

- (i) *There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $u \in C([0, T/\varepsilon]; B^{(3)})$ to (35) with initial condition u^0 .*
- (ii) *There exist $\varepsilon_0 > 0$ and $\mathbf{c}_{\text{imp}} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution \mathbf{U} to (20) provided by Theorem 1 exists on $[0, T/\varepsilon]$ and*

$$\|\mathbf{U} - \mathbf{U}_{\text{imp}}\|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_B) (1 + |\nabla u^0|_B + \mathbf{c}_{\text{imp}} |u^0|_{B^{(3)}}),$$

where $\mathbf{U}_{\text{imp}}(t, x) = u(t, x)e^{i(\mathbf{k}\cdot x - \omega t)/\varepsilon} + \text{c.c.}$

Example 7. In the framework of Example 6, i.e., if $\omega_1(\mathbf{k}) = \omega_1(k)$ with $k = |\mathbf{k}|$ and $\mathbf{k} = k\mathbf{e}_z$, (35) can be written

$$\begin{aligned} & (1 - i\varepsilon\mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \partial_t u + \omega_1'(k) \partial_z u - \varepsilon \frac{i}{2} \left(\frac{\omega_1'(k)}{k} \Delta_{\perp} + \omega_1''(k) \partial_z^2 \right) u \\ & - \varepsilon i \omega_1'(k) \mathbf{b} \cdot \nabla \partial_z u + \varepsilon^2 C_3(\nabla) u + \varepsilon^{1+p} \pi_1(\mathbf{k}) A_0 u \\ & = \varepsilon \pi_1(\mathbf{k}) F^{\text{env}}(\varepsilon, u). \end{aligned}$$

If we write $v(t, x, z) = u(\varepsilon t, x, z - \omega_1'(k)t)$ and choose $C_3(\nabla) = -\omega_1'(k) \nabla \cdot B \nabla \partial_z$, we get

$$\begin{aligned} & (1 - i\varepsilon\mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \partial_t v - \frac{i}{2} \left(\frac{\omega_1'(k)}{k} \Delta_{\perp} + \omega_1''(k) \partial_z^2 \right) v + \varepsilon^p \pi_1(\mathbf{k}) A_0 v \\ & = \pi_1(\mathbf{k}) F^{\text{env}}(\varepsilon, v). \end{aligned} \tag{37}$$

A similar equation has been proposed in [26, Sect. 8.5.3] in the framework of water wave equations.

3.6 The NLS Equation with Frequency Dependent Polarization

The idea here is to improve the rough approximation $\pi_1(\mathbf{k} + \varepsilon D) \sim \pi_1(\mathbf{k}) + O(\varepsilon)$ used to derive the NLS approximation (see Sect. 3.4). Indeed, when the description of the envelope of the laser pulse requires a broad band of frequencies as in the situations mentioned in Sect. 3.5, the variations of the polarization term $\pi_1(\mathbf{k} + \varepsilon D)$ in front of the nonlinearity in (31) become important and should be taken into account. We therefore make the following approximation,

$$\pi_1(\mathbf{k} + \varepsilon D) \sim (1 - i\varepsilon\mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla)^{-1} [\pi_1(\mathbf{k}) + \varepsilon \pi_1'(\mathbf{k}) \cdot D - i\varepsilon(\mathbf{b} \cdot \nabla) \pi_1(\mathbf{k})],$$

where \mathbf{b} and B are the same as in the NLS approximation with improved dispersion (35). In particular, if $\mathbf{b} = 0$ and $B = 0$ (standard NLS), then the above approximation coincides with the first order Taylor expansion

$$\pi_1(\mathbf{k} + \varepsilon D) = \pi_1(\mathbf{k}) + \varepsilon \pi_1'(\mathbf{k}) \cdot D.$$

The general formula has the same accuracy as this Taylor expansion as $\varepsilon \rightarrow 0$. The corresponding approximation is given by

$$\mathbf{U}_{\text{pol}}(t, x) = u_{\text{pol}}(t, x) e^{i(\mathbf{k} \cdot x - \omega t)/\varepsilon} + \text{c.c.}, \tag{38}$$

where $u = u_{\text{pol}}$ solves the *NLS equation with frequency dependent polarization*

$$\left\{ \begin{array}{l} (1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \partial_t u + \mathbf{c}_g \cdot \nabla u \\ - \varepsilon \frac{i}{2} \nabla \cdot (H_{\mathbf{k}} + 2\mathbf{c}_g \otimes \mathbf{b}) \nabla u + \varepsilon^2 C_3(\nabla) u + \varepsilon^{1+p} \pi_1(\mathbf{k}) A_0 u \\ = \varepsilon [\pi_1(\mathbf{k}) + \varepsilon \pi_1(\mathbf{k}) \pi_1'(\mathbf{k}) \cdot D - i\varepsilon (\mathbf{b} \cdot \nabla) \pi_1(\mathbf{k})] F^{\text{env}}(\varepsilon, \pi_1(\mathbf{k}) u) \\ u|_{t=0}(x) = u^0(x), \end{array} \right. \quad (39)$$

where \mathbf{b} , B , and $C_3(\nabla)$ are the same as in (31).

Contrary to all the previous models, the nonlinearity in (39) seems to be of quasilinear rather than nonlinear nature. It turns out, however, that the presence of the operator $(1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla)$ in front of the time derivative plays a smoothing role allowing the control of one or several first order derivatives (see the discussion in point 4 in p. 22). If the first order derivatives involved in the nonlinearity are all controlled by this smoothing operator, then the nonlinearity remains semilinear in nature. As shown in the proof below, the component $-i\varepsilon (\mathbf{b} \cdot \nabla) \pi_1(\mathbf{k}) F^{\text{env}}(\varepsilon, \pi_1(\mathbf{k}) u)$ of the nonlinearity is always semilinear in this sense. This is not the case for the component $\varepsilon \pi_1(\mathbf{k}) \pi_1'(\mathbf{k}) \cdot D F^{\text{env}}(\varepsilon, \pi_1(\mathbf{k}) u)$ that may be of quasilinear nature, in which case a symmetry assumption is needed on the nonlinearity to ensure local well-posedness. In order to state this assumption, it is convenient to introduce the norm $|\cdot|_*$ defined as

$$|u|_*^2 = (u, (1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) u).$$

Assumption 5. For all $v \in W^{1,\infty}(\mathbb{R}^d)^n$ and $u \in L^2(\mathbb{R}^d)^n$ such that $\pi_1(\mathbf{k})u = u$, one has

$$\forall 1 \leq j \leq d, \quad \Re(\pi_1(\mathbf{k}) \pi_1'(\mathbf{k}) \cdot \mathbf{e}_j d_v F^{\text{env}} D_j u, u) \leq \text{Cst} |v|_{W^{1,\infty}} |u|_*^2,$$

where \mathbf{e}_j is the unit vector in the j th direction of \mathbb{R}^d and $d_v F^{\text{env}}$ is the derivative at v of the mapping $u \mapsto F^{\text{env}}(\varepsilon, u)$.

Example 8. The computations performed in Appendix 2 show that this assumption is satisfied by the dimensionless Maxwell equations (73).

The approximation furnished by (39) is justified by the following corollary (the difference in the estimate with respect to Corollary 3 is just a better nonlinear constant, denoted C_{pol} to insist on this point). For the sake of simplicity, we take $B = H^{t_0}(\mathbb{R}^d)^n$ ($t_0 > d/2$) here, but adaptation to Wiener spaces is possible.

Corollary 4 (Schrödinger Approximation with Frequency Dependent Polarization). Let the assumptions of Theorem 2 be satisfied and assume moreover that Assumption 5 holds with $B = H^{t_0}(\mathbb{R}^d)$ ($t_0 > d/2$). Then for all $u_0 \in B^{(3)}$ such that $\pi_1(\mathbf{k})u_0 = u_0$, one has

- (i) There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $u \in C([0, T/\varepsilon]; B^{(3)})$ to (39) with initial condition u^0 .
- (ii) There exist $\varepsilon_0 > 0$ and $c_{\text{imp}} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution \mathbf{U} to (20) provided by Theorem 1 exists on $[0, T/\varepsilon]$ and

$$\|\mathbf{U} - \mathbf{U}_{\text{pol}}\|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C_{\text{pol}}(T, |u^0|_B)(1 + |\nabla u^0|_B + c_{\text{imp}}|u^0|_{B^{(3)}}),$$

where $\mathbf{U}_{\text{pol}}(t, x) = u(t, x)e^{i(\mathbf{k} \cdot x - \omega t)/\varepsilon} + \text{c.c.}$.

Remark 10. We have introduced the variation of the polarization on the NLS equation with improved dispersion (35), but the two steps are independent (i.e., one can take $\mathbf{b} = 0$, $B = 0$ and $C_3(\nabla) = 0$ in (39)).

Remark 11. Note that in (39), we have applied $\pi_1(\mathbf{k})$ to the full nonlinearity (hence the term $\pi_1(\mathbf{k})\pi_1'(\mathbf{k}) \cdot D$ instead of $\pi_1'(\mathbf{k}) \cdot D$). This means that we keep the effect of the frequency dependent polarization on the main polarized space $\text{Range}(\pi_1(\mathbf{k}))$ only. Similarly, we have replaced $F^{\text{env}}(\varepsilon, u)$ by $F^{\text{env}}(\varepsilon, \pi_1(\mathbf{k})u)$. This latter substitution would not change anything to the previous NLS models since we have seen that polarized initial conditions remain polarized. Its purpose in (39) is to make Assumption 5 much easier to check.

Proof. The justification of (39) is performed as for the other models. The only difference here is that local well-posedness for a time scale of order $O(1/\varepsilon)$ must be established. We therefore show here that such a local well-posedness result holds if $u_0 \in H^{s+1}(\mathbb{R}^d)^n$ with $s > t_0 + 1$. We just prove a priori energy estimates for (39); existence, uniqueness, and stability with respect to perturbations can be deduced classically.

The natural energy associated with (39) is given for all $s \geq 0$ by

$$E^s(u) = \frac{1}{2}((1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \Lambda^s u, \Lambda^s u) = \frac{1}{2} |\Lambda^s u|_*^2.$$

Under Assumptions (36) on B and \mathbf{b} , $E^s(u)^{1/2}$ defines a norm that controls uniformly the H^s -norm. It may also control a certain number of first order derivatives. The important point for the local well-posedness of (39) is that it always controls first order derivatives in the direction $\mathbf{b} \cdot \nabla$; more precisely, we claim that there exists $c > 0$ independent of ε such that for all u ,

$$E^s(u) \geq c(|u|_{H^s}^2 + \varepsilon^2 |\mathbf{b} \cdot \nabla u|_{H^s}^2). \quad (40)$$

A quick look on the Fourier side shows that it is equivalent to prove that

$$\forall X \in \mathbb{R}^d, \quad 1 + \mathbf{b} \cdot X + X \cdot B X \geq c(1 + \mathbf{b} \cdot X)^2,$$

which is a consequence of (36).

Multiplying $\Lambda^s(39)$ by $\Lambda^s \bar{u}$ and integrating by parts, we get

$$\begin{aligned} \frac{d}{dt} E^s(u) &= -\varepsilon^{1+p} \Re(\pi_1(\mathbf{k}) A_0 \Lambda^s u, \Lambda^s u) + \varepsilon (\pi_1(\mathbf{k}) \Lambda^s F(u), \Lambda^s u) \\ &\quad + \varepsilon^2 \sum_{j=1}^d \Re(\pi_1(\mathbf{k}) \pi_1(\mathbf{k})' \cdot \mathbf{e}_j \Lambda^s d_u F D_j u, \Lambda^s u) \\ &\quad - \varepsilon^2 \Re(i \pi_1(\mathbf{k}) \Lambda^s (\mathbf{b} \cdot \nabla) F(u), \Lambda^s u) \\ &:= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

where we denoted $F(u) = F^{\text{env}}(\varepsilon, \pi_1(\mathbf{k})u)$. It is straightforward to control I_1 , and Moser's estimate gives a control of I_2 ,

$$I_1 \lesssim \varepsilon^{1+p} |u|_{H^s}^2, \quad I_2 \leq \varepsilon C(|u|_\infty) |u|_{H^s}^2.$$

In order to control I_3 , we must split it into two parts,

$$\begin{aligned} I_3 &= \varepsilon^2 \sum_{j=1}^d \Re(\pi_1(\mathbf{k}) \pi_1(\mathbf{k})' \cdot \mathbf{e}_j d_u F D_j \Lambda^s u, \Lambda^s u) \\ &\quad + \varepsilon^2 \sum_{j=1}^d \Re(\pi_1(\mathbf{k}) \pi_1(\mathbf{k})' \cdot \mathbf{e}_j [\Lambda^s, d_u F] D_j u, \Lambda^s u); \end{aligned}$$

the first component is controlled using Assumption 5 while the Kato–Ponce and Moser estimate give a control of the second one,

$$I_3 \leq \varepsilon^2 C \text{st} |u|_{W^{1,\infty}} |u|_{H^s}^2 + \varepsilon^2 C(|u|_{W^{1,\infty}}) |u|_{H^s}^2.$$

