Well-Posedness and Shallow-Water Stability for a New Hamiltonian Formulation of the Water Waves Equations with Vorticity

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ABSTRACT. In this paper, we derive a new formulation of the water waves equations with vorticity that generalizes the well-known Zakharov-Craig-Sulem formulation used in the irrotational case. We prove the local well-posedness of this formulation, and show that it is formally Hamiltonian. This new formulation is cast in Eulerian variables, and in finite depth; we show that it can be used to provide uniform bounds on the lifespan and on the norms of the solutions in the singular shallow-water regime. As an application to these results, we derive and provide the first rigorous justification of a shallow-water model for water waves in the presence of vorticity; we show in particular that a third equation must be added to the standard model to recover the velocity at the surface from the averaged velocity. The estimates of the present paper also justify the formal computations of [15], where higher-order shallow-water models with vorticity (of Green-Naghdi type) are derived.

1. INTRODUCTION

1.1. General setting. The equations governing the motion of the surface of a homogeneous, inviscid fluid of density ρ under the influence of gravity (assumed to be constant and vertical, $\mathbf{g} = -g\mathbf{e}_z$, $g > 0$) are known as the water waves equations, or free surface Euler equations. In the case where the surface of the fluid is delimited above by the graph of a function $\zeta(t, X)$ over its rest state $z = 0$ (with t the time variable, $X \in \mathbb{R}^d$ the horizontal space variables, and z the vertical variable), and below by a flat bottom $z = -H_0$, and if we denote by $\mathbf{U}$ and P the

Indiana University Mathematics Journal ©, Vol. 64, No. 4 (2015)

velocity and pressure fields, these equations can be written as

$$\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} = -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z, \tag{1.1}$$

$$\nabla_{X,z} \cdot \mathbf{U} = 0 \tag{1.2}$$

in the fluid domain $\Omega_t = \{(X,z) \in \mathbb{R}^{d+1},\ -H_0 < z < \zeta(t,X)\}$; they are complemented with the boundary conditions

$$\partial_t \zeta - \mathbf{U}_{|_{\mathrm{surf}}} \cdot N = 0 \quad (\text{with } N = (-\nabla\zeta^T, 1)^T), \tag{1.3}$$

$$P_{|_{\mathrm{surf}}} = \text{constant} \tag{1.4}$$

at the surface, and

$$\mathbf{U}_{|_{\mathrm{bott}}} \cdot N_b = 0 \quad (\text{with } N_b = \mathbf{e}_z) \tag{1.5}$$

at the bottom. In many physical situations, the motion of the fluid is, in addition, irrotational, and another equation can be added to (1.1)–(1.5), namely

$$\operatorname{curl} \mathbf{U} = 0 \quad \text{in } \Omega.$$

This additional assumption yields considerable simplifications, since all the relevant information to describe the fluid motion is then concentrated on the interface. This can be exploited in many ways (see, for instance, [9, 19, 20, 45, 52, 55, 56] for local well-posedness results, [57, 58] for global well-posedness, and [12, 13, 19, 21] for the existence of turning waves and splash singularities). Let us describe briefly here the approach initiated by Zakharov [59] and Craig-Sulem [24], which is one of the most seminal, and the starting point for the present paper.

From the irrotationality assumption, one can infer the existence of a velocity potential Φ such that $\mathbf{U} = \nabla_{X,z}\Phi$; the incompressibility conditions imply that Φ must be harmonic, and the bottom boundary condition implies that its normal derivative vanishes at the bottom. It follows that Φ is fully determined by its trace at the surface $\Phi_{|_{\mathrm{surf}}} = \psi$. Zakharov noticed that the irrotational equation could be reduced to the Hamiltonian equation

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = J \operatorname{grad}_{\zeta,\psi} H, \quad \text{with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and where H is the total energy $H = \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 + \frac{1}{2}\int_{\Omega} |\mathbf{U}|^2$. Introducing the Dirichlet-Neumann operator $\mathcal{G}[\zeta]\psi = \sqrt{1+|\nabla\zeta|^2}\,\partial_n \Phi_{|_{\mathrm{surf}}}$, Craig and Sulem rewrote these equations as a closed set of two evolution equations on ζ and ψ,

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta]\psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{(\mathcal{G}[\zeta]\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0. \end{cases}$$

This is a convenient formulation for well-posedness issues (see, e.g., [38] for local well-posedness, [1, 2, 4, 48] for low regularity solutions, [5, 28, 34] for global existence, etc.). It has also been used for numerical computations [24, 29], weak turbulence modeling [60], analysis of periodic wave patterns or solitary waves [6, 35, 50], etc. More relevant to our present motivations, it is probably the most commonly used approach to derive and justify asymptotic models describing the solutions to the water waves equations in various physical regimes (e.g., shallow water or deep water). The derivation of such models follows directly from an asymptotic expansion of the Dirichlet-Neumann operators (e.g., [10, 16, 23, 25, 33, 42, 49]), while the key point for their justification is a local well-posedness theorem for a dimensionless version of the equations, and over a time scale whose dependance on the various dimensionless parameters is controlled. In the shallow-water regime—of great importance for applications in oceanography—this induces an extra difficulty, because the shallow-water limit is singular; the relevant existence results have been shown in [8, 32] (see also [22, 31, 44, 51, 53] for the justification of various asymptotic models using other approaches). We refer to the book [39] for a more comprehensive description of these aspects.

However, a limitation of the Zakharov-Craig-Sulem approach is that it is restricted to irrotational flows. This is a relevant framework for most applications in oceanography, but several important phenomena, such as rip currents, for instance, can only be understood by taking into account vorticity effects. Rip currents are only one particular example of wave-currents interactions; the understanding of the energy exchanges at stake in such interactions is an important challenge in oceanography, and vorticity is one of the key mechanisms involved. Several asymptotic models have been derived in the physics literature to take into account vorticity effects in shallow-water models; however, these derivations rely on assumptions on the structure of the flow (e.g., columnar motion) that are in general not satisfied, or only at a very low order of precision. In particular, to our knowledge there does not exist any good description of the nonlinear dynamics of the vorticity in the shallow-water regime; for instance, it is not known whether horizontal vorticity may be created from vertical vorticity.

From a mathematical viewpoint, various authors considered the local well-posedness theory for the water waves problem in the presence of vorticity [20, 45, 47, 52, 61]; these results are, however, not adapted to answer the above preoccupations, because they consider different physical configurations (drop of fluid, infinite depth, etc.) and use mathematical techniques that make the singular shallow-water limit very delicate to handle (see, e.g., the comments of [41] on the incompatibility of standard symbolic analysis and of the shallow water limit); moreover, the influence of vorticity on the flow is generally treated implicitly. The rigorous qualitative analysis of water waves seems to have essentially been restricted, when vorticity is present, to one-dimensional surfaces, and to periodic or standing waves; we refer to the recent book [18] for an extensive review and more references on these aspects. The motivation for the present work is therefore twofold:

(1) Find a formulation of the water waves equations allowing for the presence of vorticity, adapted to the physical configurations we have in mind for applications to oceanography, and making the influence of the vorticity on the flow as explicit as possible.
(2) Show the well-posedness of this formulation, and control the life span and the size of the solution in the so-called shallow-water limit, in order to pave the way for deriving shallow-water models in the presence of vorticity.

These goals are achieved by deriving first a generalization of the above classical Zakharov-Craig-Sulem formulation, as a set of three evolution equations on $(\zeta, \psi, \boldsymbol{\omega})$, where $\boldsymbol{\omega} = \operatorname{curl} \mathbf{U}$ is the vorticity. Of course, ψ cannot be defined as in the irrotational case as the trace at the surface of the velocity potential Φ; instead, we define $\nabla\psi$ as the projection onto gradient vector fields of the horizontal component of the tangential velocity $U_{\|}$ at the surface. The equations then read

$$\begin{cases} \partial_t \zeta + \underline{V} \cdot \nabla\zeta - \underline{w} = 0, \\ \partial_t \psi + g\zeta + \dfrac{1}{2}|U_{\|}|^2 - \dfrac{1}{2}(1 + |\nabla\zeta|^2)\underline{w}^2 - \dfrac{\nabla^{\perp}}{\Delta} \cdot (\underline{\omega} \cdot N\underline{V}) = 0, \\ \partial_t \boldsymbol{\omega} + \mathbf{U} \cdot \nabla_{X,z}\boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla_{X,z}\mathbf{U}, \end{cases}$$

where $\underline{V}$ and $\underline{w}$ denote the horizontal and vertical components of the velocity at the interface, and $\underline{\omega} = \boldsymbol{\omega}_{|_{\text{surf}}}$. In the irrotational case ($\boldsymbol{\omega} = 0$), these equations coincide exactly with the ones derived by Zakharov-Craig-Sulem. We will thus show that this is a closed set of equations, and we establish local well-posedness. We will also show that this new formulation is formally Hamiltonian (with a non-canonical Poisson bracket).

As the classical irrotational Zakharov-Craig-Sulem formulation, but contrary to the aforementioned works in the rotational case, our equations are cast in Eulerian variables, in a configuration relevant to applications in oceanography (finite depth); moreover, our equations can easily be used to derive asymptotic models. We consider in particular the shallow-water regime, which is of great importance in oceanography. We first rewrite the equation in dimensionless form, and prove the existence time is of order T/ε, with T independent of $\varepsilon, \mu \in (0,1)$, where

$$\varepsilon = \frac{\text{typical amplitude of the wave}}{\text{depth}} \quad \text{and} \quad \mu = \left(\frac{\text{depth}}{\text{typical horizontal length}}\right)^2;$$

this estimate on the lifespan of solutions goes with uniform bounds on the solutions, which make possible the derivation of asymptotic models. As an illustration, we provide the first rigorous derivation and justification of the nonlinear shallow-water equations with vorticity, which is an approximation of order $O(\mu)$ of the water waves equations when $\varepsilon = O(1)$. The more technical derivation of an $O(\mu^2)$ model of Green-Naghdi type is done in the companion paper [15]; let us just mention that the shallow-water nonlinear dynamics of the vorticity studied here allow us to exhibit a so-far unknown mechanism of generation of horizontal vorticity from purely vertical rotational effects. More generally, the asymptotic bounds derived in this paper justify the assumptions made in [15] to derive various asymptotic shallow-water models in the presence of vorticity.

The paper is organized as follows. Section 2 is devoted to deriving our new formulation of the water waves problem with vorticity (see above). The fact that this formulation is a closed set of equations follows from the resolution of a div-curl problem, allowing us to reconstruct the velocity $\mathbf{U}$ in the fluid domain from ζ, ψ, and $\boldsymbol{\omega}$. This div-curl problem is studied in full in Section 3. The local well-posedness is then addressed in Section 4: the equations are "quasilinearized," and *a priori* estimates are derived. Using these estimates, a solution is then constructed by an iterative scheme (which is nontrivial because of the fact the vorticity is defined on a domain depending on the surface elevation). The main result is then given in Theorem 4.7.

The proof of Theorem 4.7 has been tailored to allow its implementation in the shallow-water setting. However, handling the shallow-water limit induces some difficulties that are not relevant for a standard local well-posedness result such as Theorem 4.7 (e.g., the control of the bottom vorticity). For readers who are not interested in the shallow-water analysis, we have therefore opted to treat this aspect separately. This is done in Section 5. The first step is to write a dimensionless version of the equations; the associated well-posedness result is then given in Theorem 5.1. Note that the nondimensionalization of the vorticity is not obvious, but that this theorem justifies *a posteriori* the choice we have made. As an application of this result, we derive and justify in Section 5.7 a first-order nonlinear shallow-water model in the presence of vorticity.

Finally, we investigate in Section 6 the Hamiltonian structure of our new formulation of the water waves equations with vorticity.

1.2. Notation.

- $X = (x, y) \in \mathbb{R}^2$ denotes the horizontal variables. We also denote by z the vertical variable.
- ∇ is the gradient with respect to the horizontal variables; $\nabla_{X,z}$ is the full three-dimensional gradient operator. The curl and divergence operators are defined as

$$\operatorname{curl} \mathbf{A} = \nabla_{X,z} \times \mathbf{A} \quad \text{and} \quad \operatorname{div} \mathbf{A} = \nabla_{X,z} \cdot \mathbf{A}.$$

- We denote by $d = 1, 2$ the horizontal dimension. When $d = 1$, we often identify functions on $\mathbb{R}$ as functions on $\mathbb{R}^2$ independent of the y variable. In particular, when $d = 1$, the gradient, divergence, and curl operators take the form

$$\nabla_{X,z} f = \begin{pmatrix} \partial_x f \\ 0 \\ \partial_z f \end{pmatrix}, \ \operatorname{curl} \mathbf{A} = \begin{pmatrix} -\partial_z \mathbf{A}_2 \\ \partial_z \mathbf{A}_1 - \mathrm{d}x \mathbf{A}_3 \\ \partial_x \mathbf{A}_2 \end{pmatrix}, \ \operatorname{div} \mathbf{A} = \partial_x \mathbf{A}_1 + \partial_z \mathbf{A}_3.$$

- S is the flat strip $\mathbb{R}^d \times (-H_0, 0)$.
- We denote by $(X, \zeta(t,X))$ a parametrization of the free surface at time t, and by Ω_t the fluid domain delimited at time t by this free surface and a flat bottom at depth $z = -H_0$,

$$\Omega_t = \{(X,z) \in \mathbb{R}^3, \ -H_0 < z < \zeta(t,X)\};$$

  when the dependence on time is not important, we just write Ω instead of Ω_t.
- When $d = 1$, Ω is invariant along the y axis, and we identify it with a two-dimensional domain; in particular,

$$\begin{aligned} \|f\|_{L^2(\Omega)} &= \int_{\mathbb{R}} \int_{-H_0}^{\zeta(x)} |f(x,z)|^2 \,\mathrm{d}z \,\mathrm{d}x && \text{if } d = 1, \\ \|f\|_{L^2(\Omega)} &= \int_{\mathbb{R}^2} \int_{-H_0}^{\zeta(x,y)} |f(x,y,z)|^2 \,\mathrm{d}z \,\mathrm{d}x \,\mathrm{d}y && \text{if } d = 2. \end{aligned}$$

- We write $\mathbf{U}$ for the velocity field; its horizontal component is denoted $\mathbf{V}$, and its vertical component $\mathbf{w}$.
- For a vector $\mathbf{A} \in \mathbb{R}^3$, we often denote by $\mathbf{A}_h$ its horizontal component and by $\mathbf{A}_v$ its vertical component.
- If $\mathbf{A}$ is a vector field defined on Ω_t, we write $\underline{A}$ the function

$$\underline{A}(t,X) = \mathbf{A}_{|_{\mathrm{surf}}}(t,X) = \mathbf{A}(t,X,\zeta(t,X));$$

  consistently, if $\mathbf{A}$ is defined on the flat strip S, then $\underline{A}(t,X) = \mathbf{A}(t,X,0)$. We also denote by A_b its trace at the bottom

$$A_b(t,X) = \mathbf{A}_{|_{\mathrm{bott}}}(t,X) = \mathbf{A}(t,X,-H_0).$$

- $\mathbf{n}$ is the unit upward normal vector at the surface, $\mathbf{n} = N/|N|$, with $N = (-\nabla\zeta^T, 1)^T$.
- $\mathbf{n}_b$ is the *upward* normal vector at the (flat) bottom, $\mathbf{n}_b = N_b = \mathbf{e}_z$.
- If $\mathbf{V} \in \mathbb{R}^2$, we write $\mathbf{V}^\perp = (-\mathbf{V}_2, \mathbf{V}_1)^\perp$.
- For any vector field $\mathbf{A}$ defined on Ω and with values in $\mathbb{R}^3$, let us define $A_\parallel \in \mathbb{R}^2$ as the horizontal component of the tangential part of $\mathbf{A}$ at the surface,

$$A_\parallel = \underline{A}_h + \underline{A}_v \nabla\zeta, \tag{1.6}$$

so that

$$\underline{A} \times N = \begin{pmatrix} -A_{\|}^{\perp} \\ -A_{\|}^{\perp} \cdot \nabla \zeta \end{pmatrix}.$$

- We always use simple bars to denote functional norms on $\mathbb{R}^d$, and double bars to denote functional norms on the $d+1$ dimensional domains Ω and S; for instance,

$$|f|_p = |f|_{L^p(\mathbb{R}^d)}, \ |f|_{H^s} = |f|_{H^s(\mathbb{R}^d)}, \ \|f\|_p = \|f\|_{L^p(\Omega)} \text{ (or } \|f\|_{L^p(S)}\text{), etc.}$$

- We use the Fourier multiplier notation $f(\mathrm{D})u = \mathcal{F}^{-1}(\xi \mapsto f(\xi)\hat{u}(\xi))$, and denote by $\Lambda = (1-\Delta)^{1/2} = (1+|\mathrm{D}|^2)^{1/2}$ the fractional derivative operator.
- We define, for all $s \in \mathbb{R}$, $k \in \mathbb{N}$ the space $H^{s,k} = H^{s,k}(S)$ by

$$(1.7) \quad H^{s,k} = \bigcap_{j=0}^{k} H^j((-H_0,0);H^{s-j}(\mathbb{R}^d)), \quad \text{with } \|u\|_{H^{s,k}} = \sum_{j=0}^{k} \|\Lambda^{s-j}\partial_z^j u\|_2.$$

- We shall have to handle functions whose gradient are in some Sobolev space, but that are not in $L^2(\mathbb{R}^d)$. We therefore introduce the Beppo-Levi spaces [27]

$$\forall\, s \geq 0, \quad \dot{H}^s(\mathbb{R}^d) = \{f \in L^2_{\mathrm{loc}}(\mathbb{R}^d),\ \nabla f \in H^{s-1}(\mathbb{R}^d)^2\};$$

similarly, for functions defined on the fluid domain Ω, we write

$$\forall\, k \in \mathbb{N}^*, \quad \dot{H}^k(\Omega) = \{f \in L^2_{\mathrm{loc}}(\Omega),\ \nabla_{X,z} f \in H^{k-1}(\Omega)^3\}.$$

- The "dual spaces" are $H_0^s(\mathbb{R}^d)$ defined as

$$(1.8) \qquad H_0^s(\mathbb{R}^d) = \{u \in H^s(\mathbb{R}^d) : \exists\, v \in H^{s+1}(\mathbb{R}^d),\ u = |\mathrm{D}|v\},$$

and we write $|u|_{H_0^s} = |(1/|\mathrm{D}|)u|_{H^{s+1}}$.
- We write $s \vee t = \max\{s,t\}$.
- We generically denote by $C(\cdot)$ some positive function that has a nondecreasing dependance on its arguments.
- We write $[\partial^\alpha, f, g]$ for the symmetric commutator

$$[\partial^\alpha, f, g] = \partial^\alpha(fg) - \partial^\alpha f g - f\partial^\alpha g.$$

2. A New Formulation for the Equations

2.1. A first reduction. Taking the trace of (1.1) at the free surface and then taking the vectorial product of the resulting equation with N, we obtain, with the notation (1.6),

$$-\partial_t U_{\|}^{\perp} - g\nabla^{\perp}\zeta - \frac{1}{2}\nabla^{\perp}|U_{\|}|^2 + \frac{1}{2}\nabla^{\perp}((1+|\nabla\zeta|^2)\underline{w}^2) + \underline{\omega}\cdot N\underline{V} = 0,$$

where we also used the notation $\boldsymbol{\omega} = \operatorname{curl}\mathbf{U}$, $\underline{\omega} = \boldsymbol{\omega}_{|_{\rm surf}}$; we therefore get

$$\partial_t U_\parallel + g\nabla\zeta + \frac{1}{2}\nabla|U_\parallel|^2 - \frac{1}{2}\nabla((1+|\nabla\zeta|^2)\underline{w}^2) + \underline{\omega}\cdot N\underline{V}^\perp = 0. \tag{2.1}$$

Denoting by Π the projector onto gradient vector fields, and by $\Pi_\perp$ the projector onto orthogonal gradient vector fields, we get

$$\Pi = \frac{\nabla\nabla^T}{\Delta},\ \Pi_\perp = \frac{\nabla^\perp(\nabla^\perp)^T}{\Delta}.$$

We can decompose $U_\parallel$ under the form $U_\parallel = \Pi U_\parallel + \Pi_\perp U_\parallel = \nabla\psi + \nabla^\perp\tilde{\psi}$ for some scalar functions ψ and $\tilde{\psi}$, and similarly we can write

$$\begin{aligned}\underline{\omega}\cdot N\underline{V}^\perp &= \Pi(\underline{\omega}\cdot N\underline{V}^\perp) + \Pi_\perp(\underline{\omega}\cdot N\underline{V}^\perp)\\ &= \nabla\left[\frac{\nabla}{\Delta}\cdot(\underline{\omega}\cdot N\underline{V}^\perp)\right] + \Pi_\perp(\underline{\omega}\cdot N\underline{V}^\perp).\end{aligned}$$

Applying Π to the equation on $U_\parallel$, we therefore find the following equation on ψ:

$$\partial_t\psi + g\zeta + \frac{1}{2}|U_\parallel|^2 - \frac{1}{2}((1+|\nabla\zeta|^2)\underline{w}^2) + \frac{\nabla}{\Delta}\cdot(\underline{\omega}\cdot N\underline{V}^\perp) = 0. \tag{2.2}$$

There is no need to derive an equation on $\Pi_\perp U_\parallel = \nabla^\perp\tilde{\psi}$, since this component of $U_\parallel$ is fully determined by the knowledge of $\boldsymbol{\omega}$ and ζ; indeed, using the differential identity

$$(\nabla\times\mathbf{A})_{|_{\rm surf}}\cdot N = \nabla^\perp\cdot A_\parallel, \tag{2.3}$$

we compute easily that $\underline{\omega}\cdot N = \nabla^\perp\cdot U_\parallel$, and therefore

$$\Pi_\perp U_\parallel = \nabla^\perp\tilde{\psi},$$

where $\tilde{\psi}$ is the unique solution[1] in the Beppo-Levi space $\dot{H}^{3/2}(\mathbb{R}^d)$ of $\Delta\tilde{\psi} = \underline{\omega}\cdot N$.

Taking now the curl of (1.1), we classically obtain the vorticity equation

$$\partial_t\boldsymbol{\omega} + \mathbf{U}\cdot\nabla_{X,z}\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla_{X,z}\mathbf{U} \quad \text{in } \Omega_t, \tag{2.4}$$

with $\boldsymbol{\omega} = \operatorname{curl}\mathbf{U}$.

Our claim is that the kinematic equation (1.3), together with (2.2) and (2.4), forms a closed system of equations on $(\zeta, \psi, \boldsymbol{\omega})$. We have therefore to prove that these quantities fully determine the velocity field in the whole fluid domain. This is done in the next subsection.

[1] We assume here that $\boldsymbol{\omega} \in L^2(\Omega)$ and divergence free; see Lemma 3.7 for the existence and uniqueness of $\tilde{\psi}$.

Remark 2.1. Applying $\Pi_\perp$ to the equation on $U_\parallel$ does not bring any further information; this leads to

$$\partial_t \Pi_\perp U_\parallel + \Pi_\perp(\underline{\omega} \cdot N \underline{V}^\perp) = 0,$$

and therefore

$$\partial_t(\nabla^\perp \cdot U_\parallel) + \nabla \cdot (\underline{\omega} \cdot N \underline{V}) = 0.$$

Evaluating the vorticity equation at the surface, we also get

$$\partial_t \underline{\omega} + \underline{V} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{U} + \underline{\omega} \cdot N\, \partial_z \mathbf{U}_{|_{\mathrm{surf}}},$$

from which one readily deduces that $\partial_t(\underline{\omega} \cdot N) + \nabla \cdot (\underline{\omega} \cdot N \underline{V}) = 0$. We thus get

$$\partial_t(\underline{\omega} \cdot N - \nabla^\perp \cdot U_\parallel) = 0,$$

which is always true since, as seen above, $\underline{\omega} \cdot N = \nabla^\perp \cdot U_\parallel$.

Remark 2.2. We have an analogous equation to (2.3) at the bottom, which yields the following relation for the bottom vorticity:

$$\boldsymbol{\omega}_{|_{\mathrm{bott}}} \cdot N_b = \nabla^\perp \cdot V_{|_{\mathrm{bott}}}.$$

In particular, if $\mathbf{V} \in H^1(\Omega)^2$, then $\omega_b \cdot N_b \in H_0^{-1/2}(\mathbb{R}^d)$ with $H_0^{-1/2}(\mathbb{R}^d)$ as defined in (1.8).

2.2. A div-curl problem. Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$, and denote by Ω the associated fluid domain,

$$\Omega = \{(X, z) \in \mathbb{R}^{d+1},\ -H_0 < z < \zeta(X)\}; \tag{2.5}$$

we assume that the fluid domain is strictly connected in the sense that

$$\exists\, h_{\min},\ \forall\, X \in \mathbb{R}^d, \quad H_0 + \zeta(X) \geq h_{\min}. \tag{2.6}$$

From the discussion of the previous section, we have

$$\Pi_\perp U_\parallel = \nabla^\perp \tilde{\psi} = \nabla^\perp \Delta^{-1}(\underline{\omega} \cdot N).$$

This is thus fully determined by the knowledge of (the normal component at the surface of) the vorticity $\boldsymbol{\omega}$. The following theorem shows it is possible to reconstruct the whole velocity field $\mathbf{U}$ in the fluid domain in terms of $\boldsymbol{\omega}$, $\Pi U_\parallel = \nabla \psi$, and ζ; more precisely, there is a unique solution $\mathbf{U} \in H^1(\Omega)^3$ to the boundary value problem

$$(2.7)\qquad \begin{cases} \operatorname{curl}\mathbf{U} = \boldsymbol{\omega} & \text{in } \Omega,\\ \operatorname{div}\mathbf{U} = 0 & \text{in } \Omega,\\ U_{\parallel} = \nabla\psi + \nabla^{\perp}\Delta^{-1}(\underline{\omega}\cdot N) & \text{at the surface},\\ U_b\cdot N_b = 0 & \text{at the bottom}, \end{cases}$$

where we recall that $U_{\parallel}$ is defined in (1.6), that $\underline{\omega}$ stands for the trace of $\boldsymbol{\omega}$ at the surface, and that U_b is the trace of $\mathbf{U}$ at the bottom. In the statement of the theorem, we use the following definition to denote divergence-free vector fields defined on the fluid domain Ω. The second point of the definition is motivated by Remark 2.2 (where the space $H_0^{-1/2}$ is also introduced).

Definition 2.3. Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (2.6) is satisfied, and Ω be as in (2.5). We proceed as follows:

(i) We define the subspace of $L^2(\Omega)^3$ of divergence-free vector fields as

$$H(\operatorname{div}_0,\Omega) = \{\mathbf{B}\in L^2(\Omega)^3,\ \operatorname{div}\mathbf{B} = 0\}.$$

(ii) The set of such functions satisfying $B_b\cdot N_b \in H_0^{-1/2}(\mathbb{R}^d)$ is denoted by

$$H_b(\operatorname{div}_0,\Omega) = \{\mathbf{B}\in H(\operatorname{div}_0,\Omega),\ B_b\cdot N_b \in H_0^{-1/2}(\mathbb{R}^d)\},$$

which we equip with the norm

$$\|\mathbf{B}\|_{2,b} = \|\mathbf{B}\|_2 + |B_b\cdot N_b|_{H_0^{-1/2}}.$$

For the sake of clarity, the proof of the following theorem is postponed to Section 3.1.

Theorem 2.4. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (2.6) is satisfied, and Ω be as in (2.5). Let also $\boldsymbol{\omega} \in H_b(\operatorname{div}_0,\Omega)^3$ and $\psi \in \dot H^{3/2}(\mathbb{R}^d)$.*

Then, there exists a unique solution $\mathbf{U}\in H^1(\Omega)^3$ to the boundary value problem (2.7), and one has $\mathbf{U} = \operatorname{curl}\mathbf{A} + \nabla_{X,z}\Phi$ with $\mathbf{A}\in H^2(\Omega)^3$ solving

$$(2.8)\qquad \begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{A} = \boldsymbol{\omega} & \text{in } \Omega,\\ \operatorname{div}\mathbf{A} = 0 & \text{in } \Omega,\\ N_b\times A_b = 0, & \\ N\cdot\underline{A} = 0, & \\ (\operatorname{curl}\mathbf{A})_{\parallel} = \nabla^{\perp}\Delta^{-1}(\underline{\omega}\cdot N), & \\ N_b\cdot(\operatorname{curl}\mathbf{A})_{|_{\rm bott}} = 0, & \end{cases}$$

while $\Phi \in \dot{H}^2(\Omega)$ *solves*

$$\begin{cases} \Delta_{X,z}\Phi = 0 & \text{in } \Omega, \\ \Phi_{|_{\mathrm{surf}}} = \psi, \\ \partial_n \Phi_{|_{\mathrm{bott}}} = 0. \end{cases} \tag{2.9}$$

Moreover, one has

$$\|\mathbf{U}\|_2 + \|\nabla_{X,z}\mathbf{U}\|_2 \leq C\left(\frac{1}{h_{\min}}, H_0, |\zeta|_{W^{2,\infty}}\right)(\|\boldsymbol{\omega}\|_{2,b} + |\nabla\psi|_{H^{1/2}}).$$

The theorem furnishes a Hodge-Weyl decomposition[2] of the velocity field, $\mathbf{U} = \nabla_{X,z}\Phi + \operatorname{curl}\mathbf{A}$, the first component of which is the irrotational part of the velocity field, and the second one its rotational part. More precisely, the theorem allows us to give the following definition.

Definition 2.5. Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (2.6), and Ω be given by (2.5).

(1) We define the linear mapping $\mathbb{A}[\zeta]$ as follows:

$$\mathbb{A}[\zeta] : \begin{array}{l} H_b(\mathrm{div}_0, \Omega) \to H^2(\Omega)^3, \\ \boldsymbol{\omega} \mapsto \mathbf{A}, \end{array}$$

where $\mathbf{A}$ is the solution to (2.8) provided by Theorem 2.4.

(2) We define the linear mappings $\mathbb{U}_I[\zeta]$ and $\mathbb{U}_{II}[\zeta]$ by

$$\mathbb{U}_I[\zeta] : \begin{array}{l} \dot{H}^{3/2}(\mathbb{R}^d) \to H^1(\Omega)^3, \\ \psi \mapsto \nabla_{X,z}\Phi, \end{array}$$

and

$$\mathbb{U}_{II}[\zeta] : \begin{array}{l} H_b(\mathrm{div}_0, \Omega) \to H^1(\Omega)^3, \\ \boldsymbol{\omega} \mapsto \operatorname{curl}(\mathbb{A}[\zeta]\boldsymbol{\omega}), \end{array}$$

where Φ solves (2.9).

(3) The linear mapping $\mathbb{U}[\zeta]$ is

$$\mathbb{U}[\zeta] : \begin{array}{l} \dot{H}^{3/2}(\mathbb{R}^d) \times H_b(\mathrm{div}_0, \Omega) \to H^1(\Omega)^3, \\ (\psi, \boldsymbol{\omega}) \mapsto \mathbb{U}_I[\zeta]\psi + \mathbb{U}_{II}[\zeta]\boldsymbol{\omega}. \end{array}$$

---

[2]Note that this decomposition is not orthogonal for the $L^2(\Omega)$ scalar product. The standard (orthogonal) Hodge-Weyl decomposition would be

$$\mathbf{U} = \mathbf{U}^\sharp + \nabla_{X,z}\Phi^\sharp,$$

with $\mathbf{U}^\sharp$ divergence free and tangential to the boundaries of Ω. This decomposition does not isolate the vorticity effects in the sense that they are present in both terms of the decomposition, while they are absent in the potential part of the decomposition we use here.

2.3. The generalized Zakharov-Craig-Sulem formulation. We use here the results of Section 2.2 to derive a closed system of equations on ζ, ψ, and $\boldsymbol{\omega}$, from which all the other physical quantities can be deduced.

According to Definition 2.5, the kinematic equation (1.3) can be written

$$\partial_t \zeta - \mathbb{U}[\zeta](\psi, \boldsymbol{\omega})_{|_{\mathrm{surf}}} \cdot N = 0.$$

Proceeding similarly with the equation (2.2) for ψ, and the vorticity equation (2.4), and introducing the mappings[3]

$$\begin{aligned} \underline{\mathbb{V}}[\zeta](\psi, \boldsymbol{\omega}) &= \mathbb{V}[\zeta](\psi, \boldsymbol{\omega})_{|_{\mathrm{surf}}}, \\ \underline{\mathbb{w}}[\zeta](\psi, \boldsymbol{\omega}) &= \mathbb{w}[\zeta](\psi, \boldsymbol{\omega})_{|_{\mathrm{surf}}}, \\ \mathbb{U}_{\|}[\zeta](\psi, \boldsymbol{\omega}) &= \underline{\mathbb{V}}[\zeta](\psi, \boldsymbol{\omega}) + \underline{\mathbb{w}}[\zeta](\psi, \boldsymbol{\omega}) \nabla \zeta, \end{aligned} \tag{2.10}$$

we derive the following generalization of the Zakharov-Craig-Sulem formulation of the water-waves equations in presence of a nonzero vorticity field:

$$\begin{cases} \partial_t \zeta - \underline{\mathbb{U}}[\zeta](\psi, \boldsymbol{\omega}) \cdot N = 0, \\ \partial_t \psi + g\zeta + \dfrac{1}{2} |\mathbb{U}_{\|}[\zeta](\psi, \boldsymbol{\omega})|^2 \\ \qquad - \dfrac{1}{2}(1 + |\nabla \zeta|^2) \underline{\mathbb{w}}[\zeta](\psi, \boldsymbol{\omega})^2 \\ \qquad - \dfrac{\nabla^{\perp}}{\Delta} \cdot (\underline{\omega} \cdot N \underline{\mathbb{V}}[\zeta](\psi, \boldsymbol{\omega})) = 0, \\ \partial_t \boldsymbol{\omega} + \mathbb{U}[\zeta](\psi, \boldsymbol{\omega}) \cdot \nabla_{X,z} \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla_{X,z} \mathbb{U}[\zeta](\psi, \boldsymbol{\omega}) \end{cases} \tag{2.11}$$

(with $\underline{\omega} = \boldsymbol{\omega}_{|_{\mathrm{surf}}}$). Note that the divergence free constraint on the vorticity

$$\operatorname{div} \boldsymbol{\omega} = 0 \quad \text{in } \Omega_t \tag{2.12}$$

should be added to these equation; we omit it, however, since it is propagated by the vorticity equation if it is initially satisfied.

Remark 2.6. For $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfying (2.6), one can define a *generalized* Dirichlet-Neumann operator $\mathcal{G}_{\mathrm{gen}}[\zeta]$ as

$$\mathcal{G}_{\mathrm{gen}}[\zeta] : \begin{aligned} &\dot{H}^{3/2}(\mathbb{R}^d) \times H_b(\mathrm{div}_0, \Omega) \to H^{1/2}(\mathbb{R}^d), \\ &(\psi, \boldsymbol{\omega}) \mapsto \mathbb{U}[\zeta](\psi, \boldsymbol{\omega})_{|_{\mathrm{surf}}} \cdot N; \end{aligned}$$

[3]Note that, according to (2.7), one has

$$\mathbb{U}_{\|}[\zeta](\psi, \boldsymbol{\omega}) = \nabla \psi + \frac{\nabla^{\perp}}{\Delta}(\underline{\omega} \cdot N).$$

the standard Dirichlet-Neumann operator used in the irrotational case for the Zakharov-Craig-Sulem formulation of the water waves corresponds to $\mathcal{G}[\zeta]\psi = \mathcal{G}_{\mathrm{gen}}[\zeta](\psi, 0)$. Remarking that

$$\underline{\mathbb{w}}[\zeta](\psi, \boldsymbol{\omega}) = \frac{\mathcal{G}_{\mathrm{gen}}[\zeta](\psi, \omega) + \nabla\zeta \cdot \mathbb{U}_{\parallel}[\zeta](\psi, \omega)}{1 + |\nabla\zeta|^2},$$

the equations (2.11) can be written under the form

$$\begin{cases} \partial_t \zeta - \mathcal{G}_{\mathrm{gen}}[\zeta](\psi, \boldsymbol{\omega}) = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\mathbb{U}_{\parallel}[\zeta](\psi, \boldsymbol{\omega})|^2 \\ \qquad - \dfrac{(\mathcal{G}_{\mathrm{gen}}[\zeta](\psi, \boldsymbol{\omega}) + \nabla\zeta \cdot \mathbb{U}_{\parallel}[\zeta](\psi, \boldsymbol{\omega}))^2}{2(1 + |\nabla\zeta|^2)} \\ \qquad - \dfrac{\nabla^{\perp}}{\Delta} \cdot (\underline{\omega} \cdot N\underline{\mathbb{V}}[\zeta](\psi, \boldsymbol{\omega})) = 0, \\ \partial_t \boldsymbol{\omega} + \mathbb{U}[\zeta](\psi, \boldsymbol{\omega}) \cdot \nabla_{X,z} \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla_{X,z} \mathbb{U}[\zeta](\psi, \boldsymbol{\omega}). \end{cases}$$

In the irrotational case, one has $\boldsymbol{\omega} = 0$ and $\mathbb{U}_{\parallel}[\zeta](\psi, \boldsymbol{\omega}) = \nabla\psi$; if we denote further $\mathcal{G}[\zeta]\psi = \mathcal{G}_{\mathrm{gen}}[\zeta](\psi, 0)$, these equations then simplify into

$$\begin{cases} \partial_t \zeta - \mathcal{G}[\zeta]\psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \dfrac{(\mathcal{G}[\zeta]\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + |\nabla\zeta|^2)} = 0, \end{cases} \tag{2.13}$$

which is the standard Zakharov-Craig-Sulem formulation.

Remark 2.7. Contrary to the irrotational case where the water waves equations (2.13) are cast on the fixed domain $\mathbb{R}^d$, our formulation (2.11) of the water waves equations with vorticity are partly cast on the moving—and unknown—fluid domain Ω_t parametrized at time t by the free surface $\zeta(t, \cdot)$ through (2.5). The functional setting for the study of the vorticity $\boldsymbol{\omega}$ is therefore less straightforward than for ζ and ψ. A convenient way to deal with this difficulty[4] is to fix the domain by using a diffeomorphism $\Sigma(t, \cdot)$ mapping at each time the flat strip $S = \mathbb{R}^d \times (-H_0, 0)$ onto the fluid domain Ω_t. The equation on the vorticity $\boldsymbol{\omega}$ in (2.11) is then replaced by an equation on the *straightened vorticity* $\omega = \boldsymbol{\omega} \circ \Sigma$, which is defined on the (fixed) strip S.

[4]Even in the irrotational case, this problem arises if one wants to give a rigorous meaning to the original water waves equations (1.1), (1.2), (1.3) (plus an irrotationality condition). One of the advantages of the Zakharov-Craig-Sulem formulation (2.13) is that such difficulties have disappeared. They must, however, be dealt with to prove rigorously that the free surface Euler equations are indeed equivalent to (2.13). This very careful analysis has been performed only recently in [3].

3. THE DIV-CURL PROBLEM

The resolution of the div-curl problem (2.7) is necessary to prove that the formulation (2.11) of the water waves equations with vorticity forms a closed set of equations. The well-posedness of this boundary value problem was stated in Theorem 2.4; its proof is given below in Section 3.1. A consequence of Theorem 2.4 is that the curl operator can be "inverted," as explained in Section 3.2.