Remarking that $\Lambda^s (\mathbf{b} \cdot \nabla) F(u) = \Lambda^s d_u F (\mathbf{b} \cdot \nabla) u$ and using the tame product estimate $|fg|_{H^s} \lesssim |f|_{L^\infty} |g|_{H^s} + |f|_{H^s} |g|_{L^\infty}$ and Moser's inequality, we get

$$I_4 \leq \varepsilon C(|u|_{W^{1,\infty}}) (|u|_{H^s} + \varepsilon |\mathbf{b} \cdot \nabla u|_{H^s}) |u|_{H^s}.$$

Gathering all the above estimates and using (40), we obtain

$$\frac{d}{dt} E^s(u) \leq \varepsilon C(|u|_{W^{1,\infty}}) E^s(u).$$

Since moreover $|u|_{W^{1,\infty}}^2 \lesssim E^s(u)$ for $s > d/2 + 1$, we deduce from Gronwall's estimate that for such s , the energy $E^s(u)$ remains bounded for times of order $O(1/\varepsilon)$. \square

Example 9. In the framework of Examples 6 and 7, we can check that $v(t, x, z) = u(\varepsilon t, x, z - \omega'_1(k)t)$ solves

$$\begin{aligned} & (1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \partial_t v - \frac{i}{2} \left(\frac{\omega'_1(k)}{k} \Delta_\perp + \omega''_1(k) \partial_z^2 \right) v + \varepsilon^p \pi_1(\mathbf{k}) A_0 v \\ & = [\pi_1(\mathbf{k}) + \varepsilon \pi'_1(\mathbf{k}) \cdot D - i\varepsilon (\mathbf{b} \cdot \nabla) \pi_1(\mathbf{k})] F^{\text{env}}(\varepsilon, \pi_1(\mathbf{k}) v). \end{aligned} \quad (41)$$

3.7 Including Ionization Processes

3.7.1 The Profile Equation

As in Sect. 3.1, we solve the initial value problem (21)–(23) for times of order $O(1/\varepsilon)$, writing the solution under a profile form,

$$(\mathbf{U}, \mathbf{W})(t, x) = (U, W) \left(t, x, \frac{\mathbf{k} \cdot x - \omega t}{\varepsilon} \right). \quad (42)$$

Again, $U(t, x, \theta)$ and $W(t, x, \theta)$ are periodic with respect to θ , and we use any $\omega \in \mathbb{R}$. The action of the Fourier multiplier $\mathcal{H}(\varepsilon(\mathbf{k}/k^2) \cdot D)$ from (19) (and (16)) is transferred at the profile level into the operator $\mathcal{H}(D_\theta + \varepsilon(\mathbf{k}/k^2) \cdot D)$, with $k = |\mathbf{k}|$,

$$\left(\mathcal{H} \left(\varepsilon \frac{\mathbf{k}}{k^2} \cdot D \right) U \right) (t, x) = \left(\mathcal{H} \left(D_\theta + \varepsilon \frac{\mathbf{k}}{k^2} \cdot D \right) U \right) \left(t, x, \frac{\mathbf{k} \cdot x - \omega t}{\varepsilon} \right),$$

where $\mathcal{H}(D_\theta + \varepsilon(\mathbf{k}/k^2) \cdot D)$ is the Fourier multiplier

$$\mathcal{H} \left(D_\theta + \varepsilon \frac{\mathbf{k}}{k^2} \cdot D \right) \sum_{n \in \mathbb{Z}} u_n(x) e^{in\theta} = \sum_{n \in \mathbb{Z}} \left(\mathcal{H} \left(n + \varepsilon \frac{\mathbf{k}}{k^2} \cdot D \right) u_n \right) (x) e^{in\theta},$$

which acts continuously on $H^k(\mathbb{T}, B)$, for any $k \in \mathbb{N}$. In order to get a solution to the original problem, it is sufficient that (U, W) solves

$$\begin{cases} \partial_t U + A(\partial) U + \frac{i}{\varepsilon} \mathcal{L}(\omega D_\theta, \mathbf{k} D_\theta) U + \varepsilon^{1+p} A_0 U \\ = \varepsilon F(\varepsilon, U) - \varepsilon \mathcal{H} \left(D_\theta + \varepsilon \frac{\mathbf{k}}{k^2} \cdot D \right) (W C_1^T C_1 U) - \varepsilon c C_1^T G(C_1 U, W), \\ \partial_t W - \frac{i}{\varepsilon} \omega D_\theta W = \varepsilon G(C_1 U, W) \cdot C_1 U, \end{cases} \quad (43)$$

with initial conditions

$$(U, W)|_{t=0}(x, \theta) = (u^0(x) e^{i\theta} + \text{c.c.}, 0). \quad (44)$$

Theorem 3. *Let $B = A^n \times A$, with $A = H^{t_0}(\mathbb{R}^d)$ or $A = W(\mathbb{R}^d)$, and $u^0 \in A^n$. Under Assumptions 1–3, there exists $T > 0$ such that for all $0 < \varepsilon \leq 1$ there exists a unique solution $\mathbf{Z} = (\mathbf{U}, \mathbf{W}) \in C([0, T/\varepsilon]; B)$ to (21)–(23). Moreover, one can write \mathbf{Z} in the form*

$$\mathbf{Z}(t, x) = Z\left(t, x, \frac{\mathbf{k} \cdot x - \omega t}{\varepsilon}\right),$$

where $Z = (U, W)$ solves the profile equation (43), with the initial condition (44).

Proof. Similar to the one of Theorem 1, by an iterative scheme in $H^k(\mathbb{T}; B)$, with $k \geq 1$.

3.7.2 The Slowly Varying Envelope Approximation

In this section, as in Sect. 3.2, we shall assume that $\omega = \omega_1(\mathbf{k})$ is some characteristic frequency for $\mathcal{L}(\cdot, \mathbf{k})$. Postulating the Ansatz

$$Z(t, x, \theta) \sim (u_{\text{env}}(t, x)e^{i\theta} + \text{c.c.}, w_{\text{env}}(t, x)), \quad (45)$$

we obtain formally the following system for $(u_{\text{env}}, w_{\text{env}})$ (denoted (u, w)),

$$\begin{cases} \partial_t u + \frac{i}{\varepsilon} \mathcal{L}(\omega, \mathbf{k} + \varepsilon D)u + \varepsilon^{1+p} A_0 u \\ = \varepsilon F^{\text{env}}(\varepsilon, u) - \varepsilon i w C_1^T C_1 u - \varepsilon c C_1^T G^{\text{env}}(C_1 u, w), \\ \partial_t w = 2\varepsilon G^{\text{env}}(C_1 u, w) \cdot \overline{C_1 u}, \end{cases} \quad (46)$$

where we used that $\mathcal{H}(1 + \varepsilon(\mathbf{k}/k^2) \cdot D) = i + O(\varepsilon)$. Here, F^{env} is given by (29) and G^{env} is defined in the same way, filtering oscillations,

$$G^{\text{env}}(u, w) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} G(ue^{i\theta} + \text{c.c.}, w) d\theta.$$

Remark 12. The equation one obtains for w from direct computations is actually

$$\begin{aligned} \partial_t w &= \frac{\varepsilon}{2\pi} \int_0^{2\pi} G(C_1 u e^{i\theta} + \text{c.c.}, w) \cdot (C_1 u e^{i\theta} + \text{c.c.}) d\theta \\ &= c_1 \varepsilon \left(2^K |C_1 u|^{2K} + 2 \sum_{k=1}^{\lfloor K/2 \rfloor} \binom{K-1}{k} \binom{K-k}{k} (2|C_1 u|^2)^{K-2k} |C_1 u \cdot C_1 u|^{2k} \right) \\ &\quad + 2c_2 \varepsilon w |C_1 u|^2 \\ &= 2\varepsilon G^{\text{env}}(C_1 u, w) \cdot \overline{C_1 u}, \end{aligned}$$

in view of Assumption 2.

The approximation (45) is justified in the following theorem.

Theorem 4. *Let $B = A^n \times A$, with $A = H^{t_0}(\mathbb{R}^d)$ or $A = W(\mathbb{R}^d)$. Let $u^0 \in A^n$, with $\nabla u^0 \in A^{nd}$, and $r \in A^n$. Let Assumptions 1, 3, and 4 be satisfied and assume moreover that $\omega \neq 0$ and that $u^0 = \pi_1(\mathbf{k})u^0 + \varepsilon r$. Then*

- (i) *There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $(u, w) \in C([0, T/\varepsilon]; B^{(1)})$ to (46) with initial condition $(u^0, 0)$.*
- (ii) *There exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution \mathbf{Z} to (43) provided by Theorem 3 exists on $[0, T/\varepsilon]$ and*

$$|\mathbf{Z} - \mathbf{Z}_{\text{SVEA}}|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_A)(1 + |\nabla u^0|_A + |r|_A),$$

where $\mathbf{Z}_{\text{SVEA}}(t, x) = (u(t, x)e^{i(\mathbf{k} \cdot x - \omega t)/\varepsilon} + \text{c.c.}, w(t, x))$.

Proof. The arguments are similar to the ones in the proof of Theorem 2.

Step 1. As in the proof of Theorems 1 and 3, we have local in time T/ε (with T independent of ε) existence of (u, w) , solution to (46), together with (uniform with respect to ε) bounds.

Step 2. Decomposing $u = u_I + u_{\text{II}}$ as in Step 2 of the proof of Theorem 2, one obtains in the same way that $|\partial_t u_I|_{L^\infty([0, T/\varepsilon], A^n)}$ is bounded with respect to ε . This is also the case for $|\partial_t w|_{L^\infty([0, T/\varepsilon], A)}$ (even of order $O(\varepsilon)$), as shows directly the third equation in (46).

Step 3. As in Step 4 of the proof of Theorem 2, we deduce that

$$\frac{1}{\varepsilon} |u_{\text{II}}|_{L^\infty([0, T/\varepsilon], A)} \leq C(T, |u^0|_A)(1 + |\nabla u^0|_A + |r|_A).$$

This is obtained by integration by parts in the integral formulation giving the u_I 's of u_{II} , using Step 2 and Assumption 4 to have non-stationary phase; we conclude by a Gronwall argument.

Step 4. Approximation of \mathbf{Z} by \mathbf{Z}_{SVEA} . Compared to Theorem 2, the new point is the component w ; for the sake of simplicity, we therefore set $F = 0$ throughout this proof. Denote

$$\begin{aligned} Z_{\text{app}}(t, x, \theta) &= (U_{\text{app}}, W_{\text{app}})(t, x, \theta) := (u(t, x)e^{i\theta} + \text{c.c.}, w(t, x)), \\ \tilde{Z} &= Z - Z_{\text{app}}, \end{aligned}$$

where $Z = (U, W)$ is the solution to the profile equation (43) provided by Theorem 3. We estimate $\tilde{Z} = (\tilde{U}, \tilde{W})$ in $H^k(\mathbb{T}; B)$ (defined in (27)), $k \geq 1$.

- For \tilde{W} . We have $\tilde{W}|_{t=0} = 0$ and

$$\begin{aligned} \partial_t \tilde{W} + \frac{i}{\varepsilon} \omega D_\theta \tilde{W} &= G(C_1 U, W) \cdot C_1 U - 2G^{\text{env}}(C_1 u, w) \cdot \overline{C_1 u} \\ &= c_1 \sum_{k \neq 0} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-i\theta} G(C_1 U, W) \cdot C_1 U \, d\theta \right) e^{ik\theta} \\ &\quad + c_2 W |C_1 U|^2 - 2c_2 w |C_1 u|^2. \end{aligned}$$

The terms in c_1 and c_2 can be treated similarly. For the sake of clarity, we therefore set $c_1 = 0$ and $c_2 = 1$ in this proof, so that the right-hand side is given by

$$W |C_1 U|^2 - W_{\text{app}} |C_1 U_{\text{app}}|^2 + 2w((C_1 u) \cdot (C_1 u)) e^{2i\theta} + \text{c.c.}.$$

Since

$$|W |C_1 U|^2 - W_{\text{app}} |C_1 U_{\text{app}}|^2| \leq \varepsilon C(T, |u^0|_A) |\tilde{Z}|,$$

we easily deduce that

$$\begin{aligned} &|\tilde{W}(t)|_{H^k(\mathbb{T}; A)} \\ &\leq \varepsilon C(T, |u^0|_A) \int_0^t |\tilde{Z}(t')|_{H^k(\mathbb{T}; A)} \, dt' + 2 \left| \int_0^t e^{-2it'\omega/\varepsilon} w(C_1 u) \cdot (C_1 u) \, dt' \right|_A. \end{aligned}$$

Splitting $u = u_1 + u_{\text{II}}$ as in Theorem 2, we have a uniform (in ε) bound for $(1/\varepsilon) |w(C_1 u) \cdot (C_1 u_{\text{II}})|_{L^\infty([0, T/\varepsilon], A^n)}$ by Steps 1 and 3. The only component left to control is therefore the one involving the product $(C_1 u_1) \cdot (C_1 u_1)$, for which we write

$$\begin{aligned} \int_0^t e^{-2it'\omega/\varepsilon} w(C_1 u_1) \cdot (C_1 u_1) \, dt' &= -i \frac{\varepsilon}{2\omega} \int_0^t e^{-2it'\omega/\varepsilon} \partial_t [w(C_1 u_1) \cdot (C_1 u_1)] \, dt' \\ &\quad + i \frac{\varepsilon}{2\omega} e^{-2it\omega/\varepsilon} [w(C_1 u_1) \cdot (C_1 u_1)](t). \end{aligned}$$