The analysis of the evolution equations (2.11) shall require additional properties on the velocity field provided by Theorem 2.4. After transforming the fluid domain into a flat strip in Section 3.3, we provide in Section 3.4 higher-order estimates on the solution. The control of time derivatives requires a specific treatment, and is performed in Section 3.5. Finally, crucial properties of the so-called "good unknowns" are provided in Section 3.6.

3.1. Proof of Theorem 2.4. We prove in this section Theorem 2.4; that is, we solve the following div-curl problem in the fluid domain Ω:

$$\begin{cases} \operatorname{curl}\mathbf{U} = \boldsymbol{\omega} & \text{in } \Omega, \\ \operatorname{div}\mathbf{U} = 0 & \text{in } \Omega, \\ U_{\parallel} = \nabla\psi + \nabla^{\perp}\tilde{\psi}, & \text{at the surface,} \\ U_b \cdot N_b = 0 & \text{at the bottom,} \end{cases} \tag{3.1}$$

with $\tilde{\psi} \in \dot{H}^{3/2}(\mathbb{R}^d)$ such that $\Delta\tilde{\psi} = \underline{\omega}\cdot N$ (see Lemma 3.7 below for the existence of $\tilde{\psi}$).

In order for the boundary conditions to make sense, some minimal regularity is needed on $\mathbf{U}$; let us recall the definitions

$$H(\operatorname{div},\Omega) = \{\mathbf{U} \in L^2(\Omega)^3,\ \operatorname{div}\mathbf{U} \in L^2(\Omega)\}, \tag{3.2}$$

$$H(\operatorname{curl},\Omega) = \{\mathbf{U} \in L^2(\Omega)^3,\ \operatorname{curl}\mathbf{U} \in L^2(\Omega)^3\}. \tag{3.3}$$

It is classical (see, e.g., Chapter 9 of [26]) that normal traces (respectively, tangential traces) at the boundary are well defined in $H(\operatorname{div},\Omega)$ (respectively, $H(\operatorname{curl},\Omega)$). It follows that if $\boldsymbol{\omega} \in L^2(\Omega)^3$ and $\mathbf{U}$ solves the first two equations of (3.1), one can take normal and tangential traces at the boundaries, so that it makes sense to impose the two boundary conditions of (3.1).

The first thing to note is that it is easy to reduce (3.1) to the case $\psi = 0$. Indeed, let $\Phi \in \dot{H}^2(\Omega)$ solve the boundary value problem

$$\begin{cases} \Delta_{X,z}\Phi = 0 & \text{in } \Omega, \\ \Phi_{|_{\text{surf}}} = \psi, \\ \partial_n\Phi_{|_{\text{bott}}} = 0 \end{cases} \tag{3.4}$$

(see, e.g., Chapter 2 of [39] for the existence and uniqueness of such a Φ); defining $\tilde{\mathbf{U}} = \mathbf{U} - \nabla_{X,z}\Phi$, we readily compute that

$$
\begin{cases}
\operatorname{curl}\tilde{\mathbf{U}} = \boldsymbol{\omega} & \text{in } \Omega, \\
\operatorname{div}\tilde{\mathbf{U}} = 0 & \text{in } \Omega, \\
\tilde{\underline{U}}_{\parallel} = \nabla^{\perp}\tilde{\psi}, & \text{at the surface}, \\
\tilde{U}_b \cdot N_b = 0 & \text{at the bottom},
\end{cases} \tag{3.5}
$$

which is the same problem as (3.1) with $\psi = 0$.

We look for a solution to (3.5) under the form $\tilde{\mathbf{U}} = \operatorname{curl}\mathbf{A}$, where the potential vector $\mathbf{A}$ satisfies the system

$$
\begin{cases}
\operatorname{curl}\operatorname{curl}\mathbf{A} = \boldsymbol{\omega}, \\
N_b \times A_b = 0, \\
N \cdot \underline{A} = 0, \\
(\underline{\operatorname{curl}\mathbf{A}})_{\parallel} = \nabla^{\perp}\tilde{\psi}.
\end{cases} \tag{3.6}
$$

It is important to notice that

$$N_b \times A_b = 0 \Longrightarrow N_b \cdot (\nabla \times A)_{|_{\mathrm{bott}}} = 0,$$

which corresponds to the last boundary condition of (3.5).

Before proving the existence of such a potential vector $\mathbf{A}$ in $H^1(\Omega)$, we need a series of preliminary lemmas. The first one is a Poincaré inequality for vector fields whose normal component vanishes at the surface, and whose tangential components vanish at the bottom.

Lemma 3.1. *Assume that $\zeta \in W^{1,\infty}(\mathbb{R}^d)$ satisfies the nonvanishing depth condition* (2.6)*. Let also $\mathbf{A} \in H(\operatorname{div}, \Omega) \cap H(\operatorname{curl}, \Omega)$ be such that*

$$A_b \times N_b = 0 \quad \text{and} \quad \underline{A} \cdot N = 0.$$

Then, one has $\|\mathbf{A}\|_2 \le C(H_0, |\zeta|_{W^{1,\infty}})\|\partial_z \mathbf{A}\|_2$.

Proof. Let $\Pi_I(X) : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto $\mathbb{R}N(X)$ parallel to $N_b^{\perp}$, and $\Pi_{II}(X) : \mathbb{R}^3 \to \mathbb{R}^3$ be the projection onto the horizontal plane $N_b^{\perp}$ parallel to $N(X)$. One can decompose $\mathbf{A}$ under the form

$$\mathbf{A} = \mathbf{A}_I + \mathbf{A}_{II} \quad \text{with } \mathbf{A}_I = \Pi_I \mathbf{A},\ \mathbf{A}_{II} = \Pi_{II}\mathbf{A}.$$

Since $\mathbf{A}_I$ vanishes at the surface and $\mathbf{A}_{II}$ vanishes at the bottom, we can use the standard Poincaré inequality to get

$$\|\mathbf{A}\|_2 \le \|\mathbf{A}_I\|_2 + \|\mathbf{A}_{II}\|_2 \le |H_0 + \zeta|_\infty (\|\partial_z \mathbf{A}_I\|_2 + \|\partial_z \mathbf{A}_{II}\|_2).$$

Since the projectors P_j ($j = I, II$) do not depend on z and have operator norm bounded by $C(|\zeta|_{W^{1,\infty}})$, we easily deduce that $\|\mathbf{A}\|_2 \le C(|\zeta|_{W^{1,\infty}}, H_0)\|\partial_z \mathbf{A}\|_2$. ◻

The second lemma shows that all the derivatives of $\mathbf{A}$ can be controlled in terms of $\operatorname{curl}\mathbf{A}$, $\operatorname{div}\mathbf{A}$, and the trace of $\mathbf{A}$ at the surface and bottom.

Lemma 3.2. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (2.6), and let $\mathbf{A} \in H(\operatorname{div},\Omega) \cap H(\operatorname{curl},\Omega)$ be such that $A_b \times N_b = 0$ and $\underline{A} \cdot N = 0$. Then, one has*

$$\|\nabla_{X,z}\mathbf{A}\|_2^2 \le \|\operatorname{curl}\mathbf{A}\|_2^2 + \|\operatorname{div}\mathbf{A}\|_2^2 + C|\zeta|_{W^{2,\infty}}\,|\underline{A}|_2^2,$$

for some numerical constant C.

Proof. The proof is a small variant of classical estimates (see, e.g., Chapter 9 in [26]). With the convention of summation of repeated indices, and with the notation $(\partial_1,\partial_2,\partial_3)^T = \nabla_{X,z}$, one has

$$\begin{aligned}\int_\Omega |\partial_j\mathbf{A}_i|^2 &= \int_\Omega \partial_j\mathbf{A}_i(\partial_j\mathbf{A}_i - \partial_i\mathbf{A}_j) + \int_\Omega \partial_j\mathbf{A}_i\,\partial_i\mathbf{A}_j\\ &= \|\operatorname{curl}\mathbf{A}\|_2^2 + \int_\Omega \partial_j\mathbf{A}_i\,\partial_i\mathbf{A}_j,\end{aligned}$$

and one can rewrite the second term of the right-hand side as

$$\begin{aligned}\int_\Omega \partial_j\mathbf{A}_i\,\partial_i\mathbf{A}_j &= \int_\Gamma \vec{n}_j\mathbf{A}_i\,\partial_i\mathbf{A}_j - \int_\Omega \mathbf{A}_i\,\partial_i\,\partial_j\mathbf{A}_j\\ &= \int_\Gamma(\vec{n}_i\mathbf{A}_j\,\partial_j\mathbf{A}_i - \vec{n}_j\,\mathbf{A}_j\,\partial_i\mathbf{A}_i) + \int_\Omega \partial_i\mathbf{A}_i\,\partial_j\mathbf{A}_j,\end{aligned}$$

where $\vec{n}$ is the outward unit normal vector to the boundary Γ of Ω (i.e., the bottom and the surface); we have therefore obtained

$$\|\nabla_{X,z}\mathbf{A}\|_2^2 = \|\operatorname{curl}\mathbf{A}\|_2^2 + \|\operatorname{div}\mathbf{A}\|_2^2 + \int_\Gamma(\vec{n}_i\mathbf{A}_j\,\partial_j\mathbf{A}_i - (\vec{n}\cdot\mathbf{A})\operatorname{div}\mathbf{A}). \tag{3.7}$$

Let us now evaluate the surface and bottom contributions of the boundary integral in the right-hand side:

Bottom contribution. Since $\mathbf{A}$ is a normal vector field at the bottom, one has

$$(\vec{n}\cdot\mathbf{A})(\operatorname{div}\mathbf{A})_{|_{\mathrm{bott}}} = 2|\mathbf{A}|^2H_b + \mathbf{A}\cdot\partial_{\vec{n}}\mathbf{A},$$

where H_b is the mean curvature of the bottom. Noting that

$$\int_{\mathrm{bott}} \vec{n}_i\mathbf{A}_j\,\partial_j\mathbf{A}_i = \int_{\mathrm{bott}} 2(\vec{n}\times\mathbf{A})\cdot\operatorname{curl}\mathbf{A} + \mathbf{A}\cdot\partial_{\vec{n}}\mathbf{A} = \int_{\mathrm{bott}} \mathbf{A}\cdot\partial_{\vec{n}}\mathbf{A}$$

(since $\mathbf{A}$ is normal to the bottom), we get the following expression for the contribution of the bottom to the boundary integral in (3.7):

$$\int_{\mathrm{bott}}(\vec{n}_i\mathbf{A}_j\,\partial_j\mathbf{A}_i - (\vec{n}\cdot\mathbf{A})\operatorname{div}\mathbf{A}) = -2\int_{\mathrm{bott}} H_b|A_b|^2, \tag{3.8}$$

which vanishes since the bottom is flat.

Surface contribution. Since $\underline{A} \cdot \vec{n} = 0$ at the surface, the contribution of the surface to the boundary integral in (3.7) is of the form

$$\int_{\mathrm{surf}} \vec{n}_i \mathbf{A}_j \, \partial_j \mathbf{A}_i = \int_{\mathrm{surf}} (\mathbf{A} \cdot \nabla_{X,z})(\mathbf{A} \cdot \vec{n}) - \mathbf{A} \cdot (\mathbf{A} \cdot \nabla_{X,z})\vec{n},$$

where we still denote by $\vec{n}$ a local extension of $\vec{n}$ inside Ω. Since $\mathbf{A}$ is tangent to the surface, the operator $\mathbf{A} \cdot \nabla_{X,z}$ is tangential, and the first component of the right-hand side vanishes. Since moreover, the extension of $\vec{n}$ can be chosen such that $\|\vec{n}\|_{W^{1,\infty}} \le C|\zeta|_{W^{2,\infty}}$, we deduce that

$$\int_{\mathrm{surf}} (\vec{n}_i \mathbf{A}_j \, \partial_j \mathbf{A}_i - (\vec{n} \cdot \mathbf{A}) \operatorname{div} \mathbf{A}) \le C|\zeta|_{W^{2,\infty}} |\underline{A}|_2^2. \tag{3.9}$$ □

Lemmas 3.1 and 3.2 imply a useful equivalence property for the H^1 norm of $\mathbf{A}$ with the L^2 norms of $\operatorname{div} \mathbf{A}$ and $\operatorname{curl} \mathbf{A}$.

Lemma 3.3. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (2.6), and $\mathbf{A} \in H(\mathrm{div}, \Omega) \cap H(\mathrm{curl}, \Omega)$ be such that $A_b \times N_b = 0$ and $\underline{A} \cdot N = 0$. Then, one has*

$$||\mathbf{A}||_2^2 + ||\nabla_{X,z}\mathbf{A}||_2^2 \le C(|\zeta|_{W^{2,\infty}}, H_0)(|| \operatorname{div} \mathbf{A}||_2^2 + || \operatorname{curl} \mathbf{A}||_2^2).$$

Proof. Recalling that $\mathbf{A}_h$ and $\mathbf{A}_v$ stand for the horizontal and vertical components of $\mathbf{A}$, we deduce from the assumption that $\mathbf{A}$ is tangential at the surface that $\underline{A}_v = \nabla\zeta \cdot \underline{A}_h$, and therefore,

$$|\underline{A}|_2 \le C(|\nabla\zeta|_\infty)|\underline{A}_h|_2.$$

Therefore, we turn to estimate the L^2-norm of $\underline{A}_h$. Using the fact that $\mathbf{A}_h{}_{|_{\mathrm{bott}}} = 0$ by assumption, we can write

$$\int_{\mathbb{R}^d} |\underline{A}_h|^2 = \int_{\mathbb{R}^d} \int_{-H_0}^{\zeta} \partial_z |\mathbf{A}_h|^2 = 2\int_\Omega \mathbf{A}_h \cdot \partial_z \mathbf{A}_h.$$

Noting that $\partial_z \mathbf{A}_h = -(\operatorname{curl} \mathbf{A})_h^\perp + \nabla \mathbf{A}_v$, we have that

$$\begin{aligned}
\int_{\mathbb{R}^d} |\underline{A}_h|^2 &= -2\int_\Omega \mathbf{A}_h \cdot (\operatorname{curl} \mathbf{A})_h^\perp + 2\int_\Omega \mathbf{A}_h \cdot \nabla \mathbf{A}_v \\
&= -2\int_\Omega \mathbf{A}_h \cdot (\operatorname{curl} \mathbf{A})_h^\perp - 2\int_\Omega (\nabla \cdot \mathbf{A}_h)\mathbf{A}_v - 2\int_{\mathbb{R}^d} \nabla\zeta \cdot \underline{A}_h \underline{A}_v \\
&= -2\int_\Omega \mathbf{A}_h \cdot (\operatorname{curl} \mathbf{A})_h^\perp - 2\int_\Omega (\operatorname{div} \mathbf{A})\mathbf{A}_v + 2\int_\Omega \partial_z \mathbf{A}_v \mathbf{A}_v \\
&\quad + 2\int_{\mathbb{R}^d} -\nabla\zeta \cdot \underline{A}_h \underline{A}_v.
\end{aligned}$$

Noting that the third term is equal to $|\underline{A}_v|_2 - |(\mathbf{A}_v)_{|_{\rm bott}}|_2$, and recalling that $\underline{A}_v = \nabla\zeta\cdot\underline{A}_h$, we deduce

$$\int_{\mathbb{R}^d}|\underline{A}_h|^2 = -2\int_\Omega \mathbf{A}_h\cdot(\operatorname{curl}\mathbf{A})_h^\perp - 2\int_\Omega(\operatorname{div}\mathbf{A})\mathbf{A}_v - \int_{\mathbb{R}^2}\mathbf{A}_v{}^2_{|_{\rm bott}} - \int_{\mathbb{R}^2}\underline{A}_v^2.$$

We can now deduce from the above that

$$\begin{aligned}|\underline{A}|_2^2 &\leq C(|\nabla\zeta|_\infty)\|\mathbf{A}\|_2(\|\operatorname{div}\mathbf{A}\|_2 + \|\operatorname{curl}\mathbf{A}\|_2)\\ &\leq C(|\zeta|_{W^{1,\infty}}, H_0)\|\nabla_{X,z}\mathbf{A}\|_2(\|\operatorname{div}\mathbf{A}\|_2 + \|\operatorname{curl}\mathbf{A}\|_2),\end{aligned}$$

where we used the Poincaré inequality provided by Lemma 3.1 to obtain the second inequality. Using Young's inequality $2ab \leq \varepsilon^2a^2+\varepsilon^{-2}b^2$ with ε small enough and the estimate furnished by Lemma 3.2, we get the result. □

We can now proceed to construct a solution to (3.6); we also impose that $\mathbf{A}$ be divergence free. We are therefore concerned with the problem

$$\begin{cases}\operatorname{curl}\operatorname{curl}\mathbf{A} = \boldsymbol{\omega}\\ \operatorname{div}\mathbf{A} = 0\end{cases}\quad \text{in } \Omega, \tag{3.10}$$

with the boundary conditions

$$\begin{cases}\underline{A}\cdot N = 0,\\ N_b\times A_b = 0,\\ (\operatorname{curl}\mathbf{A})_\parallel = \nabla^\perp\tilde{\psi},\end{cases} \tag{3.11}$$

with $\tilde{\psi}\in\dot{H}^{1/2}(\mathbb{R}^d)$. The first step is to construct a variational solution in the following sense.

Definition 3.4.

(1) We denote by $\mathfrak{X}$ the closed subspace of $H^1(\Omega)^3$ defined as

$$\mathfrak{X} = \{\mathbf{A}\in H^1(\Omega)^3 \mid \operatorname{div}\mathbf{A} = 0,\ \underline{A}\cdot N = 0,\ N_b\times A_b = 0\}.$$

(2) Let $\boldsymbol{\omega}\in L^2(\Omega)^3$ and $\tilde{\psi}\in\dot{H}^{1/2}(\mathbb{R}^d)$. Then, the vector field $\mathbf{A}\in\mathfrak{X}$ is a variational solution of the system (3.10)–(3.11) if and only if

$$\forall\,\mathbf{C}\in\mathfrak{X},\quad \int_\Omega \operatorname{curl}\mathbf{A}\cdot\operatorname{curl}\mathbf{C} = \int_\Omega\boldsymbol{\omega}\cdot\mathbf{C} + \int_{\mathbb{R}^d}\nabla\tilde{\psi}\cdot C_\parallel, \tag{3.12}$$

where we recall the notation $C_\parallel = \underline{C}_h + \nabla\zeta\underline{C}_v$.

The following lemma shows the existence and uniqueness of such a variational solution.

Lemma 3.5. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (2.6), and let also $\boldsymbol{\omega} \in L^2(\Omega)^3$ and $\tilde{\psi} \in \dot{H}^{1/2}(\mathbb{R}^d)$. Then, there exists a unique variational solution $\mathbf{A} \in \mathfrak{X}$ to (3.10)–(3.11) in the sense of Definition 3.4. One has, moreover,*

$$\|\operatorname{curl}\mathbf{A}\|_2 \leq C(H_0, |\zeta|_{W^{2,\infty}})(\|\boldsymbol{\omega}\|_2 + |\nabla\tilde{\psi}|_{H^{-1/2}}). \tag{3.13}$$

Proof. The bilinear form $a(\mathbf{A},\mathbf{C}) = \int_\Omega \operatorname{curl}\mathbf{A}\cdot\operatorname{curl}\mathbf{C}$ is obviously continuous on $\mathfrak{X}\times\mathfrak{X}$; Lemma 3.3 also shows that it is coercive. The linear form

$$L(\mathbf{A}) = \int_\Omega \boldsymbol{\omega}\cdot\mathbf{A} + \int_{\mathbb{R}^2} \nabla\tilde{\psi}\cdot A_\parallel$$

also satisfies

$$\begin{aligned} L(\mathbf{A}) &\leq \|\boldsymbol{\omega}\|_2\,\|\mathbf{A}\|_2 + |\nabla\tilde{\psi}|_{H^{-1/2}}\,|A_\parallel|_{H^{1/2}} \\ &\leq \|\boldsymbol{\omega}\|_2\,\|\mathbf{A}\|_2 + C(|\zeta|_{W^{2,\infty}})|\nabla\tilde{\psi}|_{H^{-1/2}}\,\|\mathbf{A}\|_{H^1}, \end{aligned}$$

where we used the trace lemma[5] to derive the second inequality; it is therefore continuous on $\mathfrak{X}$, and we can apply Lax-Milgram's theorem to obtain the existence and uniqueness of the variational solution.

Taking $\mathbf{C} = \mathbf{A}$ in (3.12) and using the above estimate on $L(\cdot)$, we get

$$\|\operatorname{curl}\mathbf{A}\|_2^2 \leq C(|\zeta|_{W^{2,\infty}})(\|\boldsymbol{\omega}\|_2 + |\nabla\tilde{\psi}|_{H^{-1/2}})\|\mathbf{A}\|_{H^1},$$

so the estimate of the lemma follows directly from Lemma 3.3 since $\operatorname{div}\mathbf{A} = 0$. □

Because of the divergence-free condition, the space $\mathfrak{X}$ used as the space of test functions in Definition 3.4 is too small to ensure that the variational solution provided by Lemma 3.5 satisfies the first equation of (3.10) in the sense of distributions. We therefore want to take a larger space of test functions by removing the divergence-free condition in the definition of $\mathfrak{X}$, namely, by working with test functions in the space

$$H^1_{\text{b.c.}}(\Omega) = \{\mathbf{A} \in H^1(\Omega)^3 \mid \underline{A}\cdot N = 0,\ N_b \times A_b = 0\}.$$

For all $\mathbf{C} \in H^1_{\text{b.c.}}(\Omega)$, we define φ by

$$\begin{aligned} \Delta\varphi &= \operatorname{div}\mathbf{C}, \\ \partial_n\varphi_{|\text{surf}} &= 0, \\ \varphi_{|\text{bott}} &= 0, \end{aligned}$$

[5]By invoking the "trace lemma," we refer throughout this article to the continuity of the trace operators at the surface and at the bottom, as operators defined on $H^1(\Omega)$ with values in $H^{1/2}(\mathbb{R}^d)$,

$$\forall\,\mathbf{A} \in H^1(\Omega), \quad |\underline{A}| \leq C(|\zeta|_{W^{1,\infty}})\|\mathbf{A}\|_{H^1}, \text{ and } |A_b| \leq C\|\mathbf{A}\|_{H^1}.$$

so that $\mathbf{C} - \nabla_{X,z}\varphi \in \mathfrak{X}$, and therefore

$$\int_\Omega \operatorname{curl}\mathbf{A} \cdot \operatorname{curl}\mathbf{C} = \int_\Omega \boldsymbol{\omega} \cdot (\mathbf{C} - \nabla_{X,z}\varphi) + \int_{\mathbb{R}^d} \nabla\tilde{\psi} \cdot (C_\parallel - (\nabla_{X,z}\varphi)_\parallel).$$

But, if $\operatorname{div}\boldsymbol{\omega} = 0$,

$$\int_\Omega \boldsymbol{\omega} \cdot \nabla_{X,z}\varphi = \int_{\mathbb{R}^d} \underline{\omega} \cdot N\underline{\varphi}$$
$$\int_{\mathbb{R}^d} \nabla\tilde{\psi} \cdot (\nabla_{X,z}\varphi)_\parallel = -\int_{\mathbb{R}^d} \Delta\tilde{\psi}\underline{\varphi}.$$

Thus, if $\Delta\tilde{\psi} = \underline{\omega} \cdot N$ we learn that

$$\int_\Omega \operatorname{curl}\mathbf{A} \cdot \operatorname{curl}\mathbf{C} = \int_\Omega \boldsymbol{\omega} \cdot \mathbf{C} + \int_{\mathbb{R}^d} \nabla\tilde{\psi} \cdot C_\parallel \tag{3.14}$$

for all $\mathbf{C} \in H^1_{\mathrm{b.c.}}(\Omega)$.

It is now easy to deduce that the variational solution furnished by Lemma 3.5 is a strong solution of (3.10)–(3.11).

Lemma 3.6. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (2.6), and let also $\boldsymbol{\omega} \in L^2(\Omega)^3$ and $\tilde{\psi} \in \dot{H}^{1/2}(\mathbb{R}^d)$. If, also,* $\operatorname{div}\boldsymbol{\omega} = 0$ *and $\Delta\tilde{\psi} = \underline{\omega} \cdot N$, then the variational solution $\mathbf{A} \in H^1(\Omega)^3$ furnished by Lemma 3.5 solves (3.10) with boundary conditions (3.11).*

Proof. By construction, $\operatorname{div}\mathbf{A} = 0$. The fact that $\operatorname{curl}\operatorname{curl}\mathbf{A} = \boldsymbol{\omega}$ stems from (3.14) with all $\mathbf{C} \in \mathcal{D}(\Omega) \subset H^1_{\mathrm{b.c.}}(\Omega)$ as test functions. It follows that $\operatorname{curl}\mathbf{A}$ belongs to $H(\operatorname{div},\Omega) \cap H(\operatorname{curl},\Omega)$, and therefore that the traces of $\operatorname{curl}\mathbf{A}$ at the surface and bottom make sense. The fact that these traces satisfy the last condition in (3.11) is also a consequence of (3.14). The first two conditions of (3.11) are automatically satisfied since $\mathbf{A} \in H^1_{\mathrm{b.c.}}(\Omega)$. ∎

Now we have all the ingredients to finish the proof of the theorem. With $\operatorname{div}\boldsymbol{\omega} = 0$ and $\Delta\tilde{\psi} = \underline{\omega}\cdot N$, we denote by $\mathbf{A} \in \mathfrak{X}$ the variational solution furnished by Lemma 3.5 and set $\tilde{\mathbf{U}} = \operatorname{curl}\mathbf{A}$. We obtain directly from Lemma 3.6 that $\tilde{\mathbf{U}} \in H(\operatorname{div},\Omega) \cap H(\operatorname{curl},\Omega)$ solves (3.5). We also get from (3.13) that

$$\begin{aligned}\|\tilde{\mathbf{U}}\|_2 &\le C(H_0, |\zeta|_{W^{2,\infty}})(\|\boldsymbol{\omega}\|_2 + |\nabla\tilde{\psi}|_{H^{-1/2}}) \\ &\le C\left(H_0, \frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}}\right)\|\boldsymbol{\omega}\|_{2,b},\end{aligned} \tag{3.15}$$

where we used the following lemma to derive the second inequality (we recall that ${H_0^{-1/2}}$ is defined in Remark 2.2 and that $H_b(\operatorname{div}_0,\Omega)$ and $\|\boldsymbol{\omega}\|_{2,b}$ are defined in Definition 2.3).

Lemma 3.7. *Let $\zeta \in W^{1,\infty}(\mathbb{R}^d)$ satisfy (2.6) and $\boldsymbol{\omega} \in H_b(\mathrm{div}_0, \Omega)$. Then, there exists a unique solution $\tilde{\psi} \in \dot{H}^{3/2}(\mathbb{R}^d)$ to the equation $\Delta\tilde{\psi} = \underline{\omega} \cdot N$, and one has*

$$|\nabla\tilde{\psi}|_{H^{1/2}} \leq C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{1,\infty}}\right) \|\boldsymbol{\omega}\|_{2,b}.$$

Proof. The bilinear form $a(u,v) = \nabla u \cdot \nabla v$ is continuous and coercive on the Hilbert space $\dot{H}^1(\mathbb{R}^d)$. Let us now define the linear $\ell(\cdot)$ form on $\dot{H}^1(\mathbb{R}^d)$ by

$$\forall\, v \in \dot{H}^1(\mathbb{R}^d), \quad \ell(v) := \int_{\mathbb{R}^d} \underline{\omega} \cdot N v.$$

If we can prove that ℓ is continuous, then the existence and uniqueness of a variational solution $\tilde{\psi} \in \dot{H}^1(\mathbb{R}^d)$ to $\Delta\tilde{\psi} = \underline{\omega} \cdot N$ will be a direct consequence of Lax-Milgram's theorem. We therefore show this continuity property.

Note now that $\Sigma(X, z + \sigma(X,z))$, with $\sigma = (1/H_0)(H_0 + z)\zeta$, is a diffeomorphism mapping the flat strip S to the fluid domain Ω, and denote $\omega = \boldsymbol{\omega} \circ \Sigma$. Writing $\nabla^\sigma_{X,z} = (J_\Sigma^{-1})^T \nabla_{X,z}$ (with $J_\Sigma = d_{X,z}\Sigma$ the Jacobian matrix), we can integrate by parts in the above formula to find

$$\begin{aligned}
\ell(v) &= \int_{\mathbb{R}^d} \omega_b \cdot N_b v_b^{\mathrm{ext}} + \int_S (1 + \partial_z\sigma) v^{\mathrm{ext}} \nabla^\sigma_{X,z} \cdot \omega + \int_S (1 + \partial_z\sigma) \nabla^\sigma_{X,z} v^{\mathrm{ext}} \cdot \omega \\
&= \int_{\mathbb{R}^d} (\omega_b \cdot N_b) v_b^{\mathrm{ext}} + \int_S (1 + \partial_z\sigma) \nabla^\sigma_{X,z} v^{\mathrm{ext}} \cdot \omega,
\end{aligned}$$

where for all $v \in \dot{H}^1(\mathbb{R}^d)$, v^{ext} is the solution of the boundary value problem

$$\begin{cases}
\Delta_{X,z} v^{\mathrm{ext}} = 0 & \text{in } S, \\
v^{\mathrm{ext}}_{|z=0} = v, \\
(\partial_z v^{\mathrm{ext}})_{|z=-H_0} = 0.
\end{cases}$$

That is, $v^{\mathrm{ext}} = (\cosh((z + H_0)|\mathrm{D}|)/(\cosh(H_0|\mathrm{D}|)))v$ (note that we use the Fourier multiplier notation even though v belongs to $\dot{H}^1(\mathbb{R}^d)$, which is not a space of tempered distributions; we refer to Notation 2.28 of [39] to see that this makes sense). We therefore have

$$\ell(v) \lesssim |\omega_b \cdot N_b|_{H_0^{-1/2}} |v_b^{\mathrm{ext}}|_{\dot{H}^{1/2}} + C(|\zeta|_{W^{1,\infty}}) \|\omega\|_2 \|\nabla^\sigma_{X,z} v^{\mathrm{ext}}\|_2,$$

where for the first term, we used the fact that $H_0^{-1/2}(\mathbb{R}^d)$ is the dual space of $\dot{H}^{1/2}(\mathbb{R}^d)$ (see [11] for a proof). Using the explicit expression of v^{ext}, we readily deduce that

$$\ell(v) \lesssim C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right) \|\boldsymbol{\omega}\|_{2,b} |\nabla v|_2,$$

which implies the desired continuity property, and therefore the existence and uniqueness of a variational solution $\tilde{\psi} \in \dot{H}^1(\mathbb{R}^d)$ to $\Delta\tilde{\psi} = \underline{\omega} \cdot N$. We also directly get from the above that

$$|\nabla\tilde{\psi}|_2 \le C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right) \|\boldsymbol{\omega}\|_{2,b}.$$

In order to obtain an $H^{1/2}$-estimate of $\nabla\tilde{\psi}$, let us take the L^2-scalar product of the equation $\Delta\tilde{\psi} = \underline{\omega} \cdot N$ with $\Lambda\tilde{\psi}$ (with $\Lambda = (1-\Delta)^{1/2}$). Integrating by parts, we then proceed as above to get

$$\begin{aligned} |\nabla\tilde{\psi}|_{H^{1/2}}^2 &= -\int_{\mathbb{R}^d} \Lambda\tilde{\psi}\underline{\omega} \cdot N \\ &= -\int_{\mathbb{R}^d} \Lambda\tilde{\psi}_b^{\mathrm{ext}}\omega_b \cdot N_b - \int_S (1+\partial_z\sigma)\nabla_{X,z}^{\sigma}\Lambda\tilde{\psi}^{\mathrm{ext}} \cdot \omega. \end{aligned}$$

We readily deduce from the Cauchy-Schwarz inequality that

$$\begin{aligned} |\nabla\tilde{\psi}|_{H^{1/2}}^2 &\le |\omega_b \cdot N_b|_{H_0^{-1/2}} |\Lambda\tilde{\psi}^{\mathrm{ext}}_{|\mathrm{bott}}|_{\dot{H}^{1/2}} + C(|\zeta|_{W^{1,\infty}})\|\omega\|_2 \|\nabla_{X,z}^{\sigma}\Lambda\tilde{\psi}^{\mathrm{ext}}\|_2 \\ &\le C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right) \|\boldsymbol{\omega}\|_{2,b} \|\Lambda\nabla_{X,z}\tilde{\psi}^{\mathrm{ext}}\|_2. \end{aligned}$$

Since $\|\Lambda\nabla_{X,z}\tilde{\psi}^{\mathrm{ext}}\|_2 \lesssim |\nabla\tilde{\psi}|_{H^{1/2}}$ (this is a standard smoothing property; see, e.g., Lemma 2.20 of [39]), one deduces that

$$|\nabla\tilde{\psi}|_{H^{1/2}} \le C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right) \|\boldsymbol{\omega}\|_{2,b},$$

which concludes the proof of the lemma. ◻

The following lemma complements this L^2-estimate on $\tilde{\mathbf{U}}$ provided by (3.15) by an H^1-estimate.

Lemma 3.8. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (2.6) and $\boldsymbol{\omega} \in H_b(\mathrm{div}_0, \Omega)$, and let $\tilde{\psi} \in \dot{H}^{3/2}(\mathbb{R}^d)$ be the solution to the equation $\Delta\tilde{\psi} = \underline{\omega} \cdot N$ furnished by Lemma 3.7. If $\tilde{\mathbf{U}} \in L^2(\Omega)^3$ solves (3.5), then $\tilde{\mathbf{U}} \in H^1(\Omega)^3$, and one has*

$$\|\tilde{\mathbf{U}}\|_2 + \|\nabla_{X,z}\tilde{\mathbf{U}}\|_2 \le C\left(|\zeta|_{W^{2,\infty}}, H_0, \frac{1}{h_{\min}}\right) \|\boldsymbol{\omega}\|_{2,b}.$$

Proof. First, the fact that $\mathrm{div}\,\tilde{\mathbf{U}} = 0$ implies

$$\mathrm{curl}\,\mathrm{curl}\,\tilde{\mathbf{U}} = -\Delta_{X,z}\tilde{\mathbf{U}} = \mathrm{curl}\,\boldsymbol{\omega}$$

and therefore

$$-\int_\Omega \Delta_{X,z}\tilde{\mathbf{U}} \cdot \tilde{\mathbf{U}} = \int_\Omega \mathrm{curl}\,\boldsymbol{\omega} \cdot \tilde{\mathbf{U}}.$$

Thus, after integrating by parts, we have that

$$\int_\Omega |\nabla_{X,z}\tilde{\mathbf{U}}|^2 = \int_{\partial\Omega}(\partial_{\vec{n}}\tilde{\mathbf{U}} + \vec{n}\times\boldsymbol{\omega})\cdot\tilde{\mathbf{U}} + \int_\Omega \boldsymbol{\omega}\cdot\operatorname{curl}\tilde{\mathbf{U}}$$
$$= \int_{\partial\Omega}(\partial_{\vec{n}}\tilde{\mathbf{U}} + \vec{n}\times\operatorname{curl}\tilde{\mathbf{U}})\cdot\tilde{\mathbf{U}} + \int_\Omega |\boldsymbol{\omega}|^2,$$

where $\vec{n}$ is the *outward* unit normal vector to the boundary Γ of Ω. At the bottom, $\vec{n} = -N_b$ and $w_b = 0$; since, moreover, $\partial_{\vec{n}}\tilde{\mathbf{U}}_{|_{\mathrm{bott}}} = -\partial_z\tilde{\mathbf{U}}_{|_{\mathrm{bott}}}$ and since $(\operatorname{curl}\tilde{\mathbf{U}})_{h|_{\mathrm{bott}}} = \partial_z\tilde{\mathbf{V}}^\perp_{|_{\mathrm{bott}}}$, we easily get $(\partial_{\vec{n}}\tilde{\mathbf{U}} + \vec{n}\times\operatorname{curl}\tilde{\mathbf{U}})_{|_{\mathrm{bott}}} = 0$, and therefore

$$\int_\Omega |\nabla_{X,z}\tilde{\mathbf{U}}|^2 = \int_{\mathrm{surf}}(\partial_n\tilde{\mathbf{U}} + n\times\operatorname{curl}\tilde{\mathbf{U}})\cdot\tilde{\mathbf{U}} + \int_\Omega|\boldsymbol{\omega}|^2$$
$$= \int_{\mathbb{R}^d}(N\cdot\nabla_{X,z}\tilde{\mathbf{U}} + N\times\operatorname{curl}\tilde{\mathbf{U}})_{|_{\mathrm{surf}}}\cdot\underline{\tilde{U}} + \int_\Omega|\boldsymbol{\omega}|^2. \tag{3.16}$$

In order to evaluate the boundary integral, let us note that

$$(N\cdot\nabla_{X,z}\tilde{\mathbf{U}})_{|_{\mathrm{surf}}} = \begin{pmatrix} -\nabla\zeta\cdot\nabla\underline{\tilde{V}} + (1+|\nabla\zeta|^2)\,\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}} \\ -\nabla\zeta\cdot(\nabla\tilde{\mathbf{w}})_{|_{\mathrm{surf}}} + \partial_z\tilde{\mathbf{w}}_{|_{\mathrm{surf}}}\end{pmatrix}$$

and

$$(N\times\operatorname{curl}\tilde{\mathbf{U}})_{|_{\mathrm{surf}}} = \begin{pmatrix} -\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}} + \nabla\underline{\tilde{w}} - \nabla\zeta\,\partial_z\tilde{\mathbf{w}}_{|_{\mathrm{surf}}} \\ \qquad - \nabla\cdot\underline{\tilde{V}}^\perp\nabla^\perp\zeta + \nabla\zeta\cdot\partial_z\tilde{\mathbf{V}}^\perp\nabla^\perp\zeta \\ -\nabla\zeta\cdot\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}} + \nabla\zeta\cdot(\nabla\tilde{\mathbf{w}})_{|_{\mathrm{surf}}}\end{pmatrix},$$

and use the identity

$$|\nabla\zeta|^2\,\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}} = (\nabla\zeta\cdot\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}})\nabla\zeta + (\nabla\zeta^\perp\cdot\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}})\nabla\zeta^\perp$$

to obtain

$$(N\cdot\nabla_{X,z}\tilde{\mathbf{U}} + N\times\operatorname{curl}\tilde{\mathbf{U}})_{|_{\mathrm{surf}}}$$
$$= (N\cdot\partial_z\tilde{\mathbf{U}}_{|_{\mathrm{surf}}})N + \begin{pmatrix}\nabla\underline{\tilde{w}} - (\nabla\zeta\cdot\nabla)\underline{\tilde{V}} - \nabla\cdot\underline{\tilde{V}}^\perp\nabla^\perp\zeta \\ 0\end{pmatrix}.$$