Using the equation satisfied by w to control $\partial_t w$ and Step 2 to control $\partial_t u_1$, one readily deduces that

$$|\tilde{W}(t)|_{H^k(\mathbb{T}; A)} \leq C(T, |u^0|_A) \left(1 + |\nabla u^0|_A + |r|_A + \varepsilon \int_0^t |\tilde{Z}(t')|_{H^k(\mathbb{T}; A)} \, dt' \right). \quad (47)$$

- For \tilde{U} , we have $\tilde{U}|_{t=0} = 0$ and

$$\begin{aligned} & \partial_t \tilde{U} + \frac{i}{\varepsilon} \mathcal{L}(\omega D_\theta, \mathbf{k} D_\theta + \varepsilon D) \tilde{U} + \varepsilon^{1+p} A_0 \tilde{U} \\ &= - \left(\mathcal{H} \left(D_\theta + \varepsilon \frac{\mathbf{k}}{k^2} \cdot D \right) - i \right) (W C_1^T C_1 U) \\ & \quad - i (W_{\text{app}} + \varepsilon \tilde{W}) C_1^T C_1 (U_{\text{app}} + \varepsilon \tilde{U}) + i W_{\text{app}} C_1^T C_1 U_{\text{app}}, \end{aligned}$$

where, for the sake of simplicity, we also have taken $c = 0$ because the corresponding terms do not raise any difficulty. Since the Fourier multiplier $\mathcal{H}(D_\theta + \varepsilon(\mathbf{k}/k^2) \cdot D) - i$ has a norm of order ε when acting on A and because the difference of the last two terms is easily bounded in terms of $\varepsilon \tilde{Z}$, we get that $|\tilde{U}(t)|_{H^k(\mathbb{T};A)}$ satisfies the same upper bound as $|\tilde{W}(t)|_{H^k(\mathbb{T};A)}$ in (47).

Gathering the upper bounds for \tilde{W} and \tilde{U} we therefore get

$$\begin{aligned} \sup_{t' \in [0, t]} |\tilde{Z}(t')|_{H^k(\mathbb{T};A)} &\leq C(T, |u^0|_A) (1 + |\nabla u^0|_A + |r|_A) \\ & \quad + \varepsilon C(T, |u^0|_A) \int_0^t \sup_{t'' \in [0, t']} |\tilde{Z}(t'')|_{H^k(\mathbb{T};A)} dt'. \end{aligned}$$

By a Gronwall estimate, we finally obtain a bound on \tilde{Z} ,

$$\sup_{t \in [0, T/\varepsilon]} |\tilde{Z}(t)|_{H^k(\mathbb{T};B)} \leq C(T, |u^0|_A) (1 + |\nabla u^0|_A + |r|_A),$$

which, since the A -norm controls the L^∞ -norm, immediately leads to the desired estimate on $|\mathbf{Z} - \mathbf{Z}_{\text{SVEA}}|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)}$. \square

3.7.3 The NLS Equation with Ionization

As in Sect. 3.4, a Schrödinger type equation can be derived in presence of ionization,

$$\begin{cases} \partial_t u + \mathbf{c}_g \cdot \nabla u - \varepsilon \frac{i}{2} \nabla \cdot H_{\mathbf{k}} \nabla u + \varepsilon^{1+p} \pi_1(\mathbf{k}) A_0 u \\ \quad = \varepsilon \pi_1(\mathbf{k}) (F^{\text{env}}(\varepsilon, u) - i w C_1^T C_1 u - c C_1^T G^{\text{env}}(C_1 u, w)), \\ \partial_t w = 2\varepsilon G^{\text{env}}(C_1 u, w) \cdot \overline{C_1 u}. \end{cases} \quad (48)$$

Using Theorem 4, this model can be justified with a straightforward adaptation of the case without ionization.

Corollary 5 (Schrödinger Approximation). *Under the assumptions of Theorem 4, one has for all $u^0 \in A^{(3)}$ such that $u_0 = \pi_1(\mathbf{k})u_0$:*

- (i) *There exist $T > 0$ and, for all $\varepsilon \in (0, 1]$, a unique solution $(u, w) \in C([0, T/\varepsilon]; B^{(3)})$ to (48) with initial condition $(u^0, 0)$.*
- (ii) *There exist $\varepsilon_0 > 0$ and $c_{\text{NLS}} > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the solution \mathbf{Z} to (43) provided by Theorem 3 exists on $[0, T/\varepsilon]$ and*

$$|\mathbf{Z} - \mathbf{Z}_{\text{NLS}}|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C(T, |u^0|_A)(1 + |\nabla u^0|_A + c_{\text{NLS}}|u^0|_{A^{(3)}}),$$

$$\text{where } \mathbf{Z}_{\text{NLS}}(t, x) = (u(t, x)e^{i(\mathbf{k} \cdot x - \omega t)/\varepsilon} + \text{c.c.}, w(t, x)).$$

3.7.4 The Most General Model

In (48), ionization effects have been added to the standard cubic NLS equation. It is of course possible to add them to a most sophisticated model that takes into account more general nonlinearities, improved frequency dispersion, and frequency dependent polarization. We then obtain the following generalization of (48) (for which the same justification as in Corollary 5 holds),

$$\left\{ \begin{array}{l} (1 - i\varepsilon \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \partial_t u + \mathbf{c}_g \cdot \nabla u \\ \quad - \varepsilon \frac{i}{2} \nabla \cdot (H_{\mathbf{k}} + 2\mathbf{c}_g \otimes \mathbf{b}) \nabla u + \varepsilon^2 C_3(\nabla)u + \varepsilon^{1+p} \pi_1(\mathbf{k}) A_0 u \\ = \varepsilon [\pi_1(\mathbf{k}) + \varepsilon \pi_1(\mathbf{k}) \pi_1'(\mathbf{k}) \cdot D - i\varepsilon (\mathbf{b} \cdot \nabla) \pi_1(\mathbf{k})] \\ \quad \times (F^{\text{env}}(\varepsilon, \pi_1(\mathbf{k})u) - (i w C_1^T C_1 u + c C_1^T G^{\text{env}}(C_1 u, w))), \\ \partial_t w = 2\varepsilon G^{\text{env}}(C_1 u, w) \cdot \overline{C_1 u}. \end{array} \right. \quad (49)$$

As shown in section “The Case with Charge and Current Density” in Appendix 2, this system of equations takes the following form corresponding to (4) in the case of Maxwell’s equations,

$$\left\{ \begin{array}{l} i(P_2(\varepsilon \nabla) \partial_t + c_g \partial_z)u + \varepsilon(\Delta_\perp + \alpha_1 \partial_z^2)u + i\varepsilon \alpha_2 u \\ \quad + \varepsilon(1 + i\varepsilon \mathbf{c}_3 \cdot \nabla)[(|u|^2 - \rho)u + ic(\alpha_4 |u|^{2K-2}u + \alpha_5 \rho u)] = 0, \\ \partial_t \rho = \varepsilon \alpha_4 |u|^{2K} + \varepsilon \alpha_5 \rho |u|^2. \end{array} \right. \quad (50)$$

4 Analysis of (3) and (4), and Open Problems

In this section, we analyze and formulate open problems for the NLS-type equations (3), as well as the NLS-type equations (4) taking the ionization processes into account. We first consider in Sect. 4.1 the equation (3) in the case of, respectively,

no and anomalous GVD (resp. $\alpha_1 = 0$ and $\alpha_1 = 1$), and also briefly discuss (4). Then, we discuss (3) in the case of normal GVD (i.e., $\alpha_1 = -1$) in Sect. 4.2. Finally, we formulate additional open problems in Sects. 4.3, 4.4, and 4.5.

4.1 The Case of No and Anomalous GVD (Resp. $\alpha_1 = 0$ and $\alpha_1 = 1$)

Let us first consider the case where P_2, α_2, α_3 , and f take the following values,

$$P_2 = I, \quad \alpha_2 = 0, \quad \alpha_3 = 0 \quad \text{and} \quad f = 0,$$

in which case Eq. (3) corresponds to the focusing cubic NLS (1) in dimension $d = 2$ in the case of no GVD, and in dimension $d = 3$ in the case of anomalous GVD. As recalled in the introduction, (1) is critical in dimension 2, and supercritical in dimension 3, and there exist finite-time blow-up solutions in both cases. Let us also recall that some of these blow-up dynamics are stable (see, for example, [4–8] in the critical case, and [10] in a slightly supercritical case).

Now, let us recall that the breakdown of solutions is not always observed in physical experiments on the propagation of laser beams and that instead a *filamentation* phenomenon occurs. In this section, we would like to analyze the possibility that the modified model (3) in the case of no or anomalous GVD prevents the formation of singularities. Below, we analyze the role of each parameter of (3) separately starting with the nonlinearity f .

4.1.1 The Nonlinearity

In the case where $P_2 = I, \alpha_2 = 0$, and $\alpha_3 = 0$, (3) takes the following form,

$$i\partial_t v + \Delta v + (1 + f(\varepsilon^r |v|^2))|v|^2 v = 0, \quad t > 0, \quad x \in \mathbb{R}^d, \quad (51)$$

where the dimension is $d = 2$ in the case of no GVD, and $d = 3$ in the case of anomalous GVD.

As recalled in the introduction, standard modifications of the cubic nonlinearity consist either of the cubic/quintic nonlinearity, i.e., $f(s) = -s$, or a saturated nonlinearity, i.e., f is a smooth function on \mathbb{R}^+ vanishing at the origin and such that $(1 + f(s))s$ is bounded on \mathbb{R}^+ (e.g., $f(s) = -s/(1 + s)$). Let us first consider the case of a saturated nonlinearity. In that case, the fact that $(1 + f(s))s$ is bounded on \mathbb{R}^+ implies the following control of the nonlinear term in (51):

$$\|(1 + f(\varepsilon^r |v|^2))|v|^2 v\|_{L^2} \lesssim \frac{\|v\|_{L^2}}{\varepsilon^r}.$$

Thus, running a fixed point argument yields the fact that this equation is locally well-posed in $L^2(\mathbb{R}^d)$ for any integer $d \geq 1$, with a time of existence only depending on the size of $\|u_0\|_{L^2}$. Since Eq. (51) still satisfies the conservation of mass, this immediately implies global existence for any initial data in $L^2(\mathbb{R}^d)$ and for any $d \geq 1$. Therefore, modifying the cubic nonlinearity in the standard NLS equation by a saturated nonlinearity does indeed prevent finite-time breakdown of the solutions.

Next, we consider the modification by a cubic/quintic nonlinearity. In that case, (51) becomes

$$i\partial_t v + \Delta v + |v|^2 v - \varepsilon^r |v|^4 v = 0, \quad t > 0, x \in \mathbb{R}^d. \quad (52)$$

Note that (52) still satisfies the conservation of mass, and the conserved energy is now given by

$$E(v(t)) = \frac{1}{2} \int |\nabla v(t, x)|^2 dx - \frac{1}{4} \int |v(t, x)|^4 dx + \frac{\varepsilon^r}{6} \int |v(t, x)|^6 dx = E(v_0).$$

Note also that

$$\begin{aligned} E(v(t)) + \left(\frac{1}{2} + \frac{3}{32\varepsilon^r} \right) \|v(t)\|_{L^2}^2 \\ &= \frac{1}{2} \|v(t)\|_{H^1}^2 + \frac{\varepsilon^r}{6} \int |v(t)|^2 \left(|v(t)|^2 - \frac{3}{4\varepsilon^r} \right)^2 dx \\ &\geq \frac{1}{2} \|v(t)\|_{H^1}^2 \end{aligned}$$

which together with the conservation of mass and energy yields

$$\|v(t)\|_{H^1} \leq \sqrt{2E(v_0) + \left(1 + \frac{3}{16\varepsilon^r} \right) \|v_0\|_{L^2}^2}. \quad (53)$$

Let us first consider the case of dimension 2. In this case, both the cubic focusing NLS and the quintic defocusing NLS are H^1 subcritical in the sense that the equation is locally well-posed in $H^1(\mathbb{R}^2)$ with a time of existence only depending on the size of $\|u_0\|_{H^1}$ (see [1]). The proof extends to (52) which is thus locally well-posed in $H^1(\mathbb{R}^2)$ with a time of existence only depending on the size of $\|u_0\|_{H^1}$. Together with the bound (53), this immediately implies global existence for any initial data in $H^1(\mathbb{R}^2)$. The case of dimension 3 is more involved, since the quintic defocusing NLS is H^1 critical in the sense that the equation is locally well-posed in $H^1(\mathbb{R}^3)$ with a time of existence depending on the shape of u_0 (see [27]). Thus, one cannot rely solely on the bound (53) to prove global existence. However, the case of combined nonlinearities is addressed in [28], where global existence of solutions with H^1 initial data is proved, when the nonlinearity is a sum of two H^1 -subcritical powers, or the sum of an H^1 -subcritical power and of a defocusing H^1 -critical

power. This includes Eq. (52). Therefore, modifying the cubic nonlinearity in the standard NLS equation by a quintic defocusing nonlinearity does indeed prevent finite-time breakdown of the solutions both in dimensions 2 and 3. Let us mention an interesting phenomenon regarding the qualitative behavior of the solutions. Physical experiments suggest an oscillatory behavior of the solution which focuses, then defocuses, refocuses, . . . in an almost periodic fashion (see [18] and the references therein). Such a behavior is also observed in numerical simulations and suggested by heuristic arguments (see, for example, [3, 29]).