Using the fact that $\operatorname{div}\tilde{\mathbf{U}} = 0$, we now note that

$$(N\cdot\partial_z\tilde{\mathbf{U}}_{|_{\mathrm{surf}}}) = -\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}}\cdot\nabla\zeta + \partial_z\tilde{\mathbf{w}}_{|_{\mathrm{surf}}}$$
$$= -\partial_z\tilde{\mathbf{V}}_{|_{\mathrm{surf}}}\cdot\nabla\zeta - (\nabla\cdot\tilde{\mathbf{V}})_{|_{\mathrm{surf}}} = -\nabla\cdot\underline{\tilde{V}},$$

and we can then deduce the following expression:

$$(N \cdot \nabla_{X,z} \tilde{\mathbf{U}} + N \times \operatorname{curl} \tilde{\mathbf{U}})_{|\text{surf}} = \begin{pmatrix} \nabla \underline{\tilde{w}} - (\nabla^{\perp}\zeta \cdot \nabla)\underline{\tilde{V}}^{\perp} \\ -\nabla \cdot \underline{\tilde{V}} \end{pmatrix} := F. \tag{3.17}$$

Going back to (3.16), this gives

$$\int_{\Omega} |\nabla_{X,z} \tilde{\mathbf{U}}|^2 = \int_{\mathbb{R}^d} F \cdot \underline{\tilde{U}} + \int_{\Omega} |\boldsymbol{\omega}|^2 \tag{3.18}$$

$$= \int_{\mathbb{R}^d} 2\underline{\tilde{V}} \cdot \nabla \underline{\tilde{w}} - (\nabla^{\perp}\zeta \cdot \nabla)\underline{\tilde{V}}^{\perp} \cdot \underline{\tilde{V}} + \int_{\Omega} |\boldsymbol{\omega}|^2. \tag{3.19}$$

Noting that $\underline{\tilde{V}} = \nabla^{\perp}\tilde{\psi} - \underline{\tilde{w}}\nabla\zeta$, and integrating by parts if necessary, all the components of the integrand of the first term can be put under the form

$$P(\zeta)\, \partial_i \nabla^{\perp}\tilde{\psi}\underline{\tilde{U}}_j, \; Q(\zeta)(\nabla\tilde{\psi})_i\underline{\tilde{U}}_j, \text{ or } Q(\zeta)\underline{\tilde{w}}^2 \quad (1 \leq i, j \leq d),$$

with $P(\zeta)$ (respectively, $Q(\zeta)$) a generic notation for a polynomial in the first- (respectively, second-) order derivatives of ζ. We deduce therefore by standard product estimates that

$$\begin{aligned} \|\nabla_{X,z}\tilde{\mathbf{U}}\|_2^2 &\leq C(|\zeta|_{W^{2,\infty}})(|\nabla\tilde{\psi}|_{H^{1/2}}\, |\underline{\tilde{U}}|_{H^{1/2}} + |\nabla\tilde{\psi}|_2\, |\underline{\tilde{U}}|_2 + |\underline{\tilde{w}}|_2^2) + \|\boldsymbol{\omega}\|_2^2 \\ &\leq C(|\zeta|_{W^{2,\infty}})(|\nabla\tilde{\psi}|_{H^{1/2}}\, |\underline{\tilde{U}}|_{H^{1/2}} + |\underline{\tilde{w}}|_2^2) + \|\boldsymbol{\omega}\|_2^2. \end{aligned} \tag{3.20}$$

Noting now that

$$|\underline{\tilde{w}}|_2^2 = 2\int_{\Omega} \tilde{\mathbf{w}}\, \partial_z \tilde{\mathbf{w}} \leq C(H_0, |\zeta|_\infty)\, \|\tilde{\mathbf{w}}\|_2\, \|\partial_z\tilde{\mathbf{w}}\|_2,$$

and recalling that the trace lemma and Lemma 3.1 yield

$$|\underline{\tilde{U}}|_{H^{1/2}} \leq C(|\zeta|_{W^{1,\infty}})\|\tilde{\mathbf{U}}\|_{H^1},$$

we easily deduce from (3.20) and Young's inequality that

$$\|\tilde{\mathbf{U}}\|_{H^1}^2 \leq C(H_0, |\zeta|_{W^{2,\infty}})(|\nabla\tilde{\psi}|_{H^{1/2}}^2 + \|\tilde{\mathbf{U}}\|_2^2 + \|\boldsymbol{\omega}\|_2^2).$$

Using the upper bound on $\|\tilde{\mathbf{U}}\|_2$ provided by (3.15), we get

$$\|\tilde{\mathbf{U}}\|_2 + \|\nabla_{X,z}\tilde{\mathbf{U}}\|_2 \leq C\left(|\zeta|_{W^{2,\infty}}, H_0, \frac{1}{h_{\min}}\right)(\|\boldsymbol{\omega}\|_{2,b} + |\nabla\tilde{\psi}|_{H^{1/2}}),$$

and the H^1-estimate of the lemma follows from Lemma 3.7. ◻

Letting $\mathbf{U} = \tilde{\mathbf{U}} + \nabla_{X,z}\Phi$, with Φ solving (3.4), we have therefore constructed a solution to (2.7). Uniqueness of this solution follows from Lemma 3.8, while the estimate given in the statement of the theorem follows from Lemma 3.8 and the estimate

$$\|\nabla_{X,z}\Phi\|_{H^1} \le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}}\right)|\nabla\psi|_{H^{1/2}},$$

which is a well-known estimate from the study of irrotational water waves (see, e.g., Corollary 2.40 of [39]).

3.2. Inverting the* curl *operator. Theorem 2.4 can be used to construct an inverse of the curl operator on the space of divergence-free vector fields.

Corollary 3.9. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (2.6) is satisfied, and let Ω be as in (2.5). Let also $\mathbf{C} \in H(\mathrm{div}_0, \Omega)$ be such that $C_b \cdot N_b = 0$. Then, there exists a unique solution $\mathbf{B} \in H^1(\Omega)^3$ to the boundary value problem*

$$\begin{cases} \mathrm{curl}\,\mathbf{B} = \mathbf{C} & \text{in } \Omega, \\ \mathrm{div}\,\mathbf{B} = 0 & \text{in } \Omega, \\ B_\parallel = \nabla^\perp\Delta^{-1}(\underline{C}\cdot N) & \text{at the surface}, \\ B_b = 0 & \text{at the bottom}; \end{cases}$$

we denote this solution $\mathbf{B} = \mathrm{curl}^{-1}\,\mathbf{C}$.

Proof of the corollary. A direct application of Theorem 2.4 (with $\psi = 0$) provides existence and uniqueness of a solution to the same boundary problem as in the statement of the corollary, but with the bottom boundary condition replaced by $B_b \cdot N_b = 0$. The fact that $B_b \times N_b = 0$ comes from the observation that

$$(B_b \times N_b)^\perp = \nabla^\perp\Delta^{-1}(C_b \cdot N_b),$$

which is equal to zero since $C_b \cdot N_b = 0$ by assumption. ∎

3.3. The straightened div-curl problem. We know from Theorem 2.4 that there exists a unique solution $\mathbf{U} \in H^1(\Omega)^3$ of the problem

$$\begin{cases} \mathrm{curl}\,\mathbf{U} = \boldsymbol{\omega} & \text{in } \Omega, \\ \mathrm{div}\,\mathbf{U} = 0 & \text{in } \Omega, \\ U_\parallel = \nabla\psi + \nabla^\perp\Delta^{-1}(\underline{\omega}\cdot N) & \text{at the surface}, \\ U_b \cdot N_b = 0 & \text{at the bottom}. \end{cases}$$

The study of the regularity properties of this solution is made easier by working on the flat strip $S = \mathbb{R}^d \times (-H_0, 0)$, with a transformed equivalent div-curl problem.

We therefore introduce the diffeomorphism $\Sigma : S \to \Omega$ mapping the flat strip S onto the fluid domain and defined as

$$\Sigma(t,\cdot) : \begin{array}{l} S \to \Omega, \\ (X,z) \mapsto \Sigma(t,X,z) = (X, z+\sigma(t,X,z)), \end{array}$$

where σ is the scalar function

$$\sigma(X,z) = \frac{1}{H_0}(z+H_0)\zeta.$$

Defining $U = \begin{pmatrix} V \\ w \end{pmatrix} = \mathbf{U} \circ \Sigma$, with $\omega = \boldsymbol{\omega} \circ \Sigma$, we readily get that $\mathbf{U} \in H^1(\Omega)^3$ is the unique solution to the above div-curl problem if and only if $U \in H^1(S)^3$ is the unique solution to the transformed div-curl problem

$$\begin{cases} \operatorname{curl}^\sigma U = \omega & \text{in } S, \\ \operatorname{div}^\sigma U = 0 & \text{in } S, \\ U_\parallel = \nabla\psi + \nabla^\perp\Delta^{-1}(\underline{\omega}\cdot N) & \text{at the surface}, \\ U_b \cdot N_b = 0 & \text{at the bottom}, \end{cases} \tag{3.21}$$

where we use the notation

$$\begin{aligned} \operatorname{curl}^\sigma U &= (\operatorname{curl}\mathbf{U})\circ\Sigma = \nabla^\sigma_{X,z}\times U, \\ \operatorname{div}^\sigma U &= (\operatorname{div}\mathbf{U})\circ\Sigma = \nabla^\sigma_{X,z}\cdot U, \end{aligned} \tag{3.22}$$

with $\nabla^\sigma_{X,z}$ given by

$$\nabla^\sigma_{X,z} = (J_\Sigma^{-1})^T\nabla_{X,z}, \quad \text{with } (J_\Sigma^{-1})^T = \begin{pmatrix} \mathrm{Id}_{d\times d} & \dfrac{-\nabla\sigma}{1+\partial_z\sigma} \\ 0 & \dfrac{1}{1+\partial_z\sigma} \end{pmatrix} \tag{3.23}$$

($J_\Sigma = d_{X,z}\Sigma$ is the Jacobian matrix of the diffeomorphism Σ). More generally, if $F = \mathbf{F}\circ\Sigma$, we define using the convenient notation of [47]

$$\partial_i^\sigma F = \partial_i^\sigma \mathbf{F} \circ \Sigma \quad (i = t,x,y,z), \tag{3.24}$$

and therefore

$$\partial_i^\sigma = \partial_i - \frac{\partial_i\sigma}{1+\partial_z\sigma}\partial_z, \ (i = t,x,y) \quad \text{and} \quad \partial_z^\sigma = \frac{1}{1+\partial_z\sigma}\partial_z. \tag{3.25}$$

Notation 3.10. We denote by $\mathbb{U}^\sigma[\zeta](\psi,\omega)$ the solution to the straightened div-curl problem (3.21). According to Definition 2.5, one has

$$\mathbb{U}^\sigma[\zeta](\psi,\omega) = \mathbb{U}[\zeta](\psi,\boldsymbol{\omega})\circ\Sigma,$$

and we have the decomposition

$$\mathbb{U}^\sigma[\zeta](\psi,\omega) = \mathbb{U}^\sigma_I[\zeta]\psi + \mathbb{U}^\sigma_{II}[\zeta]\omega, \tag{3.26}$$

with

$$\mathbb{U}^\sigma_I[\zeta]\psi = \mathbb{U}_I[\zeta]\psi \circ \Sigma \quad \text{and} \quad \mathbb{U}^\sigma_{II}[\zeta]\omega = \mathbb{U}_{II}[\zeta]\boldsymbol{\omega} \circ \Sigma.$$

Remark 3.11. Straightening the boundary value problems (2.8) and (3.4), $\mathbb{U}^\sigma_I[\zeta]\psi$ and $\mathbb{U}^\sigma_{II}[\zeta]\omega$ can be alternatively defined as

$$\mathbb{U}^\sigma_I[\zeta]\psi = \nabla^\sigma_{X,z}\varphi \quad \text{and} \quad \mathbb{U}^\sigma_{II}[\zeta]\omega = \operatorname{curl}^\sigma A,$$

with

$$\begin{cases} \nabla_{X,z}\cdot P(\Sigma)\nabla_{X,z}\varphi = 0 & \text{in } S,\\ \varphi_{|_{z=0}} = \psi, & \\ \mathbf{e}_z \cdot P(\Sigma)\nabla_{X,z}\varphi_{|_{z=-H_0}} = 0, & \end{cases} \tag{3.27}$$

where $P(\Sigma) = (1+\partial_z\sigma)J_\Sigma^{-1}(J_\Sigma^{-1})^T$, and

$$\begin{cases} \operatorname{curl}^\sigma\operatorname{curl}^\sigma A = \omega & \text{in } S,\\ \operatorname{div}^\sigma A = 0 & \text{in } S,\\ N_b\times A_b = 0, & \\ N\cdot \underline{A} = 0, & \\ (\operatorname{curl}^\sigma A)_\parallel = \nabla^\perp\Delta^{-1}\underline{\omega}\cdot N, & \\ N_b\cdot\operatorname{curl}^\sigma A_b = 0. & \end{cases} \tag{3.28}$$

3.4. Higher-order estimates. All the properties that can be established for $\mathbf{U}$ have of course a counterpart on the straightened velocity field $U = \mathbb{U}^\sigma[\zeta](\psi,\omega)$ introduced in the previous section. For instance, proceeding as for (3.18), we know that

$$\forall\, \mathbf{C}\in H^1(\Omega),\quad \int_\Omega \nabla_{X,z}\mathbf{U}\cdot\nabla_{X,z}\mathbf{C} = \int_\Omega \boldsymbol{\omega}\cdot\operatorname{curl}\mathbf{C} + \int_{\mathbb{R}^d} F\cdot\underline{C},$$

with $\underline{C} = \mathbf{C}_{|_{\rm surf}}$ and F as in (3.17). Working on the straightened domain S, this is equivalent to saying that

$$\forall\, C\in H^1(S),\quad \int_S \nabla_{X,z}U\cdot P(\Sigma)\nabla_{X,z}C = \int_S (1+\partial_z\sigma)\omega\cdot\operatorname{curl}^\sigma C + \int_{\mathbb{R}^d} F\cdot\underline{C}, \tag{3.29}$$

with $P(\Sigma) = (1 + \partial_z\sigma)J_\Sigma^{-1}(J_\Sigma^{-1})^T$ and

$$\nabla_{X,z}U \cdot P(\Sigma)\nabla_{X,z}C := \sum_{i,j,k} \partial_j U^i P(\Sigma)_{jk}\,\partial_k C^i.$$

The H^1-estimate on $\mathbf{U}$ given in Theorem 2.4 yields the following H^1-estimate on U:

$$\|U\|_2 + \|\nabla_{X,z}U\|_2 \le C\left(\frac{1}{h_{\min}}, H_0, |\zeta|_{W^{2,\infty}}\right)(\|\omega\|_{2,b} + |\nabla\psi|_{H^{1/2}}); \tag{3.30}$$

note that the theorem below provides higher-order estimates on U, that is, estimates on $\|\Lambda^k\nabla_{X,z}U\|$ for $k \in \mathbb{N}$. It is important to note that this control is not given in terms of $|\Lambda^k\nabla\psi|_{H^{1/2}}$ but in terms of $|\mathfrak{P}\psi_{(\alpha)}|_2$ $(0 < |\alpha| \le k+1)$, where

$$\mathfrak{P} = \frac{|\mathrm{D}|}{(1+|\mathrm{D}|)^{1/2}} \tag{3.31}$$

(so that $|\mathfrak{P}\psi_{(\alpha)}|_2 \sim |\nabla\psi_{(\alpha)}|_{H^{-1/2}}$), while the *good unknowns* $\psi_{(\alpha)}$ are defined as

$$\psi_{(\alpha)} = \partial^\alpha\psi - \underline{w}\,\partial^\alpha\zeta \tag{3.32}$$

(for $\alpha = 0$, we take $\psi_{(\alpha)} = \psi$). We also recall that the spaces $H^{s,k}$ have been defined in (1.7): the norm $\|\cdot\|_{H^{s,k}}$ controls a total number of s derivatives, including at most k vertical ones. Note also that, with the convention that a summation over an empty set is equal to zero, the estimate of the proposition coincides of course with the estimate of Theorem 2.4 when $k = 0$.

Proposition 3.12. *Let $N \in \mathbb{N}$, $N \ge 5$ and $\zeta \in H^N(\mathbb{R}^d)$ be such that (2.6) is satisfied. Under the assumptions of Theorem 2.4, there exists a unique solution $U \in H^1(S)$ to (3.21); if, moreover, $0 \le k \le N-1$ and $\Lambda^k\omega \in L^2(S)$, then the following higher-order estimates hold:*

$$\|U\|_{H^{k+1,1}} \le M_N\Big(|\mathfrak{P}\psi|_{H^1} + \sum_{\alpha\in\mathbb{N}^d,\,1<|\alpha|\le k+1} |\mathfrak{P}\psi_{(\alpha)}|_2 + \|\Lambda^k\omega\|_{2,b}\Big),$$

with $M_N = C(1/h_{\min}, H_0, |\zeta|_{H^N})$.

Proof. In order to study the regularity of U, we take[6] $C = \partial^{2\beta}U$ (with $\beta \in \mathbb{N}^d$ and $k = |\beta| \le N-1$) in (3.29) as follows:

$$\int_S \nabla_{X,z}U \cdot P(\Sigma)\nabla_{X,z}\,\partial^{2\beta}U = \int_S (1+\partial_z\sigma)\omega \cdot \mathrm{curl}^\sigma\ \partial^{2\beta}U + \int_{\mathbb{R}^d} F \cdot \partial^{2\beta}\underline{U}.$$

[6]Of course, this is not correct since $\partial^{2\beta}U$ has not the H^1-regularity of the test functions C in (3.29). One should instead consider a regularization of $\partial^{2\beta}U$, such as, for instance, $\chi(\delta|\mathrm{D}|)\,\partial^{2\beta}U$, where $\delta > 0$ and χ is a smooth, compactly supported function, which is equal to one in a neighborhood of the origin. One should prove all the estimates for this regularized version, and then deduce the result by letting $\delta \to 0$. We omit this technical step, and refer to the proof of Lemma 2.39 in [39], where the details are provided in a similar context.

Integrating by parts, and denoting $\Lambda = (1-\Delta)^{1/2}$, we get

$$\int_S \nabla_{X,z}\partial^\beta U \cdot P(\Sigma)\nabla_{X,z}\partial^\beta U = \int_S \nabla_{X,z}\partial^\beta U \cdot [\partial^\beta, P(\Sigma)]\nabla_{X,z}U$$
$$+ \int_S \Lambda^k\omega \cdot \Lambda^{-k}((1+\partial_z\sigma)\,\mathrm{curl}^\sigma\ \partial^{2\beta}U) + \int_{\mathbb{R}^d} \partial^\beta F \cdot \partial^\beta \underline{U} = I_1 + I_2 + I_3.$$

Since $P(\Sigma)$ is coercive in the sense that

$$\forall\, \Theta \in \mathbb{R}^{d+1}, \quad |\Theta|^2 \le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{1,\infty}}\right)\Theta\cdot P(\Sigma)\Theta \tag{3.33}$$

(see Lemma 2.27 of [39] or Lemma 2.5 in [38]), we have that

$$||\partial^\beta \nabla_{X,z}U||_2^2 \le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{1,\infty}}\right)(I_1+I_2+I_3), \tag{3.34}$$

and we therefore need upper bounds on I_i $(1\le i\le 3)$.

Upper bound for I_1. Denoting $Q(\Sigma) = P(\Sigma) - \mathrm{Id}$, we deduce from the standard commutator estimate

$$\forall\, f \in H^{(t_0+1)}\cap H^k(\mathbb{R}^d),\ \forall\, g\in H^{k-1}(\mathbb{R}^d), \quad |\,[\partial^\beta, f]g\,|_2 \lesssim |f|_{H^{(t_0+1)\vee k}}\,|g|_{H^{k-1}},$$

(with $t_0 > d/2$) that

$$\begin{aligned}|I_1| &\le \|\partial^\beta\nabla_{X,z}U\|_2\,\|Q(\Sigma)\|_{L^\infty H^{(t_0+1)\vee k}}\,\|\Lambda^{k-1}\nabla_{X,z}U\|_2\\ &\le M_N\|\partial^\beta\nabla_{X,z}U\|_2\,\|\Lambda^{k-1}\nabla_{X,z}U\|_2,\end{aligned} \tag{3.35}$$

where we used the fact that $N \ge t_0+2$ (if $t_0 > d/2$ is chosen close enough to $d/2$).