Open Problem 1. Establish rigorously the “oscillatory” phenomenon of the solutions to (51) observed in physical experiments (see, e.g., [18]).

Another open problem concerns the behavior of the solution as $\varepsilon \rightarrow 0$. Pick an initial data $v_0 \in H^1(\mathbb{R}^d)$, $d = 2, 3$, leading to a finite-time blow-up solution v at time $T > 0$ to the cubic NLS (1). Consider the solution v_ε of (51) with the same initial data v_0 . In the case of dimension 2, it is shown in [30] that $v_\varepsilon(t) \rightarrow v(t)$ in $H^1(\mathbb{R}^2)$ as $\varepsilon \rightarrow 0$ on $[0, T)$ and,

$$\lim_{\varepsilon \rightarrow 0} \|v_\varepsilon(T)\|_{H^1(\mathbb{R}^2)} = +\infty.$$

An interesting question is the understanding of the limit of $v_\varepsilon(t)$ as $\varepsilon \rightarrow 0$ for $t > T$. Partial results have been obtained in this direction in [31]. There, it is proved that only few scenarios are possible, but one would like to establish whether all scenarios do occur or only some of them, and which scenarios are generic.

Open Problem 2. Let $v_0 \in H^1(\mathbb{R}^d)$, $d = 2, 3$ be leading to a finite-time blow-up solution v at time $T > 0$ to the cubic NLS (1). Consider the solution v_ε of (51) with the same initial data v_0 . Describe the behavior of $v_\varepsilon(t)$ as $\varepsilon \rightarrow 0$ for $t > T$.

4.1.2 Taking the Ionization Process into Account

Let us discuss (4) and (5) in the case of no or anomalous GVD which take the ionization process into account and also correspond to a modification of the nonlinearity in the focusing cubic NLS (1). Ionization is certainly the most important phenomenon leading to the formation of laser filaments. Recall that (4) is given by

$$\begin{cases} i(\partial_t + c_g \partial_z)u + \varepsilon(\Delta_\perp + \alpha_1 \partial_z^2)u + \varepsilon(|u|^2 - \rho)u \\ \quad = -i\varepsilon c(\alpha_4 |u|^{2K-2}u + \alpha_5 \rho u), \\ \partial_t \rho = \varepsilon \alpha_4 |u|^{2K} + \varepsilon \alpha_5 \rho |u|^2, \end{cases} \quad (54)$$

where $\alpha_4, \alpha_5 \geq 0$, $c > 0$, where ρ is the density of electron created by ionization, while $\mathbf{c}_g = c_g \mathbf{e}_z$ is the group velocity associated with the laser pulse, and where $d = 2$ in the case of no GVD, and $d = 3$ in the case of anomalous GVD. Also, recall that (5) is given by

$$\begin{cases} i\partial_\tau v + (\Delta_\perp + \alpha_1 \partial_z^2)v + (|v|^2 - \tilde{\rho})v = -ic(\alpha_4 |v|^{2K-2}v + \alpha_5 \tilde{\rho}v), \\ -c_g \partial_z \tilde{\rho} = \varepsilon \alpha_4 |v|^{2K} + \varepsilon \alpha_5 \tilde{\rho} |v|^2, \end{cases} \quad (55)$$

where $\tilde{\rho}$ corresponds to ρ written in a frame moving at the group velocity $\mathbf{c}_g = c_g \mathbf{e}_z$ and with respect to a rescaled time $\tau = \varepsilon t$, i.e.,

$$\rho(t, X_\perp, z) = \tilde{\rho}(\varepsilon t, X_\perp, z - c_g t).$$

In view of the second equation of (55), a boundary condition for $\tilde{\rho}$ has to be prescribed at $z = z_0$ for some z_0 in order to obtain a well-posed problem. A natural choice, which ensures that $\tilde{\rho} \geq 0$, is to prescribe $\tilde{\rho}$ at $+\infty$:

$$\lim_{z \rightarrow +\infty} (\nabla^\perp)^l \tilde{\rho}(\tau, X_\perp, z) = 0, \quad \forall l \geq 0. \quad (56)$$

Let us first discuss the local well-posedness theory for Eqs. (54) and (55), starting with the first one. One may obtain the existence of solutions to (54) over an interval of time with size $O(\varepsilon^{-1})$. Indeed, for an integer $N > d/2$, differentiating N times with respect to space variables the equations both for v and ρ , multiplying (54), respectively, by $\nabla^N \bar{v}$ and $\nabla^N \rho$, integrating on \mathbb{R}^d , taking the imaginary part and integrating by parts for the first equation, and summing both equations yields:

$$\begin{aligned} \frac{d}{dt} [\|v\|_{H^N}^2 + \|\rho\|_{H^N}^2] &\lesssim \varepsilon (\|v\|_{H^N}^2 + \|\rho\|_{H^N}^2) (\|v\|_{L^\infty} + \|v\|_{L^\infty}^{2K-2} + \|\rho\|_{L^\infty}) \\ &\lesssim \varepsilon (\|v\|_{H^N}^2 + \|\rho\|_{H^N}^2) (1 + \|v\|_{H^N}^2 + \|\rho\|_{H^N}^2)^{K-1} \end{aligned}$$

where we used the fact that $N > d/2$ together with simple product rules and the Sobolev embedding. Integrating this differential inequality, we obtain a time of existence with size $O(\varepsilon^{-1})$.

Next, we discuss the local well-posedness theory for Eq. (55) supplemented with the boundary condition (56). For any integer N , we introduce the space \mathcal{H}^N ,

$$\mathcal{H}^N = \left\{ f: \mathbb{R}^d \rightarrow \mathbb{R}, \sum_{j=0}^N \|\nabla^j f\|_{L_{X_\perp}^2 L_z^\infty} < +\infty \right\}.$$

For an integer $N > d$, differentiating N times with respect to space variables the equations for v , multiplying the first equation of (55) by $\nabla^N \bar{v}$, integrating on \mathbb{R}^d , taking the imaginary part and integrating by parts yields:

$$\frac{d}{dt} \|v\|_{H^N}^2 \lesssim \|v\|_{H^N}^4 + \|v\|_{H^N}^{2K} + \|v\|_{H^N}^2 \|\tilde{\rho}\|_{\mathcal{H}^N}$$

where we used the fact that $N > d$ together with the Sobolev embedding in the last inequality. Next, we estimate $\|\tilde{\rho}\|_{\mathcal{H}^N}$. In view of the second equation of (55) and the boundary condition (56), we have

$$\tilde{\rho}(t, x, y, z) = \int_z^{+\infty} \alpha_4 \varepsilon |v|^{2K}(t, x, y, \sigma) \exp\left(\frac{\alpha_5 \varepsilon}{c} \int_z^\sigma |v|^2(t, x, y, s) ds\right) d\sigma.$$

We infer

$$|\tilde{\rho}(t, x, y, z)| \lesssim \left(\int_{-\infty}^{+\infty} |v|^{2K}(t, x, y, \sigma) d\sigma \right) \exp(\|v\|_{H^N}^2),$$

which yields

$$\|\tilde{\rho}\|_{\mathcal{H}^0} \lesssim \|v\|_{H^N}^{2K} \exp(\|v\|_{H^N}^2),$$

where we used the fact that $N > d$ together with the Sobolev embedding. Next, differentiating the second equation of (55) N times, we obtain

$$-c \partial_z \nabla^N \tilde{\rho} = \alpha_4 \varepsilon \nabla^N (|v|^{2K}) + \alpha_5 \varepsilon \nabla^N (|v|^2 \tilde{\rho}).$$

In view of the boundary condition (56), this yields

$$\begin{aligned} & \nabla^N \tilde{\rho}(t, x, y, z) \\ &= \int_z^{+\infty} (\alpha_4 \varepsilon \nabla^N (|v|^{2K})(t, x, y, \sigma) + \alpha_5 \varepsilon (\nabla^N (|v|^2 \tilde{\rho}) - |v|^2 \nabla^N \tilde{\rho})(t, x, y, \sigma)) \\ & \quad \times \exp\left(\frac{\alpha_5 \varepsilon}{c} \int_z^\sigma |v|^2(t, x, y, s) ds\right) d\sigma. \end{aligned}$$

Hence, we deduce

$$\begin{aligned} & |\nabla^N \tilde{\rho}(t, x, y, z)| \\ & \lesssim \left(\int_{-\infty}^{+\infty} (|\nabla^N (|v|^{2K})(t, x, y, \sigma) + |\nabla^N (|v|^2 \tilde{\rho}) - |v|^2 \nabla^N \tilde{\rho}|(t, x, y, \sigma)) d\sigma \right) \\ & \quad \times \exp(\|v\|_{H^N}^2), \end{aligned}$$

which yields

$$\|\tilde{\rho}\|_{\mathcal{H}^N} \lesssim (\|\tilde{\rho}\|_{\mathcal{H}^{N-1}} \|v\|_{H^N}^2 + \|v\|_{H^N}^{2K}) \exp(\|v\|_{H^N}^2).$$

By induction, we obtain

$$\|\tilde{\rho}\|_{\mathcal{H}^N} \lesssim (1 + \|v\|_{H^N}^{2N}) \|v\|_{H^N}^{2K} \exp((N+1)\|v\|_{H^N}^2).$$

This yields

$$\frac{d}{dt} \|v\|_{H^N}^2 \lesssim \|v\|_{H^N}^4 + \|v\|_{H^N}^{2K} + \|v\|_{H^N}^2 (1 + \|v\|_{H^N}^{2N}) \|v\|_{H^N}^{2K} \exp((N+1)\|v\|_{H^N}^2).$$

Integrating this differential inequality, we obtain local existence for $(v, \tilde{\rho})$ in $H^N \times \mathcal{H}^N$ for $N > d$.

Remark 13. Note that the factor ε in the second equation of (55), while present in the model, is not needed for the well-posedness theory.

Let us now come back to the issue of global well-posedness/finite time singularity formation. For equations (54) and (55), note that the L^2 norm of u (resp. v) is dissipated. Indeed, in the case of (55), multiplying by v , integrating on \mathbb{R}^d , taking the imaginary part and integrating by parts yields:

$$\frac{d}{dt} \|v\|_{L^2}^2 = -c \left(\alpha_4 \|v\|_{L^{2K}}^{2K} + \alpha_5 \int |v|^2 \tilde{\rho} \, dx \right),$$

from which we deduce that the L^2 norm is dissipated since $\tilde{\rho} \geq 0$. On the other hand, the energy is not conserved, nor dissipated. We thus cannot carry out the analysis of Sect. 4.1.1 for (52), even in the case of dimension 2.⁹ An interesting problem would then be to prove that taking the ionization process into account (i.e., replacing the cubic focusing NLS (1) by either (54) or (55)) does indeed prevent finite-time breakdown of the solutions in dimensions 2 and 3. This is formulated in the following open problem.

Open Problem 3. Prove that the solutions to (54) and (55) are global in dimensions 2 and 3.

4.1.3 The Damping

In the case where $P_2 = I$, $\alpha_3 = 0$, and $f = 0$, (3) takes the following form,

$$i\partial_t v + \Delta v + i\alpha_2 v + |v|^2 v = 0, \quad (57)$$

where $d = 2$ in the case of no GVD, and $d = 3$ in the case of anomalous GVD. The mass and energy are not conserved quantities anymore. The mass decreases,

$$\|v(t)\|_{L^2} = e^{-\alpha_2 t} \|v_0\|_{L^2}, \quad (58)$$

while for the energy we have

$$\frac{d}{dt} \left[\frac{1}{2} \|\nabla v\|_{L^2}^2 - \frac{1}{4} \int |v|^4 \, dx \right] = -\alpha_2 \left(\|\nabla v\|_{L^2}^2 - \int |v|^4 \, dx \right), \quad (59)$$

⁹Note that a simpler model of damped NLS where the ρ or $\tilde{\rho}$ term is not present has been investigated in [32]. The authors obtain global existence for a certain range of parameters by controlling a modified energy even if it is not conserved.

so that the energy is neither increasing nor decreasing. Equation (57) has been analyzed in several works (see, e.g., [3, 33–35]). In particular, from standard arguments using Strichartz estimates, there is in dimensions 2 and 3 a continuous and increasing function θ such that global existence holds for $\alpha_2 > \theta(\|u_0\|_{H^1})$ (see [3, 35]). Also, in the 3-dimensional case, there is a function θ such that stable blow-up solutions exist for $0 \leq \alpha_2 \leq \theta(u_0)$ (see [33, 35]). Finally, the same behavior has been observed in numerical simulations and suggested by heuristic arguments in the 2-dimensional case (see [34]). Thus, a small α_2 does not prevent finite-time blow-up. Since the constant α_2 obtained in (4) is usually small, it appears that the damping cannot by itself explain the physical observation according to which the breakdown of solutions does not occur.