Upper bound for I_2. We can write the operator $\Lambda^{-k}((1+\partial_z\sigma)\,\mathrm{curl}^\sigma\ \partial^{2\beta}\cdot)$ as a sum of vectorial operators with coordinates of the form

$$\Lambda^{-k}\,\partial^{2\beta}\,\partial_{j_2}\cdot,\ \Lambda^{-k}(\partial_{j_1}\sigma)^\ell\,\partial^{2\beta}\partial_{j_2}\cdot, \quad j_1, j_2 = x, y, z,\ \ell = 1,2.$$

We can therefore use the product estimate (see [30, p. 240])

$$\forall\, f\in H^{r_1}(\mathbb{R}^d),\ \forall\, g\in H^{r_2}(\mathbb{R}^d), \quad |fg|_{H^r} \lesssim |f|_{H^{r_1}}\,|g|_{H^{r_2}}, \tag{3.36}$$

for all $r, r_1, r_2 \in \mathbb{R}$ such that $r_1 + r_2 \ge 0$, $r \le r_j$ $(j=1,2)$ and $r < r_1 + r_2 - d/2$ to deduce (taking $r = -k$, $r_1 = k\vee t_0$, $r_2 = -k$) that

$$\|\Lambda^{-k}((1+\partial_z\sigma)\,\mathrm{curl}^\sigma\ \partial^{2\beta}U)\|_2 \le C\left(\frac{1}{h_{\min}}, |\nabla_{X,z}\sigma|_{L^\infty H^{k\vee t_0}}\right)\|\partial^\beta\nabla_{X,z}U\|_2.$$

Using a simple Cauchy-Schwarz inequality, we then get that

$$|I_2| \le M_N \|\Lambda^k \omega\|_2 \, \|\partial^\beta \nabla_{X,z} U\|_2. \tag{3.37}$$

Upper bound for I_3. Proceeding as for (3.19), we get

$$\begin{aligned} I_3 &= \int_{\mathbb{R}^d} 2\,\partial^\beta \underline{V} \cdot \nabla \partial^\beta \underline{w} - (\nabla^\perp \zeta \cdot \nabla)\,\partial^\beta \underline{V}^\perp \cdot \partial^\beta \underline{V} - [\partial^\beta, \nabla^\perp \zeta] \cdot \nabla \underline{V}^\perp \cdot \partial^\beta \underline{V} \\ &:= I_{31} + I_{32} + I_{33}, \end{aligned}$$

and we now turn to give upper bounds on the three components of the right-hand side. Substituting

$$\begin{aligned} \partial^\beta \underline{V} &= \partial^\beta U_\| - \partial^\beta (\underline{w} \nabla \zeta) \\ &= \partial^\beta (\nabla \psi + \nabla^\perp \tilde{\psi}) - (\partial^\beta \underline{w}) \nabla \zeta - \underline{w} \nabla \partial^\beta \zeta - [\partial^\beta, \underline{w}, \nabla \zeta] \\ &= \nabla \partial^\beta \psi - \underline{w}\, \partial^\beta \nabla \zeta - (\partial^\beta \underline{w}) \nabla \zeta + \partial^\beta \nabla^\perp \tilde{\psi} - [\partial^\beta, \underline{w}, \nabla \zeta], \end{aligned}$$

one gets

$$I_{31} = 2 \int_{\mathbb{R}^d} (\nabla \partial^\beta \psi - \underline{w} \nabla \partial^\beta \zeta) \cdot \nabla \partial^\beta \underline{w} + 2 \int_{\mathbb{R}^d} (-\partial^\beta \underline{w} \nabla \zeta - [\partial^\beta, \underline{w}, \nabla \zeta]) \cdot \nabla \partial^\beta \underline{w}$$

(note that the term involving $\nabla^\perp \tilde{\psi}$ vanishes). Using the product estimate

$$\forall\, f, g \in H^{1/2}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f\, \partial_j g \le |\mathfrak{P} f|_2 \, |g|_{H^{1/2}} \ (1 \le j \le d) \tag{3.38}$$

to control the first term, and integrating by parts in the second one, we obtain

$$\begin{aligned} |I_{31}| &\le 2 |\mathfrak{P}(\nabla \partial^\beta \psi - \underline{w} \nabla \partial^\beta \zeta)|_2 \, |\partial^\beta \underline{w}|_{H^{1/2}} \\ &\quad + |\Delta \zeta|_\infty \, |\partial^\beta \underline{w}|_2^2 + \big|\nabla [\partial^\beta, \underline{w}, \nabla \zeta]\big|_2 \, |\partial^\beta \underline{w}|_2 \\ &\lesssim |\mathfrak{P}(\nabla \partial^\beta \psi - \underline{w} \nabla \partial^\beta \zeta)|_2 \, \|\partial^\beta w\|_{H^1} + |\zeta|_{H^N} \, |\Lambda^k \underline{w}|_2^2, \end{aligned}$$

where the last inequality stems from the trace lemma, the assumption that $N \ge 5$, and standard product estimates; we also have that $\|\partial^\beta w\|_{H^1} \lesssim \|\nabla_{X,z} \partial^\beta w\|$ by Poincaré's inequality. For the second term of the right-hand side, we also use the fact that $\Lambda^k w$ vanishes at the bottom to write

$$|\Lambda^k \underline{w}|_2^2 = 2 \int_S \Lambda^k w \, \partial_z \Lambda^k w \le 2 \|\Lambda^k w\|_2 \, \|\partial_z \Lambda^k w\|_2.$$

We therefore get

$$|I_{31}| \le M_N \big(|\mathfrak{P}(\nabla \partial^\beta \psi - \underline{w} \nabla \partial^\beta \zeta)|_2 + \|\Lambda^{k-1} \nabla_{X,z} U\|_2 \big) \|\Lambda^k \nabla_{X,z} U\|_2.$$

For I_{32}, we make the same substitution for $\partial^\beta \underline{V}$ as above to obtain

$$I_{32} = \int_{\mathbb{R}^d} \nabla^\perp\zeta \cdot \nabla[(\partial^\beta\nabla\psi - \underline{w}\nabla\,\partial^\beta\zeta) + \partial^\beta\nabla^\perp\tilde{\psi} - [\partial^\beta, \underline{w}, \nabla\zeta]] \\ \cdot (\partial^\beta\underline{V} - \partial^\beta\underline{w}\nabla\zeta)^\perp + \int_{\mathbb{R}^d}((\nabla^\perp\zeta\cdot\nabla)\nabla\zeta\cdot\nabla^\perp\zeta)|\partial^\beta\underline{w}|^2;$$

proceeding as for I_{31}, we therefore obtain

$$\begin{aligned}|I_{32}| &\lesssim M_N\Big((|\mathfrak{P}(\nabla\,\partial^\beta\psi - \underline{w}\nabla\,\partial^\beta\zeta)|_2 + |\mathfrak{P}\,\partial^\beta\nabla^\perp\tilde{\psi}|_2 + |\mathfrak{P}\Lambda^{k-1}w|_2) \\ &\qquad \times |\partial^\beta\underline{U}|_{H^{1/2}} + |\partial^\beta\underline{w}|_2^2\Big) \\ &\leq M_N(|\mathfrak{P}(\nabla\,\partial^\beta\psi - \underline{w}\nabla\,\partial^\beta\zeta)|_2 + |\mathfrak{P}\,\partial^\beta\nabla^\perp\tilde{\psi}|_2 + \|\Lambda^{k-1}\nabla_{X,z}U\|_2) \\ &\qquad \times \|\Lambda^k\nabla_{X,z}U\|_2.\end{aligned}$$

We therefore need the following lemma.

Lemma 3.13. *Under the assumptions of the proposition, the solution $\tilde{\psi}$ to the equation $\Delta\tilde{\psi} = \underline{\omega}\cdot N$ furnished by Lemma 3.7 satisfies the estimate*

$$|\Lambda^k\nabla\tilde{\psi}|_{H^{1/2}} \leq M_N\|\Lambda^k\omega\|_{2,b}.$$

Proof. Proceeding as in the proof of Lemma 3.7 and using the same notation, we take the L^2-scalar product of the equation $\Delta\tilde{\psi} = \underline{\omega}\cdot N$ with $\Lambda^{2k+1}\tilde{\psi}$ to obtain

$$\begin{aligned}|\Lambda^k\nabla\tilde{\psi}|_{H^{1/2}} &= -\int_{\mathbb{R}^d}\Lambda^{2k+1}\tilde{\psi}_b^{\rm ext}\omega_b\cdot N_b - \int_S(1+\partial_z\sigma)\nabla^\sigma_{X,z}\Lambda^{2k+1}\tilde{\psi}^{\rm ext}\cdot\omega \\ &= -\int_{\mathbb{R}^d}\Lambda^{2k+1}\tilde{\psi}_b^{\rm ext}\omega_b\cdot N_b \\ &\quad - \int_S(1+\partial_z\sigma)(J_\Sigma^{-1})^T\nabla_{X,z}\Lambda^{k+1}\tilde{\psi}^{\rm ext}\cdot\Lambda^k\omega \\ &\quad - \int_S\nabla_{X,z}\Lambda^{k+1}\tilde{\psi}^{\rm ext}\cdot[\Lambda^k, (1+\partial_z\sigma)J_\Sigma^{-1}]\omega.\end{aligned}$$

One readily deduces that

$$|\Lambda^k\nabla\tilde{\psi}|_{H^{1/2}} \lesssim |\Lambda^k\tilde{\psi}_b^{\rm ext}|_{\dot{H}^{1/2}}\,|\Lambda^k\omega_b\cdot N_b|_{H_0^{-1/2}} + M_N\|\nabla_{X,z}\Lambda^{k+1}\tilde{\psi}^{\rm ext}\|_2\,\|\Lambda^k\omega\|_2,$$

and the result follows from the fact that

$$|\Lambda^k\tilde{\psi}_b^{\rm ext}|_{\dot{H}^{1/2}} + \|\Lambda^{k+1}\nabla_{X,z}\tilde{\psi}^{\rm ext}\|_2 \lesssim |\Lambda^k\nabla\tilde{\psi}|_{H^{1/2}}. \qquad \square$$

Using this lemma, we obtain the following bound on I_{32}:

$$|I_{32}| \le M_N(|\mathfrak{P}(\nabla\partial^\beta\psi - \underline{w}\nabla\partial^\beta\zeta)|_2 + \|\Lambda^k\omega\|_{2,b} + \|\Lambda^{k-1}\nabla_{X,z}U\|_2) \\ \times \|\Lambda^k\nabla_{X,z}U\|_2.$$

For I_{33}, we use again the product estimates (3.36) and (3.38), and the trace lemma, to get

$$\begin{aligned}|I_{33}| &\le M_N|\Lambda^{k-3/2}\nabla\underline{U}|_2\,|\Lambda^{1/2}\partial^\beta\underline{U}|_2 \\ &\le M_N\|\Lambda^{k-1}\nabla_{X,z}U\|_2\,\|\Lambda^k\nabla_{X,z}U\|_2.\end{aligned}$$

Gathering the estimates on I_{31}, I_{32} and I_{32}, we finally get

$$|I_3| \le M_N(|\mathfrak{P}(\nabla\partial^\beta\psi - \underline{w}\nabla\partial^\beta\zeta)|_2 + \|\Lambda^k\omega\|_{2,b} + \|\Lambda^{k-1}\nabla_{X,z}U\|_2) \\ \times \|\Lambda^k\nabla_{X,z}U\|_2. \tag{3.39}$$

We can then deduce from (3.34) and (3.35), (3.37) and (3.39) that, for all $0 < k \le N-1$ and all $\beta \in \mathbb{N}^d \setminus \{0\}$ such that $|\beta| \le k$, one has

$$||\partial^\beta\nabla_{X,z}U||_2^2 \le M_N\Big(\sum_{1<|\alpha|\le k+1}|\mathfrak{P}\psi_{(\alpha)}|_2 + \|\Lambda^k\omega\|_{2,b} + \|\Lambda^{k-1}\nabla_{X,z}U\|_2\Big) \\ \times \|\Lambda^k\nabla_{X,z}U\|_2.$$

Summing these inequalities for all $0 < |\beta| \le k$ yields a control on $\|\Lambda^k\nabla_{X,z}U\|_2$, and using the H^1-estimate furnished by Theorem 2.4 for the case $\beta = 0$, the result follows by a finite induction on k. □

Theorem 3.12 provides an $H^{k+1,1}$-estimate of U. We now deduce a more general $H^{k+1,\ell+1}$ estimate of U.

Corollary 3.14. *Let $N \in \mathbb{N}$, $N \ge 5$, and $\zeta \in H^N(\mathbb{R}^d)$. Under the assumptions of Theorem 2.4, there is a unique solution $U \in H^1(S)$ to (3.21); if, moreover, we have $0 \le \ell \le k \le N-1$ and $\omega \in H^{k,\ell}(S)$, then*

$$\|U\|_{H^{k+1,\ell+1}} \le M_N\Big(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha|\le k+1}|\mathfrak{P}\psi_{(\alpha)}|_2 \\ + \|\omega\|_{H^{k,\ell}} + |\Lambda^k(\omega_b\cdot N_b)|_{H_0^{-1/2}}\Big),$$

with $\omega_b = \omega_{|_{z=-H_0}}$ and M_N as in Theorem 3.12.

Proof. We can rewrite the first two equations of (3.21) under the form

$$\left[\begin{pmatrix}\nabla\\0\end{pmatrix} + \tilde{N}\partial_z^\sigma\right]\times U = \omega \quad\text{and}\quad \left[\begin{pmatrix}\nabla\\0\end{pmatrix} + \tilde{N}\partial_z^\sigma\right]\cdot U = 0,$$

with $\tilde{N} = (-\nabla\sigma^T, 1)^T$, and therefore

$$\tilde{N} \cdot \partial_z^\sigma U = -\nabla \cdot V \quad \text{and} \quad \tilde{N} \times \partial_z^\sigma U = \omega + \begin{pmatrix} \nabla^\perp w \\ -\nabla^\perp \cdot V \end{pmatrix}.$$

From the identity

$$\partial_z^\sigma U = \frac{1}{1+|\nabla\sigma|^2}(\tilde{N} \cdot \partial_z^\sigma U \tilde{N} + (\tilde{N} \times \partial_z^\sigma U) \times \tilde{N}),$$

we deduce

$$\partial_z U = \frac{1+\partial_z\sigma}{1+|\nabla\sigma|^2}\left(-(\nabla \cdot V)\tilde{N} + \omega \times \tilde{N} - \begin{pmatrix} \nabla w + (\nabla^\perp \cdot V)\nabla^\perp\sigma \\ \nabla\sigma \cdot \nabla w \end{pmatrix}\right). \tag{3.40}$$

This identity will be used to trade one vertical derivative of U with one horizontal one, using the product estimate provided by the following lemma.

Lemma 3.15. *Let $N \in \mathbb{N}$, $N \geq 5$, and $1 \leq k \leq N-1$. Then, for $f \in H^{N-1}(S)$ and $g \in H^k(S)$, one has*

$$\forall\, 0 \leq \ell \leq k, \quad \|\Lambda^{k-\ell}\partial_z^\ell(fg)\|_2 \lesssim \|f\|_{H^{N-1}}\|g\|_{H^{k,\ell}}.$$

Proof of the lemma. One can decompose $\Lambda^{k-\ell}\partial_z^\ell(fg)$ as a sum of terms of the form

$$\Lambda^{k-\ell}(\partial_z^{\ell'} f\, \partial_z^{\ell-\ell'} g), \quad 0 \leq \ell' \leq \ell.$$

Choosing $t_0 > d/2$ such that $N > 2t_0 + \frac{3}{2}$, these terms can then be bounded from above using the product estimate (3.36):

$$\begin{aligned}|\Lambda^{k-\ell}(\partial_z^{\ell'} f\, \partial_z^{\ell-\ell'} g)(z)|_2 \lesssim{}& |\partial_z^{\ell'} f(z)|_A\, |\partial_z^{\ell-\ell'} g(z)|_B \\ &+ \langle |\partial_z^{\ell'} f(z)|_B\, |\partial_z^{\ell-\ell'} g(z)|_A \rangle_{k-\ell>t_0},\end{aligned}$$

where the term between brackets should be removed from the right-hand side when $k-\ell \leq t_0$, and where $(A,B) = (H^{t_0}, H^{k-\ell})$ or $(A,B) = (H^{k-\ell}, H^{t_0})$. Integrating in z, we easily deduce

$$\begin{aligned}\|\Lambda^{k-\ell}(\partial_z^{\ell'} f\, \partial_z^{\ell-\ell'} g)\|_2 \lesssim{}& \|\partial_z^{\ell'} f(z)\|_{L^a A}\, |\partial_z^{\ell-\ell'} g(z)|_{L^b B} \\ &+ \langle |\partial_z^{\ell'} f(z)|_{L^c B}\, |\partial_z^{\ell-\ell'} g(z)|_{L^d A} \rangle_{k-\ell>t_0},\end{aligned}$$

with (a,b) and (c,d) being equal to $(2,\infty)$ or $(\infty,2)$. The choice of A, B and a, b, c, and d depends on several cases:

(1) If $t_0 + \ell' + \frac{1}{2} \leq N - 1$ and $k - \ell + \ell' + \frac{1}{2} \leq N - 1$, then $(A, B) = (H^{t_0}, H^{k-\ell})$ and $(a, b) = (c, d) = (\infty, 2)$.

(2) If $t_0 + \ell' + \frac{1}{2} \leq N - 1$ and $k - \ell + \ell' + \frac{1}{2} > N - 1$ (and therefore $k = N - 1$, $\ell = \ell'$), then $(A, B) = (H^{t_0}, H^{k-\ell})$ and $(a, b) = (\infty, 2)$, $(c, d) = (2, \infty)$.

(3) If $t_0 + \ell' + \frac{1}{2} > N - 1$ and $k - \ell + \ell' + \frac{1}{2} \leq N - 1$, then $(A, B) = (H^{k-\ell}, H^{t_0})$ and $(a, b) = (\infty, 2)$ (there is no need to specify (c, d) since one then has $k - \ell < t_0$).

(4) If $t_0 + \ell' + \frac{1}{2} > N - 1$ and $k - \ell + \ell' + \frac{1}{2} > N - 1$, then $(A, B) = (H^{k-\ell}, H^{t_0})$ and $(a, b) = (c, d) = (2, \infty)$.

The results then follow from the continuous embedding $L^\infty H^r \leq H^{r+1/2,1}$ ($s \in \mathbb{R}$; see, e.g., Proposition 2.13 of [39]). For instance, in the first case, this yields

$$\begin{aligned}
\|\Lambda^{k-\ell}(\partial_z^{\ell'} f\, \partial_z^{\ell-\ell'} g)\|_2 &\lesssim \|f\|_{H^{t_0+\ell'+1/2,\ell'+1}} \|g\|_{H^{k-\ell',\ell-\ell'}} \\
&\quad + \langle \|f\|_{H^{k-\ell+\ell'+1/2,\ell'+1}} \|g\|_{H^{\ell-\ell'+t_0,\ell-\ell'+1}} \rangle_{k-\ell>t_0} \\
&\lesssim \|f\|_{H^{N-1}} \|g\|_{H^{k,\ell}},
\end{aligned}$$

where we used the assumptions corresponding to the first case. The other cases are treated similarly. ◻

Let $1 \leq \ell \leq k$; taking the $H^{k,\ell}$ norm of (3.40), we obtain with the help of the lemma that

$$\|\partial_z U\|_{H^{k,\ell}} \leq C(\|\sigma\|_{H^N})(\|U\|_{H^{k+1,\ell}} + \|\omega\|_{H^{k,\ell}}),$$

and therefore

$$\|U\|_{H^{k+1,\ell+1}} \leq C(\|\sigma\|_{H^N})(\|U\|_{H^{k+1,\ell}} + \|\omega\|_{H^{k,\ell}}).$$

By a finite induction on ℓ, we readily obtain

$$\|U\|_{H^{k+1,\ell+1}} \leq C(\|\sigma\|_{H^N})(\|U\|_{H^{k+1,1}} + \|\omega\|_{H^{k,\ell}}),$$

and the result then follows from Theorem 3.12. ◻

3.5. Time derivatives. For the analysis of our formulation (2.11) of the water waves equations with vorticity, we shall need to control time derivatives of the solution $U = \mathbb{U}^\sigma[\zeta](\psi, \omega)$ to (3.21). Such a control cannot be obtained with the same methods as the control on space derivatives obtained in the previous section (i.e., by taking time derivatives of U as test functions in (3.29)). We deal with this issue in this section. We first need the following notation.