4.1.4 Off-Axis Variation of the Group Velocity

In the case where $\alpha_2 = 0$, $\alpha_3 = 0$, and $f = 0$, (3) takes the following form,

$$iP_2(\varepsilon\nabla)\partial_t v + \Delta v + |v|^2 v = 0, \quad (60)$$

where $d = 2$ in the case of no GVD, and $d = 3$ in the case of anomalous GVD. The energy remains unchanged and is still conserved, while the mass is replaced by the following conserved quantity,

$$(P_2(\varepsilon\nabla)v(t), v(t)) = (P_2(\varepsilon\nabla)v_0, v_0). \quad (61)$$

Let us first consider the case of full off-axis dependence, i.e., the case where

$$(P_2(\varepsilon\nabla)v, v) \gtrsim \|v\|_{L^2}^2 + \varepsilon^2 \|\nabla v\|_{L^2}^2. \quad (62)$$

The operator $P_2(\varepsilon\nabla)$ is a second-order self-adjoint operator which is invertible in view of (62). We denote by $P_2(\varepsilon\nabla)^{-1}$ its inverse. We rewrite (60) in the following form,

$$i\partial_t v + P_2(\varepsilon\nabla)^{-1}\Delta v + P_2(\varepsilon\nabla)^{-1}(|v|^2 v) = 0. \quad (63)$$

Using Duhamel's formula, and the semi-group $e^{itP_2(\varepsilon\nabla)^{-1}\Delta}$, we obtain

$$v = e^{itP_2(\varepsilon\nabla)^{-1}\Delta} v_0 + \int_0^t e^{i(t-s)P_2(\varepsilon\nabla)^{-1}\Delta} P_2(\varepsilon\nabla)^{-1}(|v|^2 v)(s) ds. \quad (64)$$

As a simple consequence of (62) and the Sobolev embedding in dimensions 2 and 3, we have

$$\|P_2(\varepsilon\nabla)^{-1}(|v|^2 v)\|_{H^1(\mathbb{R}^d)} \lesssim \frac{1}{\varepsilon} \| |v|^2 v \|_{L^2(\mathbb{R}^d)} \lesssim \frac{1}{\varepsilon} \|v\|_{H^1(\mathbb{R}^d)}^3, \quad d = 2, 3. \quad (65)$$

Now, a fixed point argument based on the formulation (64) together with the estimate (65) and the fact that the semi-group $e^{itP_2(\varepsilon\nabla)^{-1}\Delta}$ is unitary on H^1 implies that (60) is locally well-posed in H^1 for dimensions $d = 2, 3$, with a time of existence only depending on the size of $\|u_0\|_{H^1}$. Together with the lower bound (62) and the conserved quantity (61) which yield a uniform in time bound on the H^1 norm of the solution, this immediately implies global existence for any initial data in H^1 when $d = 2, 3$. Therefore, modifying the focusing cubic NLS equation by adding a full off-axis dependence does indeed prevent the finite-time breakdown of the solutions.

Next, let us consider the case of partial off-axis dependence, i.e., the case where

$$(P_2(\varepsilon\nabla)v, v) \gtrsim \|v\|_{L^2}^2 + \varepsilon^2 \sum_{k=1}^j \|v_k \cdot \nabla v\|_{L^2}^2, \quad (66)$$

for vectors v_k in \mathbb{R}^d , $k = 1, \dots, j$, where $j = 1$ if $d = 2$, and $j = 1$ or $j = 2$ if $d = 3$ so that the right-hand side of (66) does not control the full H^1 norm. Notice that the well-posedness theory has yet to be investigated in this case. Indeed, consider the nonlinear term in the formulation (64),

$$e^{itP_2(\varepsilon\nabla)^{-1}\Delta} P_2(\varepsilon\nabla)^{-1} (|v|^2 v). \quad (67)$$

In the directions v_k of (66), the operator $P_2(\varepsilon\nabla)^{-1}$ gains two derivatives, while the semi-group $e^{itP_2(\varepsilon\nabla)^{-1}\Delta}$ does not disperse (and thus does not satisfy a useful Strichartz estimate). On the other hand, in the directions orthogonal to the vectors v_k , the operator $P_2(\varepsilon\nabla)^{-1}$ does not gain any derivative, while the semi-group $e^{itP_2(\varepsilon\nabla)^{-1}\Delta}$ should satisfy a useful Strichartz estimate. Thus, in order to investigate the well-posedness theory, one should try to combine the regularization provided by the operator $P_2(\varepsilon\nabla)^{-1}$ in the directions v_k of (66) with the dispersive properties of the semi-group $e^{itP_2(\varepsilon\nabla)^{-1}\Delta}$ in the direction orthogonal to the vectors v_k . Now, it would be interesting to investigate whether a suitable well-posedness theory together with the conserved quantities given by the energy and (61) yield global existence. This suggests the following open problem.

Open Problem 4. Investigate both the local and global well-posedness for Eq. (60) in the case of partial off-axis dependence. In particular, does the modification of the focusing cubic NLS (1) by the addition of a partial off-axis dependence prevent the finite-time breakdown of the solutions?

4.1.5 Self-steepening of the Pulse

In the case where $P_2 = I$, $\alpha_2 = 0$, and $f = 0$, (3) takes the following form,

$$i\partial_t v + \Delta v + (1 + i\varepsilon\alpha_3 \cdot \nabla)|v|^2 v = 0, \quad (68)$$

where $d = 2$ in the case of no GVD, and $d = 3$ in the case of anomalous GVD. Adding the operator $(1 + i\varepsilon\alpha_3 \cdot \nabla)$ in front of the nonlinearity of the focusing cubic NLS does certainly not prevent finite-time breakdown of the solutions. Indeed, one expects to obtain even more blow-up solutions since the formation of optical shocks is expected in this case. The term shock is used in view of the similarity with the Burgers equation—both in terms of the nonlinearity of the equation and the profiles of the solutions observed in some numerical simulations (see [36] for more details on optical shocks). The reason we included this modification in the discussion is because it may account for the physical observation of the self-steepening of the pulse: an initial pulse which is symmetric becomes asymmetric after propagating over a large distance and its profile seems to develop a shock. This phenomenon has been widely observed, and we refer the reader to [18] and the references therein. Thus, an interesting open problem would be to exhibit solutions of (68) for which the corresponding profile develops a shock. We formulate a slightly more general open problem below.

Open Problem 5. Describe the blow-up scenarios for the finite-time blow-up solutions to EQ. (68).

4.2 The Case of Normal GVD (i.e., $\alpha_1 = -1$)

Let us consider the case where $P_2, \alpha_1, \alpha_2, \alpha_3$, and f take the following value,

$$P_2 = I, \quad \alpha_1 = -1, \quad \alpha_2 = 0, \quad \alpha_3 = 0 \quad \text{and} \quad f = 0,$$

in which case Eq. (3) corresponds to the following equation in dimension 2 or 3,

$$i\partial_t v + (\Delta_\perp - \partial_z^2)v + |v|^2 v = 0. \quad (69)$$

This equation is sometimes referred to as hyperbolic cubic NLS, or non-elliptic cubic NLS. Since Strichartz estimates only depend on the curvature of the corresponding characteristic manifold (see [37]), (69) satisfies the same Strichartz estimates as the cubic focusing NLS. Thus, the result of Ginibre and Velo [1] immediately extends to (69) which is locally well-posed in $H^1 = H^1(\mathbb{R}^d)$ with $d = 2, 3$. Therefore, for $v_0 \in H^1$, there exists $0 < T \leq +\infty$ and a unique solution $v(t) \in \mathcal{C}([0, T], H^1)$ to (69) and either $T = +\infty$, and the solution is global, or the solution blows up in finite time $T < +\infty$ and then $\lim_{t \uparrow T} \|\nabla u(t)\|_{L^2} = +\infty$.

Equation (69) admits the following conservation laws in the energy space H^1 ,

L^2 -norm: $\|v(t)\|_{L^2}^2 = \|v_0\|_{L^2}^2$;
Energy: $E(v(t)) = \frac{1}{2} \int |\nabla_\perp v(t, x)|^2 dx - \frac{1}{2} \int |\partial_z v|^2 dx - \frac{1}{4} \int |v(t, x)|^4 dx = E(v_0)$;
Momentum: $\text{Im}(\int \nabla v(t, x) \overline{v(t, x)} dx) = \text{Im}(\int \nabla v_0(x) \overline{v_0(x)} dx)$.

A large group of H^1 symmetries leaves the equation invariant: if $u(t, x)$ solves (69), then $\forall (\lambda_0, \tau_0, x_0, \gamma_0) \in \mathbb{R}_*^+ \times \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}$, so does

$$u(t, x) = \lambda_0 v(\lambda_0^2 t + \tau_0, \lambda_0 x + x_0) e^{i\gamma_0}. \quad (70)$$

Note that (69) is not invariant under the usual Galilean transform. However, it is invariant under a twisted Galilean transform. For $\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3$, we define

$$\hat{\beta} = (\beta_1, \beta_2, -\beta_3) \in \mathbb{R}^3.$$

Then (69) is invariant under the following twisted Galilean transform,

$$v_\beta(t, x) = v(t, x - \beta t) e^{i\hat{\beta}/2 \cdot (x - (\beta/2)t)}. \quad (71)$$

Note also that the scaling symmetry $u(t, x) = \lambda_0 v(\lambda_0^2 t, \lambda_0 x)$ leaves the space $L^2(\mathbb{R}^2)$ invariant so that (69) is critical with respect to the conservation of mass in dimension 2, while it leaves the homogeneous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$ invariant so that (69) is supercritical with respect to the conservation of mass in dimension 3.

In contrast to the cubic focusing NLS, the existence or absence of finite-time blow-up solutions for (69) is widely open. While there exists a counterpart to the virial for (69), it is too weak to provide the existence of finite-time blow up dynamics (see the discussion in [3]). Also, there are no standing waves in the form $v(t, x) = Q(x) e^{i\omega t}$, $\omega \in \mathbb{R}$ in the energy space for the Eq. (69) (see [38]), while for the focusing cubic NLS, such an object exists and is fundamental in the analysis of the blow-up dynamics (see, e.g., [4–9] in dimension 2, and [10] in dimension 3). Now, numerical simulations suggest that the solutions to (69) do not break down in finite time (see, e.g., [3, 29] and the references therein). In particular, the simulations exhibit a phenomenon called pulse splitting where the pulse focuses until it reaches a certain threshold. Once this threshold is attained, the pulse splits into two pulses of less amplitude moving away from each other. This phenomenon might repeat itself (multiple splitting) and ultimately prevent finite-time blow-up, but this has not been clearly backed up by numerics so far. Thus an interesting problem would be to prove that the cubic NLS equation (69) (i.e., in the case of normal GVD) does not have finite-time blow-up solutions, whether this is due to pulse splitting, or to some other phenomenon. This is formulated in the two open problems below.

Open Problem 6. Prove that the solutions to (69) are global in dimensions 2 and 3.

Open Problem 7. Describe rigorously the phenomenon of pulse splitting for (69).

Remark 14. We only considered in this section the case where

$$P_2 = I, \quad \alpha_2 = 0, \quad \alpha_3 = 0 \quad \text{and} \quad f = 0$$

which is (69). While it is of interest to investigate the qualitative role of the parameters P_2, α_2, α_3 and f of (3) in the case of normal GVD (i.e., the case

$\alpha_1 = -1$), we chose not to investigate their effects since we are interested in this paper primary on the non-existence of focusing dynamics, and (69) is expected to have only global solutions.

Remark 15. In a recent preprint [39], standing waves and self-similar solutions have been exhibited for (69) in dimension 2. However, these solutions do not belong to the energy space. In particular, they are not in C^2 and do not decay along the two diagonals in \mathbb{R}^2 .

4.3 Mixing Several Phenomena

In Sects. 4.1 and 4.2, we have investigated the effect of each of the parameters $P_2, \alpha_1, \alpha_2, \alpha_3$, and f on the solutions to Eq. (3), and in particular whether these modifications of the cubic NLS (1) prevent the existence of finite-time blow-up solutions. We have also formulated several open problems. Now, instead of studying all these effects separately, one may investigate the case when all these phenomena are present at the same time, and ask which are the dominant ones? Consider, for instance, the case where there is partial off-axis variation of the group velocity in the direction z , and at the same time a potential self-steepening of the pulse in the same direction, i.e., $P_2 = 1 - \partial_z^2$ and $\alpha_3 = e_z$,

$$i(1 - \varepsilon^2 \partial_z^2) \partial_t v + \Delta v + (1 + i\varepsilon \partial_z)(|v|^2 v) = 0. \quad (72)$$

Then, one may wonder whether the partial off-axis variation of the group velocity prevents the self-steepening from taking place. This is formulated in the open problem below.

Open Problem 8. Do the solutions to (72) exhibit the phenomenon of self-steepening?

4.4 The Vectorial Case

Recall that (3) is a particular case of the vectorial equation (3)_{vect},

$$iP_2(\varepsilon \nabla) \partial_t \mathbf{v} + (\Delta_{\perp} + \alpha_1 \partial_z^2) \mathbf{v} + i\alpha_2 \mathbf{v} + \frac{1}{3}(1 + i\varepsilon \alpha_3 \cdot \nabla)[(\mathbf{v} \cdot \mathbf{v}) \bar{\mathbf{v}} + 2|\mathbf{v}|^2 \mathbf{v}] = 0,$$

where \mathbf{v} is now a \mathbb{C}^2 -valued function, and where we consider for simplicity the equation corresponding to the cubic case (i.e., $f = 0$ in (3)). In fact, (3) is a particular case of (3)_{vect} corresponding to initial data living on a one-dimensional subspace of \mathbb{C}^2 (see Remark. 2). Now, one may of course consider the previous questions formulated for (3), and investigate the same problems for the vectorial case. This is formulated in the open problem below.