Notation 3.16. We say that $\zeta \in C([0,T];W^{1,\infty}(\mathbb{R}^d))$ satisfy $(2.6)_T$ if (2.6) is uniformly satisfied by all $\zeta(t,\cdot)$ with $t\in[0,T]$.

Proposition 3.17. *Let $T>0$ and $\zeta \in C^1([0,T];W^{2,\infty}(\mathbb{R}^d))$ satisfy $(2.6)_T$. Let also $\psi \in C^1([0,T];\dot{H}^{3/2}(\mathbb{R}^d))$ and $\omega \in C^1([0,T];L^2(\mathcal{S})^{d+1})$ be such that $(\nabla^\sigma_{X,z}\cdot\omega)(t)=0$ for all $t\in[0,T]$ and $\omega_b\cdot N_b \in C^1([0,T];H_0^{-1/2}(\mathbb{R}^d))$. Then, one has*

$$\partial_t(\mathbb{U}^\sigma[\cdot](\psi,\omega)) = \mathbb{U}^\sigma[\zeta]\Big(\partial_t\psi - \underline{w}\,\partial_t\zeta + \frac{\nabla}{\Delta}\cdot(\underline{\omega}_h^\perp\,\partial_t\zeta),\partial_t^\sigma\omega\Big) + \partial_t\sigma\,\partial_z^\sigma\mathbb{U}^\sigma[\zeta](\psi,\omega),$$

where $(\underline{V}^T,\underline{w})^T = \mathbb{U}^\sigma[\zeta](\psi,\omega)_{|_{z=0}}$.

Proof. The main ingredient in the proof is the following identity:

$$\delta(\partial_j^\sigma f) = \partial_j^\sigma(\delta f - \delta\sigma\,\partial_z^\sigma f) + \delta\sigma\,\partial_z^\sigma\,\partial_j^\sigma f \quad (j=x,y,z), \tag{3.41}$$

where δ can be any linearization operator ($\delta=\partial_t$ here). Note that the quantity $\delta f - \delta\sigma\,\partial_z^\sigma f$ is called Alinhac's good unknown after [7]. Its role in the water waves equations was noticed in [38], but it was in [6] that its interpretation as Alinhac's good unknown was understood (see also the discussion in [47]).

Decomposing $\mathbb{U}^\sigma[\zeta](\psi,\omega)$ as in Remark 3.11, we are led to compute the time derivatives of $\mathbb{U}^\sigma_I[\zeta]\psi$ and $\mathbb{U}^\sigma_{II}[\zeta]\omega$:

Computation of $\partial_t\mathbb{U}_I[\zeta]\psi$. Recalling $\mathbb{U}^\sigma_I[\zeta]\psi = \nabla^\sigma_{X,z}\varphi$ with φ solving (3.27), we have, according to (3.41),

$$\partial_t\mathbb{U}^\sigma_I[\cdot]\psi = \nabla^\sigma_{X,z}(\partial_t\varphi - \partial_t\sigma\,\partial_z^\sigma\varphi) + \partial_t\sigma\,\partial_z^\sigma\nabla^\sigma_{X,z}\varphi.$$

On the other hand, and after remarking that

$$(1+\partial_z\sigma)\nabla^\sigma_{X,z}\cdot\nabla^\sigma_{X,z} = \nabla_{X,z}\cdot P(\Sigma)\nabla_{X,z},$$

we can differentiate (3.27) with respect to time to obtain

$$\begin{cases} \nabla_{X,z}\cdot P(\Sigma)\nabla_{X,z}(\partial_t\varphi - \partial_t\sigma\,\partial_z^\sigma\varphi) = 0 & \text{in } \mathcal{S}, \\ (\partial_t\varphi - \partial_t\sigma\,\partial_z^\sigma\varphi)_{|_{z=0}} = (\partial_t\varphi - \partial_t\zeta\,\partial_z^\sigma\varphi)_{|_{z=0}}, \\ \partial_z(\partial_t\varphi - \partial_t\sigma\,\partial_z^\sigma\varphi)_{|_{z=-H_0}} = 0, \end{cases}$$

where $P(\Sigma) = (1+\partial_z\sigma)J_\Sigma^{-1}(J_\Sigma^{-1})^T$, and where we used the fact that $\partial_t\sigma_{z=0} = \partial_t\zeta$ and $\partial_t\sigma_{|_{z=-H_0}} = 0$. It follows that

$$\partial_t\mathbb{U}^\sigma_I[\cdot]\psi = \mathbb{U}^\sigma_I[\zeta](\partial_t\psi - \underline{w}_I\,\partial_t\zeta) + \partial_t\sigma\,\partial_z^\sigma\mathbb{U}^\sigma_I[\zeta]\psi, \tag{3.42}$$

where $\underline{w}_I$ is the vertical component of $\mathbb{U}^\sigma_I[\zeta]\psi$ evaluated at the surface.

Computation of $\partial_t \mathbb{U}_{II}[\zeta]\omega$. Recalling that $\mathbb{U}^\sigma_{II}[\zeta]\omega = \mathrm{curl}^\sigma A$ with A solving (3.28), we have, according to (3.41),

$$\partial_t \mathbb{U}^\sigma_{II}[\cdot]\omega = \mathrm{curl}^\sigma(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A) + \partial_t\sigma\,\partial^\sigma_z \mathrm{curl}^\sigma A.$$

Differentiating (3.28) with respect to time, we also have

$$\begin{cases} \mathrm{curl}^\sigma \mathrm{curl}^\sigma(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A) = \partial_t\omega - \partial_t\sigma\,\partial^\sigma_z\omega, \\ \mathrm{div}^\sigma(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A) = 0, \end{cases}$$

inside the flat strip $\mathcal{S}$, together with the boundary conditions (with the notation $U_{II} = (V_{II}^T, w_{II})^T := \mathrm{curl}^\sigma A$)

$$\begin{cases} N_b \times (\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=-H_0} = 0, \\ N\cdot(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=0} = \nabla\partial_t\zeta\cdot\underline{A}_h - \partial_t\sigma N\cdot\partial^\sigma_z A_{|z=0}, \\ (\mathrm{curl}^\sigma(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=0})_{\parallel} = \nabla^\perp\partial_t\tilde{\psi} - \partial_t\zeta(\partial_z U_{II})_{\parallel} - \underline{w}_{II}\nabla\partial_t\zeta, \\ N_b\cdot\mathrm{curl}^\sigma(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=-H_0} = 0. \end{cases}$$

In order to simplify the boundary conditions, let us observe that when evaluated at the surface, the equations $\mathrm{div}^\sigma A = 0$ and $\mathrm{curl}^\sigma U_{II} = \omega$ give

$$N\cdot\partial^\sigma_z A_{|z=0} = -\nabla\cdot\underline{A}_h \quad\text{and}\quad -(\partial^\sigma_z U_{II})_{\parallel} = -\nabla\underline{w}_{II} + \underline{\omega}_h^\perp,$$

and that

$$\begin{aligned} \Delta\partial_t\tilde{\psi} &= \partial_t\underline{\omega}\cdot N - \underline{\omega}_h\cdot\nabla\partial_t\zeta \\ &= (\partial_t\underline{\omega} - \partial_t\sigma\,\partial^\sigma_z\omega_{|z=0})\cdot N + (\partial^\sigma_z\omega)_{|z=0}\cdot N\,\partial_t\zeta - \underline{\omega}_h\cdot\nabla\partial_t\zeta \\ &= (\partial_t\underline{\omega} - \partial_t\sigma\,\partial^\sigma_z\omega_{|z=0})\cdot N - \nabla\cdot(\underline{\omega}_h\,\partial_t\zeta), \end{aligned}$$

where we used the fact that $\mathrm{div}^\sigma\omega = 0$ to derive the last equation. It follows that

$$\nabla^\perp\partial_t\tilde{\psi} = \frac{\nabla^\perp}{\Delta}((\partial_t\underline{\omega} - \partial_t\sigma\,\partial^\sigma_z\omega_{|z=0})\cdot N) - \Pi_\perp(\underline{\omega}_h^\perp\,\partial_t\zeta),$$

where we recall that $\Pi = \nabla\nabla^T/\Delta$ and $\Pi_\perp = \nabla^\perp(\nabla^\perp)^T/\Delta$ are, respectively, the orthogonal projectors onto gradient and orthogonal gradient vector fields. These three identities imply that the boundary conditions simplify into

$$\begin{cases} N_b \times (\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=-H_0} = 0, \\ N\cdot(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=0} = \nabla\cdot(\partial_t\zeta\cdot\underline{A}_h), \\ (\mathrm{curl}^\sigma(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=0})_{\parallel} \\ \qquad = \Pi(\underline{\omega}_h^\perp\,\partial_t\zeta) - \nabla(\underline{w}_{II}\,\partial_t\zeta) + \dfrac{\nabla^\perp}{\Delta}(\partial^\sigma_t\omega_{|\mathrm{surf}}\cdot N), \\ N_b\cdot\mathrm{curl}^\sigma(\partial_t A - \partial_t\sigma\,\partial^\sigma_z A)_{|z=-H_0} = 0. \end{cases}$$

Let us now decompose $(\partial_t A - \partial_t\sigma\, \partial_z^\sigma A)$ into

$$(\partial_t A - \partial_t\sigma\, \partial_z^\sigma A) = B + \nabla_{X,z}^\sigma \varphi,$$

where φ solves

$$\begin{cases} \nabla_{X,z} \cdot P(\Sigma)\nabla_{X,z}\varphi = 0 & \text{in } S, \\ N \cdot \nabla_{X,z}^\sigma{}_{|z=0}\varphi = \nabla \cdot (\partial_t \zeta \cdot \underline{A}_h), \\ N_b \cdot \nabla_{X,z}^\sigma{}_{|z=-H_0}\varphi = 0, \end{cases}$$

and where B solves therefore the same equations as $\partial_t A - \partial_t\sigma\, \partial_z^\sigma A$, but where the second boundary condition is now homogeneous. It follows that

$$\begin{aligned} \operatorname{curl}^\sigma (\partial_t A - \partial_t\sigma\, \partial_z^\sigma A) &= \operatorname{curl}^\sigma B \\ &= \mathbb{U}^\sigma[\zeta]\left(\frac{\nabla}{\Delta} \cdot (\underline{\omega}_h^\perp\, \partial_t\zeta) - \underline{w}_{II}\, \partial_t\zeta, \partial_t^\sigma \omega\right), \end{aligned}$$

and therefore

(3.43)
$$\partial_t \mathbb{U}_{II}^\sigma[\cdot]\omega = \mathbb{U}^\sigma[\zeta]\left(\frac{\nabla}{\Delta} \cdot (\underline{\omega}_h^\perp\, \partial_t\zeta) - \underline{w}_{II}\, \partial_t\zeta, \partial_t^\sigma \omega\right) + \partial_t\sigma\, \partial_z^\sigma\, \mathbb{U}_{II}^\sigma[\zeta]\omega.$$

The proposition is then a direct consequence of (3.26), (3.42), and (3.43). □

The proposition also allows us to derive the following H^{N-1} control on $\partial_t U$.

Corollary 3.18. *Let the assumptions of Proposition 3.17 be satisfied, and let also $N \in \mathbb{N}$, $N \geq 5$. Then, one has*

$$\begin{aligned} \|\partial_t U\|_{H^{N-1,1}} \leq C(M_N, |\partial_t\zeta|_{H^{N-1}})\Big(&|\nabla\, \partial_t\psi|_{H^{1/2}} + \sum_{1<|\alpha|\leq N-1} |\mathfrak{P}\, \partial_t\psi_{(\alpha)}|_2 \\ &+ \|\Lambda^{N-2}\, \partial_t\omega\|_{2,b} + |\nabla\psi|_{H^{1/2}} + \sum_{1<|\alpha|\leq N} |\mathfrak{P}\psi_{(\alpha)}|_2 \\ &+ \|\omega\|_{H^{N-1,1}} + |\omega_b \cdot N_b|_{H_0^{-1/2}}\Big). \end{aligned}$$

Proof. Introducing the notation

$$\psi^t = \partial_t\psi - \underline{w}\, \partial_t\zeta + \frac{\nabla}{\Delta} \cdot (\underline{\omega}_h^\perp\, \partial_t\zeta)$$

and

$$\psi_{(\alpha)}^t = \partial^\alpha\psi^t - \mathrm{w}^\sigma[\zeta](\psi^t, \partial_t^\sigma\omega)_{|z=0}\, \partial^\alpha\zeta$$

(where $\alpha \in \mathbb{N}^d$ and $\mathrm{w}^\sigma[\zeta]$ is the vertical component of the mapping $\mathbb{U}^\sigma[\zeta]$ defined in Notation 3.10), we get from Proposition 3.17 that

$$\partial_t U = \mathbb{U}^\sigma[\zeta](\psi^t, \partial_t^\sigma\omega) + \partial_t\sigma\, \partial_z^\sigma U = A + B, \tag{3.44}$$

and we therefore turn to control A and B as follows.

Control of A. Using Proposition 3.12,

$$\|A\|_{H^{N-1,1}} \leq M_N\Big(|\nabla\psi^t|_{H^{1/2}} + \sum_{1<|\alpha|\leq N-1} |\mathfrak{P}\psi^t_{(\alpha)|}|_2 + \|\Lambda^{N-2}\partial_t^\sigma\omega\|_{2,b}\Big). \tag{3.45}$$

We now make the following observations:

$$\begin{aligned}|\nabla\psi^t|_{H^{1/2}} &\leq |\nabla\partial_t\psi|_{H^{1/2}} + C(|\partial_t\zeta|_{W^{2,\infty}})(|\underline{w}|_{H^{3/2}} + |\underline{\omega}_h|_{H^{1/2}})\\ &\leq |\nabla\partial_t\psi|_{H^{1/2}} + C(|\partial_t\zeta|_{W^{2,\infty}})(\|U\|_{H^{2,1}} + \|\omega\|_{H^{1,1}})\end{aligned} \tag{3.46}$$

(the second inequality stemming from the trace lemma); and, for all $\alpha \in \mathbb{N}^d$, $|\alpha| \leq N-1$,

$$\begin{aligned}\psi^t_{(\alpha)} &= \partial_t\psi_{(\alpha)} + (\partial_t\underline{w} - \mathrm{w}^\sigma[\zeta](\psi^t, \partial_t^\sigma\omega)_{|_{z=0}})\,\partial^\alpha\zeta - [\partial^\alpha, \underline{w}]\,\partial_t\zeta\\ &\quad + \partial^\alpha\frac{\nabla}{\Delta}\cdot(\underline{\omega}_h^\perp\,\partial_t\zeta)\\ &= \partial_t\psi_{(\alpha)} + (\partial_t\zeta\,\partial_z^\sigma w_{|_{z=0}})\,\partial^\alpha\zeta - [\partial^\alpha, \underline{w}]\,\partial_t\zeta + \partial^\alpha\frac{\nabla}{\Delta}\cdot(\underline{\omega}_h^\perp\,\partial_t\zeta),\end{aligned}$$

where the formula of Proposition 3.17 has been used to derive the last identity. By standard product estimates and the trace lemma, this yields

$$\begin{aligned}|\mathfrak{P}\psi^t_{(\alpha)}|_2 &\leq |\mathfrak{P}\,\partial_t\psi_{(\alpha)}|_2 + C(|\zeta|_{H^N}, |\partial_t\zeta|_{H^{N-3/2}})\\ &\qquad\times(|\partial_z^\sigma w_{|_{z=0}}|_{H^{3/2}} + |\underline{w}|_{H^{N-1/2}} + |\underline{\omega}_h^\perp|_{H^{N-3/2}})\\ &\leq |\mathfrak{P}\,\partial_t\psi_{(\alpha)}|_2 + C(|\zeta|_{H^N}, |\partial_t\zeta|_{H^{N-3/2}})\\ &\qquad\times(\|U\|_{H^{N,1}} + \|\omega\|_{H^{N-1,1}}).\end{aligned} \tag{3.47}$$

Using Proposition 3.12 to control $\|U\|_{H^{N,1}}$, from (3.45), (3.46), and (3.47) we get

$$\begin{aligned}&\|A\|_{H^{N-1,1}}\\ &\quad\leq M_N\Big(|\nabla\partial_t\psi|_{H^{1/2}} + \sum_{1<|\alpha|\leq N-1}|\mathfrak{P}\,\partial_t\psi_{(\alpha)}|_2 + \|\Lambda^{N-2}\partial_t^\sigma\omega\|_{2,b}\Big)\\ &\qquad + C(|\zeta|_{H^N}, |\partial_t\zeta|_{H^{N-3/2}})\\ &\qquad\times\Big(|\nabla\psi|_{H^{1/2}} + \sum_{1<|\alpha|\leq N}|\mathfrak{P}\psi_{(\alpha)}|_2 + \|\omega\|_{H^{N-1,1}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big).\end{aligned}$$

Control of B. We get from the product estimates of Lemma 3.15 that

$$\begin{aligned}\|B\|_{H^{N-1,1}} &\leq \|\partial_t\sigma\|_{H^{N-1,1}}\,\|\partial_z^\sigma U\|_{H^{N-1,1}} \leq C(M_N, |\partial_t\zeta|_{H^{N-1}})\\ &\qquad\times\Big(|\nabla\psi|_{H^{1/2}} + \sum_{1<|\alpha|\leq N}|\mathfrak{P}\psi_{(\alpha)}|_2 + \|\omega\|_{H^{N-1,1}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big),\end{aligned}$$

where we used Corollary 3.14 to derive the second inequality.

The estimate of the corollary is then a consequence of these two controls and of the observation that

$$\|\Lambda^{N-2}\partial_t^\sigma \omega\|_{2,b} \leq \|\Lambda^{N-2}\partial_t\omega\|_{2,b} + C(M_N, |\partial_t\zeta|_{H^{N-3/2}})\|\omega\|_{H^{N-1,1}}. \qquad \square$$

Another corollary of Proposition 3.17 is that $\mathbb{U}^\sigma[\zeta](\psi,\omega)$ has a Lipschitz dependence on its coefficients.