Open Problem 9. Investigate the vectorial counterparts for (3)_{vect} of the various open problems formulated for the scalar equation (3) in Sects. 4.1–4.3, and 4.5.

4.5 *The Approximation of the Maxwell Equations over Longer Times*

We have provided in Sect. 3 a rigorous justification for all the NLS-type models derived in this paper. More precisely, we have proved that there exists $T > 0$ such that

1. The exact solution \mathbf{U} of (20)–(22) exists on $[0, T/\varepsilon]$.
2. The NLS-type model under consideration (e.g., (39)) admits a unique solution u_{app} with initial condition u^0 on the same time interval.
3. The approximation $\mathbf{U}_{\text{app}}(t, x) = u_{\text{app}}(t, x)e^{i(\mathbf{k}\cdot x - \omega t)/\varepsilon} + \text{c.c.}$ remains close to \mathbf{U} on this time interval,¹⁰

$$|\mathbf{U} - \mathbf{U}_{\text{app}}|_{L^\infty([0, T/\varepsilon] \times \mathbb{R}^d)} \leq \varepsilon C(T, |u_{\text{app}}|_{L^\infty([0, T/\varepsilon]; B)})(1 + |\nabla u^0|_B + |u^0|_{B^{(3)}}).$$

Such a justification is far from being sharp. In particular, for the standard cubic focusing NLS equation, one has $T < T_{\text{foc}}$, where T_{foc} is the time when focusing occurs. Indeed, the above justification process requires that u_{app} remains bounded on $[0, T/\varepsilon]$ (after rescaling, this is equivalent to require that the solution v to (3) is bounded on $[0, T]$). In the error estimate above, the constant $C(T, |u_{\text{app}}|_{L^\infty([0, T/\varepsilon]; B)})$ therefore blows up as $T \rightarrow T_{\text{foc}}$.

It follows that all the NLS-type equations derived here are justified far enough from the focusing time. Now, relevant differences between the different models can only be observed close enough to focusing (they all differ by formally $O(\varepsilon^2)$ terms that become relevant only near the focusing point). So, roughly speaking, we have only proved that *all the models are justified on a time interval for which they are basically identical*.

As seen above, existence beyond the focusing time of the standard cubic NLS equation is proved or expected for many of the variants considered here. For such models, the above argument does not work, i.e., the constant $C(T, |u_{\text{app}}|_{L^\infty([0, T/\varepsilon]; B)})$ of the error estimate does not blow up as $T \rightarrow T_{\text{foc}}$. This is, for instance, the case of the cubic/quintic NLS equation (52) that admits a global solution v^ε . However (see Open Problem 2), we have

$$\lim_{\varepsilon \rightarrow 0} \lim_{T \rightarrow T_{\text{foc}}} |v^\varepsilon(t, \cdot)|_{H^1} = +\infty,$$

¹⁰In the error estimates of Sect. 3, the constant on the right-hand side is $C(T, |u^0|_B)$ rather than $C(T, |u_{\text{app}}|_{L^\infty([0, T/\varepsilon]; B)})$. Since $|u_{\text{app}}|_{L^\infty([0, T/\varepsilon]; B)} = C(T, |u^0|_B)$, this is of course equivalent, but the first form is more convenient for the present discussion.

so that there is no reason to expect the error term $\varepsilon C(T, |v^\varepsilon|_{L^\infty([0, T/\varepsilon]; B)})$ to be small near the focusing point and for small ε .

For the moment, the merits of the NLS-type models derived here to describe correctly the mechanisms at stake during focusing can only be assessed numerically. Hence the following interesting open problem,

Open Problem 10. Rigorously justify one of the NLS-type models derived here on a time scale $[0, T/\varepsilon]$ with $T \geq T_{\text{foc}}$.

Acknowledgements David Lannes acknowledges support from the ANR-13-BS01-0003-01 DYFICOLTI, the ANR BOND.

The authors warmly thank Christof Sparber for pointing [28] out.

Appendix 1: Nondimensionalization of the Equations

The Case Without Charge nor Current Density

There are two characteristic times for the situation considered here. The first one, denoted \bar{t} , is the inverse of the frequency of the laser pulse; the second one, denoted \bar{T} , is the duration of the pulse. In the regimes considered here,¹¹ one has $\bar{T} \gg \bar{t}$, and the small parameter ε is defined as

$$\varepsilon = \frac{\bar{t}}{\bar{T}}.$$

The time variable is naturally nondimensionalized with \bar{T} , while the space variables are nondimensionalized by $L = c\bar{T}$, namely,

$$t = \bar{T}\tilde{t}, \quad x = L\tilde{x},$$

where dimensionless quantities are denoted with a tilde. We also nondimensionalize the unknowns E , B , and P with typical orders E_0 , B_0 , and P_0 . The typical scale for the electric and magnetic fields is directly deduced from P_0 ,

$$E_0 = \frac{1}{\epsilon_0} P_0, \quad B_0 = \frac{1}{c} E_0.$$

In order to choose the remaining P_0 , some consideration on the polarization equation is helpful. We recall that (11) is given by

$$\partial_{\tilde{t}}^2 P + \Omega_1 \partial_{\tilde{t}} P + \Omega_0^2 P - \nabla V_{\text{NL}}(P) = \epsilon_0 b E.$$

¹¹We refer to [40] for variants of this nondimensionalization.

For all the applications we have in mind, one has $\nabla V_{\text{NL}}(\mathbb{P}) = a_3 |\mathbb{P}|^2 \mathbb{P} + \text{h.o.t.}$, so that we nondimensionlize the nonlinear term as

$$\nabla_{\mathbb{P}} V_{\text{NL}}(\mathbb{P}) = a_3 \mathbb{P}_0^3 \nabla_{\tilde{\mathbb{P}}} \tilde{V}_{\text{NL}}(\tilde{\mathbb{P}}),$$

where \tilde{V}_{NL} is the dimensionless potential

$$\tilde{V}_{\text{NL}}(\tilde{\mathbb{P}}) = \frac{1}{a_3 \mathbb{P}_0^4} V_{\text{NL}}(\mathbb{P}).$$

Example 10. For the three cases considered in Example 1, we find, denoting $\tilde{a}\varepsilon^r = a_5 \mathbb{P}_0^2 / a_3$, where $r > 0$ is chosen such that $\tilde{a} = O(1)$,

(i) Cubic nonlinearity:

$$\nabla_{\tilde{\mathbb{P}}} \tilde{V}_{\text{NL}}(\tilde{\mathbb{P}}) = |\tilde{\mathbb{P}}|^2 \tilde{\mathbb{P}}.$$

(ii) Cubic/quintic nonlinearity:

$$\nabla_{\tilde{\mathbb{P}}} \tilde{V}_{\text{NL}}(\tilde{\mathbb{P}}) = (1 - \tilde{a}\varepsilon^r |\tilde{\mathbb{P}}|^2) |\tilde{\mathbb{P}}|^2 \tilde{\mathbb{P}}.$$

(iii) Saturated nonlinearity:

$$\nabla_{\tilde{\mathbb{P}}} \tilde{V}_{\text{NL}}(\tilde{\mathbb{P}}) = \frac{1 + (\tilde{a}\varepsilon^r/3) |\tilde{\mathbb{P}}|^2}{(1 + (2\tilde{a}\varepsilon^r/3) |\tilde{\mathbb{P}}|^2)^2} |\tilde{\mathbb{P}}|^2 \tilde{\mathbb{P}}.$$

All these examples can therefore be put under the form

$$\nabla_{\tilde{\mathbb{P}}} \tilde{V}_{\text{NL}}(\tilde{\mathbb{P}}) = (1 + \tilde{f}(\tilde{a}\varepsilon^r |\tilde{\mathbb{P}}|^2)) |\tilde{\mathbb{P}}|^2 \tilde{\mathbb{P}},$$

where $\tilde{f} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a smooth function such that $\tilde{f}(0) = 0$.

In dimensionless form, the polarization equation becomes therefore

$$\partial_{\tilde{t}}^2 \tilde{\mathbb{P}} + \bar{T} \Omega_1 \partial_{\tilde{t}} \tilde{\mathbb{P}} + \bar{T}^2 \Omega_0^2 \tilde{\mathbb{P}} - a_3 \bar{T}^2 \mathbb{P}_0^2 \nabla_{\tilde{\mathbb{P}}} \tilde{V}_{\text{NL}}(\tilde{\mathbb{P}}) = \epsilon_0 b \frac{\bar{T}^2 \mathbb{E}_0}{\mathbb{P}_0} \tilde{\mathbb{E}}.$$

We therefore choose \mathbb{P}_0 to be the typical nonlinear scale for which the nonlinear terms in the above equations are of the same order as the second order time derivative, namely,

$$\mathbb{P}_0 = \frac{1}{\bar{T} \sqrt{a_3}}.$$

Finally, we need some information on the size of Ω_0 , Ω_1 and b to give our final dimensionless form of the equations. The resonance frequency of the harmonic oscillator is typically of the same order as the frequency of the laser pulse,

$$\Omega_0 = \bar{t}^{-1} \omega_0;$$

the damping frequency is written under the form

$$\Omega_1 = \varepsilon^{\alpha+1} \bar{t}^{-1} \omega_1,$$

where the dimensionless damping frequency ω_1 has size $O(1)$ while the coefficient α depends on the medium in which the laser propagates. Typically, $\alpha = 1$ or $\alpha = 2$. Writing the coupling constant b as

$$b = \frac{\gamma}{\bar{t}^2},$$

the dimensionless polarization equations can therefore be written as

$$\partial_{\tilde{t}}^2 \tilde{\mathbf{P}} + \varepsilon^\alpha \omega_1 \partial_{\tilde{t}} \tilde{\mathbf{P}} + \varepsilon^{-2} \omega_0^2 \tilde{\mathbf{P}} - \nabla_{\tilde{\mathbf{P}}} \tilde{V}_{\text{NL}}(\tilde{\mathbf{P}}) = \varepsilon^{-2} \gamma \tilde{\mathbf{E}}.$$

Omitting the tildes for the sake of clarity, Maxwell's equations (10)–(11) read therefore (without charge nor current density),

$$\begin{cases} \partial_t \mathbf{B} + \text{curl } \mathbf{E} = 0, \\ \partial_t \mathbf{E} - \text{curl } \mathbf{B} = -\partial_t \mathbf{P}, \\ \partial_t^2 \mathbf{P} + \varepsilon^\alpha \omega_1 \partial_t \mathbf{P} + \varepsilon^{-2} \omega_0^2 \mathbf{P} - \nabla V_{\text{NL}}(\mathbf{P}) = \varepsilon^{-2} \gamma \mathbf{E}. \end{cases}$$

Introducing the auxiliary unknowns,

$$\mathbf{P}^\sharp = \frac{\omega_0}{\sqrt{\gamma}} \mathbf{P}, \quad \mathbf{Q}^\sharp = \frac{\varepsilon}{\omega_0} \partial_t \mathbf{P}^\sharp = \frac{\varepsilon}{\sqrt{\gamma}} \partial_t \mathbf{P},$$

and working with nonlinearities as those considered in Example 10, this system can be written as a first order system,

$$\begin{cases} \partial_t \mathbf{B} + \text{curl } \mathbf{E} = 0, \\ \partial_t \mathbf{E} - \text{curl } \mathbf{B} + \frac{1}{\varepsilon} \sqrt{\gamma} \mathbf{Q}^\sharp = 0, \\ \partial_t \mathbf{Q}^\sharp + \varepsilon^\alpha \omega_1 \mathbf{Q}^\sharp - \frac{1}{\varepsilon} (\sqrt{\gamma} \mathbf{E} - \omega_0 \mathbf{P}^\sharp) = \varepsilon \frac{\gamma}{\omega_0^3} (1 + f(\varepsilon' |\mathbf{P}^\sharp|^2)) |\mathbf{P}^\sharp|^2 \mathbf{P}^\sharp, \\ \partial_t \mathbf{P}^\sharp - \frac{1}{\varepsilon} \omega_0 \mathbf{Q}^\sharp = 0, \end{cases} \quad (73)$$

where we wrote $f(x) = \tilde{f}(\tilde{a}(\gamma/\omega_0^2)x)$.

This system has the form (20) with $n = 12$, $\mathbf{U} = (\mathbf{B}, \mathbf{E}, \mathbf{Q}^\sharp, \mathbf{P}^\sharp)^T$ and

$$A(\partial) = \begin{pmatrix} 0 & \text{curl} & 0 & 0 \\ -\text{curl} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\gamma}I & 0 \\ 0 & -\sqrt{\gamma}I & 0 & \omega_0 I \\ 0 & 0 & -\omega_0 I & 0 \end{pmatrix}$$

while A_0 is the block diagonal matrix $A_0 = \text{diag}(0, 0, \omega_1 I, 0)$, so that the first two points of Assumption 1 are satisfied. For the third one, remark that the nonlinearity is given by

$$F(\varepsilon, \mathbf{U}) = \left(0, 0, \frac{\gamma}{\omega_0^3} (1 + f(\varepsilon^r |\mathbf{P}^\sharp|^2)) |\mathbf{P}^\sharp|^2 \mathbf{P}^\sharp, 0 \right)^T, \quad (74)$$

which is of the same form as in Assumption 1 with $Q(U) = |\mathbf{P}^\sharp|^2$ and $T(U, \bar{U}, U) = (0, 0, (\gamma/\omega_0^3) |\mathbf{P}^\sharp|^2 \mathbf{P}^\sharp, 0)^T$. One can check that such nonlinearities also satisfy Assumption 3.

The Case with Charge and Current Density

Now, we deal with the nondimensionalization of (10), (11), (17). We proceed in the same way as in the previous paragraph, simply adding the characteristic sizes J_0 and ρ_0 of the free electron current density \mathcal{J}_e and charge density ρ . Considerations on the polarization equation are the same.

We choose ρ_0 from the equation for the charge density,

$$\partial_{\tilde{t}} \tilde{\rho} = \sigma_K \rho_{\text{nt}} \frac{E_0^{2K}}{\rho_0} \bar{T} |\tilde{\mathbf{E}}|^{2K} + \frac{\sigma}{U_i} E_0^2 \bar{T} \tilde{\rho} |\tilde{\mathbf{E}}|^2.$$

Many configurations are possible, according to the numerical value of the various coefficients involved (see, for instance, [18] for experimental data). We choose here a configuration that leads to the richest model, where the coupling constants of the two nonlinear terms are of the same order; more precisely

$$\partial_{\tilde{t}} \tilde{\rho} = \varepsilon c_1 |\tilde{\mathbf{E}}|^{2K} + \varepsilon c_2 \tilde{\rho} |\tilde{\mathbf{E}}|^2,$$

for nonnegative constants c_1, c_2 . This corresponds to the following choice for ρ_0 ,

$$\rho_0 = \frac{\sigma_K \rho_{\text{nt}} E_0^{2K}}{\varepsilon c_1}.$$

Knowing ρ_0 , we can then determine \mathcal{J}_0 from the equation for the free electron current density,

$$\mathcal{J}_e = \frac{q_e^2}{\omega_1 m_e} \rho_0 E_0 \tilde{\rho} \mathcal{H}(\varepsilon D_{\tilde{z}}) \tilde{\mathbf{E}},$$

which naturally leads to

$$\mathcal{J}_0 = \frac{q_e^2}{\omega_1 m_e} \rho_0 E_0 = \varepsilon \frac{q_e^2 \bar{T} \rho_0 E_0}{m_e}.$$

Going back to Maxwell's equations, we get

$$\begin{cases} \partial_{\tilde{t}} \tilde{\mathbf{B}} + \text{curl } \tilde{\mathbf{E}} = 0, \\ \partial_{\tilde{t}} \tilde{\mathbf{E}} - \text{curl } \tilde{\mathbf{B}} = -\partial_{\tilde{t}} \tilde{\mathbf{P}} - \frac{1}{\varepsilon_0} \frac{\bar{T}}{E_0} \mathcal{J}_0 \tilde{\mathbf{J}}_e - \frac{U_i \rho_0}{\varepsilon_0 E_0} (\varepsilon c_1 |\tilde{\mathbf{E}}|^{2K-2} + \varepsilon c_2 \tilde{\rho}) \tilde{\mathbf{E}}. \end{cases}$$

Here again, many configurations can be found, and we choose one that leads to the richest models, namely,

$$\frac{1}{\varepsilon_0} \frac{\bar{T}}{E_0} \mathcal{J}_0 = \varepsilon c_3, \quad \frac{U_i \rho_0}{\varepsilon_0 E_0} = c_0$$

where c_0, c_3 are dimensionless positive constants. Replacing $\tilde{\mathbf{J}}_e$ by $c_3 \tilde{\mathbf{J}}_e$, $\tilde{\rho}$ by $c_3 \tilde{\rho}$ and c_1 by $c_1 c_3$, we can assume that $c_3 = 1$.

Finally, dropping the tildes, introducing the unknowns Q^\sharp and P^\sharp , and using the same notations as in (73), we obtain

$$\begin{cases} \partial_t B + \text{curl } E = 0, \\ \partial_t E - \text{curl } B + \frac{1}{\varepsilon} \sqrt{\gamma} Q^\sharp = -\varepsilon \rho \mathcal{H}(\varepsilon D_z) E - \varepsilon c_0 (c_1 |E|^{2K-2} + c_2 \rho) E, \\ \partial_t Q^\sharp + \varepsilon^\alpha \omega_1 Q^\sharp - \frac{1}{\varepsilon} (\sqrt{\gamma} E - \omega_0 P^\sharp) = \varepsilon \frac{\gamma}{\omega_0^3} (1 + f(\varepsilon' |P^\sharp|^2)) |P^\sharp|^2 P^\sharp, \\ \partial_t P^\sharp - \frac{1}{\varepsilon} \omega_0 Q^\sharp = 0, \\ \partial_t \rho = \varepsilon c_1 |E|^{2K} + \varepsilon c_2 \rho |E|^2. \end{cases} \quad (75)$$

Appendix 2: Explicit Computations for Maxwell's Equations

The Case Without Charge nor Current Density

We derive here the variants of the NLS equations derived in this paper in the particular case of the Maxwell equations (73). We derive all the versions of the NLS equations that do not take into account the frequency dependence of the polarization (this corresponds to (37)). This latter effect, which leads to the generalization (41) of the previous model, is examined separately. For the sake of simplicity, we only consider cubic nonlinearities here; we show that in this context, (41) reduces to the vectorial family of NLS equation (3)_{vect}.

Without Frequency Dependent Polarization

In order to check that Assumption 4 is satisfied by the dimensionless Maxwell equations (73), we need to compute the eigenvalues and eigenvectors of the matrix

$$\mathcal{L}(0, \mathbf{k}) = A(\mathbf{k}) + \frac{1}{i}E,$$

where $A(\mathbf{k})$ and E are given in section “The Case with Charge and Current Density” in Appendix 1, which is equivalent to find all nontrivial solutions to the equation

$$\mathcal{L}(\omega, \mathbf{k})u = 0, \quad \text{with } \mathcal{L}(\omega, \mathbf{k}) = -\omega I + A(\mathbf{k}) + \frac{1}{i}E,$$

and $\omega \in \mathbb{R}$, $u = (\mathbf{b}, e, \mathbf{q}^\sharp, \mathbf{p}^\sharp) \in \mathbb{C}^{12}$.

From the last two equations of (73), we can always write \mathbf{p}^\sharp and \mathbf{q}^\sharp in terms of e ,

$$\mathbf{p}^\sharp = -\frac{\omega_0 \sqrt{\gamma}}{\omega^2 - \omega_0^2} e, \quad \mathbf{q}^\sharp = i \frac{\omega \sqrt{\gamma}}{\omega^2 - \omega_0^2} e. \quad (76)$$

Let us now consider several cases:

- If $e \parallel \mathbf{k}$, then by the first equation, either $\omega = 0$ and $\mathbf{b} \parallel \mathbf{k}$ or $\mathbf{b} = 0$ and $\omega^2 = \omega_0^2 + \gamma$.
- If $e \not\parallel \mathbf{k}$, then $\omega \neq 0$ and combining the first two equations, we get

$$\left[\omega^2 \frac{\omega^2 - (\omega_0^2 + \gamma)}{\omega^2 - \omega_0^2} - k^2 \right] e = -(\mathbf{k} \cdot e) \mathbf{k},$$

where we denoted as usual by k the modulus of \mathbf{k} , $k = |\mathbf{k}|$; this in turns implies (since \mathbf{k} is not parallel to e) that $\mathbf{k} \perp e$ and

$$\omega^4 - \omega^2((\omega_0^2 + \gamma) + k^2) + \omega_0^2 k^2 = 0;$$

denoting $\Delta = ((\omega_0^2 + \gamma) + k^2)^2 - 4\omega_0^2 k^2 > 0$, this equation admits four real solutions

$$\omega_{\pm,\pm}(k) = \pm \frac{1}{\sqrt{2}} \sqrt{\omega_0^2 + \gamma + k^2 \pm \sqrt{\Delta}}.$$

There are $m = 7$ different branches of $\mathcal{C}_{\mathcal{X}}$: three constant sheets given by

$$\omega = 0 \quad \text{and} \quad \omega = \pm \sqrt{\gamma + \omega_0^2}$$

(the first one being of multiplicity two, the two other ones of multiplicity one), and four curved sheets (each one being of multiplicity two) given by $\omega = \omega_{\pm,\pm}$. We denote by $\pi_{\pm,\pm}$ the associated eigenprojectors (which are therefore of rank 2). Remarking that $\pi_{\pm,\pm}(\mathbf{k})u = u$ if and only if

$$\mathbf{e} \cdot \mathbf{k} = 0, \quad \mathbf{b} = -\frac{1}{\omega} \mathbf{k} \wedge \mathbf{e},$$

and \mathfrak{p}^\sharp and \mathfrak{q}^\sharp are given in terms of \mathbf{e} by (76), one readily finds that $\pi_{\pm,\pm}(\mathbf{k})$ can be written as a block matrix

$$\pi_{\pm,\pm}(\mathbf{k}) = \frac{1}{N(k)^2} (P_{ij})_{1 \leq i,j \leq 4} \quad \text{with} \quad N(k)^2 = \frac{k^2}{\omega(k)^2} + 1 + \gamma \frac{\omega(k)^2 + \omega_0^2}{(\omega(k)^2 - \omega_0^2)^2}$$

(for the sake of clarity, we write $\omega(k) = \omega_{\pm,\pm}(k)$), and where the 3×3 blocks P_{1j} are given by

$$\begin{aligned} P_{11} &= \frac{k^2}{\omega(k)} \Pi_{\mathbf{k}^\perp}, & P_{12} &= -\frac{1}{\omega(k)} \mathbf{k} \wedge, \\ P_{13} &= i \frac{\sqrt{\gamma}}{\omega(k)^2 - \omega_0^2} \mathbf{k} \wedge, & P_{14} &= \frac{\omega_0}{\omega(k)} \frac{\sqrt{\gamma}}{\omega(k)^2 - \omega_0^2} \mathbf{k} \wedge, \\ P_{22} &= \Pi_{\mathbf{k}^\perp}, & P_{23} &= -i \frac{\omega(k) \sqrt{\gamma}}{\omega(k)^2 - \omega_0^2} \Pi_{\mathbf{k}^\perp}, \\ P_{24} &= -\frac{\omega_0 \sqrt{\gamma}}{\omega(k)^2 - \omega_0^2} \Pi_{\mathbf{k}^\perp}, & P_{33} &= \frac{\omega(k)^2 \gamma}{(\omega(k)^2 - \omega_0^2)^2} \Pi_{\mathbf{k}^\perp}, \\ P_{34} &= -i \frac{\omega(k) \omega_0 \gamma}{(\omega(k)^2 - \omega_0^2)^2} \Pi_{\mathbf{k}^\perp}, & P_{44} &= \frac{\omega_0^2 \gamma}{(\omega(k)^2 - \omega_0^2)^2} \Pi_{\mathbf{k}^\perp}, \end{aligned}$$

where $\Pi_{\mathbf{k}^\perp}$ is the orthogonal projector onto the orthogonal plane to \mathbf{k} ; the remaining blocks stem from the symmetry relations $P_{ij} = P_{ji}^*$.

It follows from these computations that in the case of a cubic nonlinearity¹² (i.e., $f = 0$ in (74)), $F^{\text{env}}(\varepsilon, u)$ is given for all u such that $u = \pi_{\pm, \pm}(\mathbf{k})u$ by

$$F^{\text{env}}(\varepsilon, u) = \left(0, 0, -\frac{\gamma^{5/2}}{(\omega^2 - \omega_0^2)^3}[(\mathbf{e} \cdot \mathbf{e})\bar{\mathbf{e}} + 2|\mathbf{e}|^2\mathbf{e}], 0 \right)^T;$$

similarly,

$$\pi_{\pm, \pm}(\mathbf{k})F^{\text{env}}(\varepsilon, u) = \left(*, \frac{i}{N(k)^2} \frac{\gamma^3 \omega}{(\omega^2 - \omega_0^2)^4}[(\mathbf{e} \cdot \mathbf{e})\bar{\mathbf{e}} + 2|\mathbf{e}|^2\mathbf{e}], *, * \right)^T.$$

and

$$\pi_{\pm, \pm}A_0u = \left(*, \omega_1 \frac{\gamma \omega^2}{(\omega^2 - \omega_0^2)^2} \mathbf{e}, *, * \right)^T.$$

We can now write explicitly the general NLS equation (37) derived in Example 7 in the particular case of the Maxwell equations (73). Choosing $\omega_1(\cdot) = \omega_{\pm, \pm}(\cdot)$ and therefore $\pi_1(\cdot) = \pi_{\pm, \pm}(\cdot) = \omega(\cdot)$, and remarking that the solution v to (37) remains polarized along $\pi_{\pm, \pm}(\mathbf{k})$ if it is initially polarized, it is enough to give an equation on its electric field component \mathbf{e} (the components \mathbf{b} , \mathbf{p}^\sharp , and \mathbf{q}^\sharp being recovered as indicated above). Moreover, we can assume the $\mathbf{k} = k\mathbf{e}_z$, so that the electric field \mathbf{e} only has transverse components $\boldsymbol{\varepsilon} \in \mathbb{C}^2$ (i.e., $\mathbf{e} = (\boldsymbol{\varepsilon}^T, 0)^T$) and (37) finally reduces to

$$\begin{aligned} & (1 - i\varepsilon\mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot B \nabla) \partial_t \boldsymbol{\varepsilon} - \frac{i}{2} \left(\frac{\omega'(k)}{k} \Delta_\perp + \omega''(k) \partial_z^2 \right) \boldsymbol{\varepsilon} \\ & + \varepsilon^p \omega_1 \frac{\gamma \omega(k)^2}{(\omega(k)^2 - \omega_0^2)^2} \boldsymbol{\varepsilon} = \frac{i}{N(k)^2} \frac{\gamma^3 \omega(k)}{(\omega(k)^2 - \omega_0^2)^4} [(\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon})\bar{\boldsymbol{\varepsilon}} + 2|\boldsymbol{\varepsilon}|^2 \boldsymbol{\varepsilon}]. \end{aligned} \quad (77)$$

Proceeding to the following rescaling of the space variables

$$(x, y) = \left(\frac{\omega'(k)}{2k} \right)^{1/2} (x', y') \quad \text{and} \quad z = |\frac{1}{2}\omega''(k)|^{1/2} z'$$

(the latter one only if $\omega''(k) \neq 0$), and rescaling $\boldsymbol{\varepsilon}$ as

$$\boldsymbol{\varepsilon} = \left(\frac{1}{N(k)^2} \frac{\gamma^3 \omega(k)}{3(\omega(k)^2 - \omega_0^2)^4} \right)^{1/2} \boldsymbol{\varepsilon}',$$

¹²The treatment of other kinds of nonlinearities is absolutely similar.

the above equation becomes

$$iP_2(\varepsilon\nabla)\partial_t\mathbf{e} + (\Delta_\perp + \alpha_1\partial_z^2)\mathbf{e} + i\alpha_2\mathbf{e} + \frac{1}{3}[(\mathbf{e} \cdot \mathbf{e})\bar{\mathbf{e}} + 2|\mathbf{e}|^2\mathbf{e}] = 0, \quad (78)$$

where $\alpha_1 = \text{sgn}(\omega''_{\pm,\pm}(k)) \in \{0, \pm 1\}$, $\alpha_2 = \varepsilon^p \omega_1 \gamma \omega(k)^2 / (\omega(k)^2 - \omega_0^2)^2$ and

$$P_2(\varepsilon\nabla) = 1 - i\varepsilon M \mathbf{b} \cdot \nabla - \varepsilon^2 \nabla \cdot M B M \nabla,$$

where $M = \text{diag}\left((\omega'(k)/(2k))^{-1/2}, (\omega'(k)/(2k))^{-1/2}, |\frac{1}{2}\omega''(k)|^{-1/2}\right)$. This equation is exactly in the form (3)_{vect} with $\alpha_3 = 0$ (i.e., without the terms modeling the frequency dependence of the polarization).

With Frequency Dependent Polarization

Let us now describe the modifications that have to be made in order to take into account a frequency dependent polarization, as in (41). Since initial data polarized along $\pi_{\pm,\pm}(\mathbf{k})$ conserve this polarization during the evolution in time, the only difference between (41) and (37) is the presence of the terms $\varepsilon\pi_{\pm,\pm}(\mathbf{k})\pi'_{\pm,\pm}(\mathbf{k}) \cdot DF^{\text{env}}(\varepsilon, u)$ and $-i\varepsilon(\mathbf{b} \cdot \nabla)\pi_{\pm,\pm}(\mathbf{k})F^{\text{env}}(\varepsilon, u)$. Only the first one requires a nontrivial computation. Thanks to the expression for $\pi_{\pm,\pm}(\mathbf{k})$ given above, we can write for all $F = (0, 0, f, 0)$,

$$\pi_{\pm,\pm}(\mathbf{k})F = (ig_1(k)\mathbf{k} \wedge f, -ig_2(k)\Pi_{\mathbf{k}\perp}f, g_3(k)\Pi_{\mathbf{k}\perp}f, ig_4(k)\Pi_{\mathbf{k}\perp}f)^T,$$

with

$$\begin{aligned} g_1(k) &= \frac{1}{N(k)^2} \frac{\sqrt{\gamma}}{\omega(k)^2 - \omega_0^2}, & g_2(k) &= \omega(k)g_1(k), \\ g_3(k) &= \frac{\omega(k)\sqrt{\gamma}}{\omega(k)^2 - \omega_0^2}g_2(k), & g_4(k) &= \frac{\omega_0\sqrt{\gamma}}{\omega(k)^2 - \omega_0^2}g_2(k). \end{aligned}$$

One has therefore, with $h = \frac{\mathbf{k}}{k} \cdot Df$ and $l = d_{\mathbf{k}}(\mathbf{k}' \mapsto \Pi_{\mathbf{k}'\perp}) \cdot Df$,

$$\begin{aligned} \pi'_{\pm,\pm}(\mathbf{k}) \cdot DF &= (ig'_1(k)\mathbf{k} \wedge h, -ig'_2(k)\Pi_{\mathbf{k}\perp}h, g'_3(k)\Pi_{\mathbf{k}\perp}h, ig'_4(k)\Pi_{\mathbf{k}\perp}h)^T \\ &\quad + (ig_1(k)D \wedge f, -ig_2(k)l, g_3(k)l, ig_4(k)l)^T. \end{aligned}$$

If $f \cdot \mathbf{k} = 0$, we can use the identity $\Pi_{\mathbf{k}\perp}d_{\mathbf{k}}(\mathbf{k}' \mapsto \Pi_{\mathbf{k}'\perp}) \cdot D\Pi_{\mathbf{k}\perp} = 0$ to deduce that

$$\pi_{\pm,\pm}(\mathbf{k})\pi'_{\pm,\pm}(\mathbf{k}) \cdot DF = \left(*, -\frac{m(k)}{N(k)^2} \frac{\mathbf{k}}{k} \cdot \nabla f, *, * \right)^T,$$

where the $*$ can be easily computed, and where

$$m(k) = \frac{k}{\omega(k)}(g_1(k) + kg'_1(k)) + g'_2(k) + \frac{\sqrt{\gamma}}{\omega(k)^2 - \omega_0^2}(\omega(k)g'_3(k) - \omega_0g'_4(k)).$$

The nonlinearity in the right-hand side of (77) must therefore be replaced by

$$\frac{i}{N(k)^2} \frac{\gamma^3 \omega(k)}{(\omega(k)^2 - \omega_0^2)^4} (1 - i\varepsilon \mathbf{b} \cdot \nabla + i\alpha_3 \partial_z)[(\mathbf{e} \cdot \mathbf{e})\bar{\mathbf{e}} + 2|\mathbf{e}|^2 \mathbf{e}],$$

with $\alpha_3 = -(\omega^2 - \omega_0^2)/(\sqrt{\gamma}\omega(k))$. The reduction to (3)_{vect} then follows the same steps as for (78).

The Case with Charge and Current Density

According to (48) and more generally (49), ionizations effects are taken into account by adding

$$I := -\varepsilon\pi_1(\mathbf{k})(i\omega C_1^T C_1 u + c C_1^T G^{\text{env}}(C_1 u, w))$$

to the right-hand side of the NLS equation, and by considering the following equation for w ,

$$\partial_t w = 2\varepsilon G^{\text{env}}(C_1 u, w) \cdot \overline{C_1 u}.$$

In the particular case of the Maxwell equations (19), one has $w = \rho$ and

$$I = -\varepsilon\pi_1(\mathbf{k})(0, i\rho\mathbf{e} + c_0(c_1|\mathbf{e}|^{2K-2} + c_2\rho)\mathbf{e}, 0, 0)^T;$$

consequently, one must add to the right-hand side of (77) the ionization term

$$-i\rho\mathbf{e} - c_0(c_1|\mathbf{e}|^{2K-2} + c_2\rho)\mathbf{e},$$

so that after rescaling as for (78), the equation takes the form (54) (when written in a fixed frame rather than the frame moving at the group velocity).

References

1. J. Ginibre, G. Velo, J. Funct. Anal. **32**(1), 1 (1979)
2. V.E. Zakharov, A.B. Shabat, Sov. Phys. JETP **34**(1), 62 (1972)

3. C. Sulem, P.L. Sulem, *The Nonlinear Schrödinger Equation: Self-Focusing and Wave Collapse. Applied Mathematical Sciences*, vol. 139 (Springer, New York, 1999)
4. F. Merle, P. Raphaël, *Ann. Math.* **161**(1), 157 (2005)
5. F. Merle, P. Raphaël, *Geom. Funct. Anal.* **13**(3), 591 (2003)
6. F. Merle, P. Raphaël, *Invent. Math.* **156**(3), 565 (2004)
7. F. Merle, P. Raphaël, *J. Am. Math. Soc.* **19**(1), 37 (2006)
8. F. Merle, P. Raphaël, *Commun. Math. Phys.* **253**(3), 675 (2005)
9. F. Merle, P. Raphaël, *J. Hyperbol. Differ. Eq.* **2**(4), 919 (2005)
10. F. Merle, P. Raphaël, J. Szeftel, *Geom. Funct. Anal.* **20**(4), 1028 (2010)
11. J.E. Rothenberg, *Opt. Lett.* **17**(19), 1340 (1992)
12. H.A. Lorentz, *Theory of Electrons* (Teubner, New York, 1909). Reprint (Dover, Englewood Cliffs, NJ, 1952)
13. N. Bloembergen, *Nonlinear Optics. Frontiers in Physics*, vol. 21 (W. A. Benjamin, Reading, MA, 1977)
14. A. Owyong, The origins of the nonlinear refractive indices of liquids and gases. Ph.D. thesis, California Institute of Technology (1971)
15. P. Donnat, J.L. Joly, G. Métivier, J. Rauch, *Équations aux dérivées partielles 1995–1996* (École Polytechnique, Palaiseau, 1996). Exp. No. XVII
16. J.L. Joly, G. Métivier, J. Rauch, *Indiana U. Math. J.* **47**(4), 1167 (1998)
17. D. Lannes, *Asymptot. Anal.* **18**(1–2), 111 (1998)
18. L. Bergé, S. Skupin, *Discrete Contin. Dyn. Syst.* **23**(4), 1099 (2009)
19. L. Bergé, S. Skupin, R. Nuter, J. Kasparian, J.P. Wolf, *Rep. Prog. Phys.* **70**(10), 1633 (2007)
20. M. Colin, D. Lannes, *SIAM J. Math. Anal.* **41**(2), 708 (2009)
21. D. Lannes, *Proc. R. Soc. Edinb. A Math.* **141**(2), 253 (2011)
22. L.A. Kalyakin, *Math. USSR Sb.* **60**(2), 457 (1988)
23. G. Schneider, *Nonlinear Differ. Equ. Appl.* **5**(1), 69 (1998)
24. T. Colin, G. Gallice, K. Laurieux, *SIAM J. Math. Anal.* **36**(5), 1664 (2005)
25. M. Kolesik, J.V. Moloney, *Phys. Rev. E* **70**(3), 036604 (2004)
26. D. Lannes, *The Water Waves Problem: Mathematical Analysis and Asymptotics. Mathematical Surveys and Monographs*, vol. 188 (American Mathematical Society, Providence, RI, 2013)
27. T. Cazenave, F.B. Weissler, *Nonlinear Anal.* **14**(10), 807 (1990)
28. T. Tao, M. Visan, X. Zhang, *Commun. Partial Differ. Equ.* **32**(8), 1281 (2007)
29. G. Fibich, G. Papanicolaou, *SIAM J. Appl. Math.* **60**(1), 183 (1999)
30. F. Merle, *C. R. Acad. Sci. Paris Sér. I Math.* **304**(16), 479 (1987)
31. F. Merle, *Commun. Math. Phys.* **149**(2), 377 (1992)
32. P. Antonelli, R. Carles, C. Sparber, *Int. Math. Res. Not.* **2015**(3), 740 (2015)
33. M. Tsutsumi, *SIAM J. Math. Anal.* **15**(2), 357 (1984)
34. G. Fibich, *SIAM J. Appl. Math.* **61**(5), 1680 (2001)
35. M. Ohta, G. Todorova, *Discrete Contin. Dyn. Syst.* **23**(4), 1313 (2009)
36. D. Anderson, M. Lisak, *Phys. Rev. A* **27**(3), 1393 (1983)
37. R.S. Strichartz, *Duke Math. J.* **44**(3), 705 (1977)
38. J.M. Ghidaglia, J.C. Saut, *J. Nonlinear Sci.* **6**(2), 139 (1996)
39. P. Kevrekidis, A.R. Nahmod, C. Zeng, *Nonlinearity* **24**(5), 1523 (2011)
40. P. Donnat, Quelques contributions mathématiques en optique non linéaire. Ph.D. thesis, École Polytechnique, Paris (1994)