Corollary 3.19. *Let $N \in \mathbb{N}$, $N \geq 5$. Let also $(\zeta_j,\psi_j,\omega_j) \in H^N(\mathbb{R}^d) \times \dot{H}^N(\mathbb{R}^d) \times H^{N-2}(S)$ be such that $\nabla^{\sigma_j}_{X,z}\cdot\omega_j = 0$ for $j = 1,2$. Then, one has*

$$\begin{aligned}\|\mathbb{U}^{\sigma_2}[\zeta_2](\psi_2,\omega_2) - \mathbb{U}^{\sigma_1}[\zeta_1](\psi_1,\omega_1)\|_{H^{N-2}} &\leq C(|\zeta|_{H^N}, |\psi|_{\dot{H}^N}, \|\omega\|_{H^{N-2}})\\ &\quad\times\left(|\zeta_2-\zeta_1|_{H^N} + |\psi_2-\psi_1|_{\dot{H}^N} + \|\omega_2-\omega_1\|_{H^{N-2}}\right).\end{aligned}$$

Proof. Let us define time dependent functions on $[0,1]$ as

$$\forall\, t\in[0,1],\quad \zeta^{(t)} = \zeta_1 + t(\zeta_2-\zeta_1),\ \psi^{(t)} = \psi_1 + t(\psi_2-\psi_1).$$

For every value of $\zeta^{(t)}$, one can define an explicit diffeomorphism $\Sigma^{(t)}$ as in Section 3.3; we then define

$$\omega^{(t)} = (\boldsymbol{\omega}_1 + t(\boldsymbol{\omega}_2-\boldsymbol{\omega}_1))\circ\Sigma^{(t)}, \quad \text{with } \boldsymbol{\omega}_j = \omega_j\circ\Sigma_j^{-1}\ (j=1,2);$$

by construction, one has $\nabla^{\sigma^{(t)}}_{X,z}\cdot\omega^{(t)} = 0$. We can therefore write

$$\mathbb{U}^{\sigma_2}[\zeta](\psi_2,\omega_2) - \mathbb{U}^{\sigma_1}[\zeta_1](\psi_1,\omega_1) = \int_0^1 \partial_t(\mathbb{U}^{(\sigma^{(t)})}[\zeta^{(t)}])(\psi^{(t)},\omega^{(t)})\,\mathrm{d}t$$

and use Proposition 3.17 to express the integrand in terms of the time derivatives of $\zeta^{(t)}$, $\psi^{(t)}$, and $\omega^{(t)}$. The desired estimate is then a direct consequence of Corollary 3.14. $\square$

3.6. Almost-incompressibility of the good unknown. We have already mentioned in the proof of Proposition 3.17 the role of Alinhac's good unknown. As we shall see later, it shall also play an important role in the energy estimates where we shall typically have to control terms of the form

$$(\varphi,(\partial^\alpha\underline{U})\cdot N) = \int_S \varphi^\dagger(\nabla^\sigma_{X,z}\cdot\partial^\alpha U)(1+\partial_z\sigma) + \int_S \nabla_{X,z}\varphi^\dagger\cdot\partial^\alpha U(1+\partial_z\sigma),$$

where $\varphi^\dagger$ is defined in S and satisfies $\varphi^\dagger_{|_{z=0}} = \varphi$, and with $U = \mathbb{U}^\varphi[\zeta](\psi,\omega)$. When $\alpha = 0$, the first term in the right-hand side vanishes since U is incompressible by construction. When $\alpha \neq 0$, this is no longer true, and this component cannot be controlled by the $L^2(S)$-norm of $|\alpha|$ derivatives of U. It is, however, possible to get rid of this difficulty by working with the *good unknown* $U_{(\alpha)}$ instead of $\partial^\alpha U$ (and this is actually what we do in Section 4.3 below), with

$$\forall\, \alpha \in \mathbb{N}^d \setminus \{0\}, \quad U_{(\alpha)} = \partial^\alpha U - \partial^\alpha \sigma\, \partial_z^\sigma U, \tag{3.48}$$

while for $\alpha = 0$, we simply take $U_{(0)} = U$. Indeed, the good unknown is almost incompressible, as noted in [47] and stated in the proposition below. We also give an estimate on the curl of the good unknown.

Proposition 3.20. *Let $N \in \mathbb{N}$, $N \geq 5$, and $\zeta \in H^N(\mathbb{R}^d)$. Under the assumptions of Theorem 2.4, and denoting by $\mathbf{U} = \mathbb{U}[\zeta](\psi, \boldsymbol{\omega})$ the solution to (2.7), and by $U = \mathbf{U} \circ \Sigma$ its straightened version, we have, for all $\alpha \in \mathbb{N}^d$, $0 < |\alpha| \leq N$,*

$$\begin{aligned} &\|\nabla_{X,z}^\sigma \cdot U_{(\alpha)}\|_2 + \|\nabla_{X,z}^\sigma \times U_{(\alpha)} - \partial^\alpha \omega\|_2 \\ &\quad\leq M_N \Big(|\nabla\psi|_{H^{1/2}} + \sum_{1<|\alpha'|\leq|\alpha|} |\mathfrak{P}\psi_{(\alpha')}|_2 + \|\omega\|_{H^{|\alpha|-1}} + |\omega_b \cdot N_b|_{H_0^{-1/2}} \Big), \end{aligned}$$

with M_N, ψ_α, $U_{(\alpha)}$ as in Theorem 2.4, (3.32), and (3.48), respectively.

Proof. For the estimate on the divergence, we reproduce here the proof of [47], which is based on the following identity, with $i = x, y, z$:

$$\partial^\alpha \partial_i^\sigma f = \partial_i^\sigma \partial^\alpha f - \partial_z^\sigma f\, \partial_i^\sigma \partial^\alpha \sigma + C_i(f),$$

and where

$$\begin{aligned} C_i(f) = &- \Big[\partial^\alpha, \frac{\partial_i \sigma}{1+\partial_z\sigma}, \partial_z f\Big] - \Big[\partial^\alpha, \partial_i\sigma, \frac{1}{1+\partial_z\sigma}\Big]\partial_z f \\ &- \partial_i\sigma \Big[\partial^\alpha\Big(\frac{1}{1+\partial_z\sigma}\Big) + \frac{\partial_z\partial^\alpha\sigma}{(1+\partial_z\sigma)^2}\Big]\partial_z f. \end{aligned}$$

It follows that

$$0 = \partial^\alpha \nabla_{X,z}^\sigma \cdot U = \nabla_{X,z} \cdot U_{(\alpha)} + C(U),$$

with $C(U) = C_1(V_1) + C_2(V_2) + C_3(w)$. By using the product estimates of Lemma 3.15, we have in particular

$$\|C(U)\|_2 \leq C(\|\sigma\|_{H^N}, \|U\|_{H^N}),$$

and the result is therefore a consequence of Corollary 3.14. The estimate on the vorticity is obtained along the same lines. ∎

We can deduce the following property that, together with its proof, will play an important role in the derivation of the energy estimates in Section 4.3 below.

Corollary 3.21. *Under the assumptions of Proposition 3.20, one has, for all $\varphi \in H^{1/2}(\mathbb{R}^d)$, and for all $k = x, y$, $|\beta| \le N-1$, and α such that $\partial^\alpha = \partial_k \partial^\beta$,*

$$(\varphi, \partial_k \underline{U}_{(\beta)} \cdot N) \le M_N \Big(|\nabla\psi|_{H^{1/2}} + \sum_{1<|\alpha'|\le|\alpha|} |\mathfrak{P}\psi_{(\alpha')}|_2 + \|\omega\|_{H^{|\alpha|-1}} + |\omega_b \cdot N_b|_{H_0^{-1/2}} \Big) |\varphi|_{H^{1/2}}.$$

Proof. Remarking that, when $\beta \neq 0$,

$$\partial_k U_{(\beta)} = U_{(\alpha)} - \partial^\beta \sigma \, \partial_k \, \partial_z^\sigma U$$

(the adaptations to the case $\beta = 0$ are straightforward), it is enough to prove the estimate of the corollary on $(\varphi, \underline{U}_{(\alpha)} \cdot N)$. Let us first give the following integration by parts formula that will be used several times in the sequel:

$$\int_S (\nabla_{X,z}^\sigma f) \cdot \mathbf{g} h = -\int_S f \nabla_{X,z}^\sigma \cdot \mathbf{g} h + \int_{z=0} f \mathbf{g} \cdot N - \int_{z=-1} f \mathbf{g} \cdot N_b, \tag{3.49}$$

where $h = 1+\zeta$ (note simply that $h = 1+\zeta = 1+\partial_z\sigma$ is the Jacobian determinant of the diffeomorphism $\Sigma : S \to \Omega$, so that this formula is just the pullback in S of the standard integration by parts formula in Ω). It follows from this formula that

$$(\varphi, \underline{U}_{(\alpha)} \cdot N) = \int_S (1 + \partial_z\sigma)\varphi^\dagger \nabla_{X,z}^\sigma \cdot U_{(\alpha)} + \int_S (1 + \partial_z\sigma)\nabla_{X,z}^\sigma \varphi^\dagger \cdot U_{(\alpha)},$$

with $\varphi^\dagger$ as the extension of φ to S given by $\varphi^\dagger = \chi(z|\mathrm{D}|)\varphi$, and with χ a smooth, compactly supported, and even function equal to 1 in a neighborhood of the origin. Consequently,

$$(\varphi, \underline{U}_{(\alpha)} \cdot N) \le C\Big(\frac{1}{h_{\min}}, |\zeta|_{W^{1,\infty}}\Big) \|\varphi^\dagger\|_{H^1} (\|U_{(\alpha)}\|_2 + \|\nabla_{X,z}^\sigma \cdot U_{(\alpha)}\|_2);$$

since $\|\varphi^\dagger\|_{H^1} \lesssim |\varphi|_{H^{1/2}}$, the result follows from Propositions 3.12 and 3.20. ◻

In Section 4.3 below, we shall derive a priori estimates on ω and on $\|\partial_k U_\beta\|_2$ ($k = x, y$, $|\beta| \le N-1$); the corollary below shall play a crucial role in deducing *a priori* estimates on the quantities $|\mathfrak{P}\psi_{(\alpha)}|_2$ ($|\alpha| \le N$) more closely related to the formulation (2.11) of the water waves equations with vorticity.

Corollary 3.22. *Under the assumptions of Proposition 3.20, one has*

$$|\mathfrak{P}\psi_{(\alpha)}|_2 \le M_N \Big(|\mathfrak{P}\psi|_{H^3} + \sum_{\substack{k=x,y \\ 1\le|\beta|\le|\alpha|-1}} \|\partial_k U_{(\beta)}\|_2 + \|\omega\|_{H^{N-1}} + |\omega_b \cdot N_b|_{H_0^{-1/2}} \Big).$$

Proof. Since $|\mathfrak{P}\psi_{(\alpha)}|_2 \le |\mathfrak{P}\psi_{(\beta)}|_{H^1} + |\zeta|_{H^N}\,|\underline{w}|_{W^{1,\infty}}$, we first need an upper bound for $|\mathfrak{P}\psi_{(\beta)}|_{H^1}$. Let us remark that, for all $\beta \in \mathbb{N}^d$, $|\beta| \le N-1$,

$$\begin{aligned} U_{(\beta)\|} &= (\partial^\beta U)_\| - \partial^\beta\zeta(\partial_z^\sigma U)_\| \\ &= \partial^\beta U_\| - \underline{w}\nabla\,\partial^\beta\zeta - [\partial^\beta, \underline{w}, \nabla\zeta] - \partial^\beta\zeta(\partial_z^\sigma U)_\|, \end{aligned}$$

and therefore, substituting $U_\| = \nabla\psi + \nabla^\perp\tilde{\psi}$,

$$U_{(\beta)\|} = \nabla\psi_{(\beta)} + \nabla^\perp\partial^\beta\tilde{\psi} + \nabla\underline{w}\,\partial^\beta\zeta - [\partial^\beta, \underline{w}, \nabla\zeta] - \partial^\beta\zeta(\partial_z^\sigma U)_\|, \tag{3.50}$$

from which we get

$$\begin{aligned} |\mathfrak{P}\psi_{(\beta)}|_{H^1} &\le \left|\mathfrak{P}\frac{\nabla}{\Delta}\cdot U_{(\beta)\|}\right|_{H^1} + |\nabla\underline{w}\,\partial^\beta\zeta - [\partial^\beta, \underline{w}, \nabla\zeta] - \partial^\beta\zeta(\partial_z^\sigma U)_\||_{H^{1/2}} \\ &\le \left|\mathfrak{P}\frac{\nabla}{\Delta}\cdot U_{(\beta)\|}\right|_{H^1} + M_N\|U\|_{H^{|\beta|}}, \end{aligned}$$

the last line being a consequence of the product estimates (3.36) and the trace lemma. By Corollary 3.14, we then get

$$\begin{aligned} |\mathfrak{P}\psi_{(\beta)}|_{H^1} \le{}& \left|\mathfrak{P}\frac{\nabla}{\Delta}\cdot U_{(\beta)\|}\right|_{H^1} \\ &+ M_N\Big(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\beta'|\le|\beta|}|\mathfrak{P}\psi_{(\beta')}|_2 + \|\omega\|_{H^{|\beta|-1}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big). \end{aligned} \tag{3.51}$$

By the trace lemma, one gets a control of $|\mathfrak{P}(\nabla/\Delta)\cdot U_{(\beta)\|}|_{H^1}$ in terms of the H^1-norm of $U_{(\beta)}$. The lemma below shall be used together with Proposition 3.20 to get a control involving only horizontal derivatives of $U_{(\beta)}$, and therefore more adapted to the energy estimates of Section 4.3.

Lemma 3.23. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ satisfy (2.6). Let also $U \in H^1(S)^3$. Then, one has*

$$\begin{aligned} \left|\mathfrak{P}\frac{\nabla}{\Delta}\cdot U_\|\right|_{H^1} \le{}& C\left(\frac{1}{h_{\min}}, H_0, |\zeta|_{W^{2,\infty}}\right) \\ &\times(\|\Lambda U\|_2 + \|\nabla_{X,z}^\sigma\cdot U\|_2 + \|\operatorname{curl}^\sigma U\|_{2,b}). \end{aligned}$$

Proof. Let us denote by u the solution to the boundary problem

$$\begin{cases} \nabla_{X,z}\cdot P(\Sigma)\nabla_{X,z}u = (1+\partial_z\sigma)\nabla_{X,z}^\sigma\cdot U, \\ u_{|z=0} = 0, \\ \mathbf{e}_z\cdot P(\Sigma)\nabla_{X,z}u_{|z=-H_0} = 0, \end{cases} \tag{3.52}$$

recalling that $(1+\partial_z\sigma)\nabla^\sigma_{X,z}\cdot\nabla^\sigma_{X,z} = \nabla_{X,z}\cdot P(\Sigma)\nabla_{X,z}\sigma$. The quantity

$$\tilde{U} = U - \nabla^\sigma_{X,z}u \tag{3.53}$$

solves the div-curl problem

$$\begin{cases} \nabla^\sigma_{X,z}\times\tilde{U} = \nabla^\sigma_{X,z}\times U, \\ \nabla^\sigma_{X,z}\cdot\tilde{U} = 0, \\ \tilde{U}_\parallel = \nabla\left(\dfrac{\nabla}{\Delta}\cdot U_\parallel\right) + \nabla^\perp\left(\dfrac{\nabla^\perp}{\Delta}\cdot U_\parallel\right), \\ \tilde{U}_{|z=-H_0}\cdot N_b = 0, \end{cases}$$

and therefore,

$$\begin{aligned}\tilde{U} &= \mathbb{U}^\sigma[\zeta]\left(\frac{\nabla}{\Delta}\cdot U_\parallel, \nabla^\sigma_{X,z}\times U\right) = \mathbb{U}^\sigma_I[\zeta]\frac{\nabla}{\Delta}\cdot U_\parallel + \mathbb{U}^\sigma_{II}[\zeta]\nabla^\sigma_{X,z}\times U \\ &:= \tilde{U}_I + \tilde{U}_{II}\end{aligned}$$

(using the notation of Section 3.3). One also gets from the irrotational theory (e.g., Proposition 3.19 in [39]) that

$$\left|\mathfrak{P}\frac{\nabla}{\Delta}\cdot U_\parallel\right|_{H^1} \le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}}\right)\|\Lambda\tilde{U}_I\|_2;$$

therefore,

$$\begin{aligned}\left|\mathfrak{P}\frac{\nabla}{\Delta}\cdot U_\parallel\right|_{H^1} &\le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}}\right)(\|\Lambda\tilde{U}\|_2 + \|\Lambda\tilde{U}_{II}\|_2) \\ &\le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}}\right)(\|\Lambda U\|_2 + \|\Lambda\nabla^\sigma_{X,z}u\|_2 + \|\Lambda\tilde{U}_{II}\|_2),\end{aligned}$$

the last line stemming from (3.53). The result follows therefore from the estimates

$$\|\Lambda\nabla^\sigma_{X,z}u\|_2 \le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}}\right)\|\nabla^\sigma_{X,z}\cdot U\|_2,$$

which stems from a standard elliptic estimate on (3.52), and

$$\|\Lambda\tilde{U}_{II}\|_2 \le C\left(\frac{1}{h_{\min}}, H_0, |\zeta|_{W^{2,\infty}}\right)\|\operatorname{curl}^\sigma U\|_{2,b},$$

which is a direct consequence of Theorem 2.4. ◻

Using the lemma with $U = U_{(\beta)}$, we get that

$$\begin{aligned}
&\left|\mathfrak{P}\frac{\nabla}{\Delta}\cdot U_{(\beta)\parallel}\right|_{H^1}\\
&\le C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{2,\infty}}\right)\left(\|\Lambda U_{(\beta)}\|_2 + \|\nabla^{\sigma}_{X,z}\cdot U_{(\beta)}\|_2 + \|\operatorname{curl}^{\sigma} U_{(\beta)}\|_{2,b}\right)\\
&\le M_N\Big(\|\Lambda U_{(\beta)}\|_2 + |\nabla\psi|_{H^{1/2}} + \sum_{1<|\beta'|\le|\beta|}|\mathfrak{P}\psi_{(\beta')}|_2 + \|\omega\|_{H^{|\beta|}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big),
\end{aligned}$$

where Proposition 3.20 has been used to derive the second inequality (without the cancellations obtained by working with the good unknown, the sum in the right-hand side would be over all $1 \le |\beta'| \le |\beta| + 1$, and the induction strategy used below could not be implemented); because of (3.51), the same control holds on $|\mathfrak{P}\psi_{(\beta)}|_{H^1}$. To get the result stated in the corollary, notice that, for all $\alpha \in \mathbb{N}^d$, $\alpha \neq 0$, we can write $\partial^{\alpha} = \partial_k\partial^{\beta}$ for some $k = x, y$ and $|\beta| = |\alpha| - 1$, so that

$$\psi_{(\alpha)} = \partial_k\psi_{(\beta)} + \partial^{\beta}\zeta\,\partial_k\underline{w},$$

and therefore

$$\begin{aligned}
|\mathfrak{P}\psi_{(\alpha)}|_2 &\le |\mathfrak{P}\psi_{(\beta)}|_{H^1} + |\zeta|_{H^N}|\underline{w}|_{W^{2,\infty}}\\
&\le M_N\Big(\|\Lambda U_{(\beta)}\|_2 + |\mathfrak{P}\psi|_{H^3} + \sum_{1<|\beta'|\le|\beta|}|\mathfrak{P}\psi_{(\beta')}|_2\\
&\qquad + \|\omega\|_{H^{N-1}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big),
\end{aligned}$$

where we also used Proposition 3.12 as well as the Sobolev embedding $H^N(S) \subset W^{2,\infty}(S)$ to derive the second line. Since, for all $|\beta| \ge 2$, one can write

$$U_{(\beta)} = \partial_{k'}U_{(\beta')} + \partial_{k'}\partial_z^{\sigma}U\,\partial^{\beta'}\zeta,$$

for some $k' = x, y$, and $\beta' \in \mathbb{N}^d$, one has

$$\begin{aligned}
\|\Lambda U_{(\beta)}\|_2 &\le \sum_{k=x,y,\,1\le|\beta'|\le|\beta|}\|\partial_k U_{(\beta')}\|_2 + M_N\|U\|_{H^{|\beta|}}\\
&\le M_N\Big(\sum_{k=x,y,\,1\le|\beta'|\le|\beta|}\|\partial_k U_{(\beta')}\|_2 + \sum_{1<|\beta'|\le|\beta|}|\mathfrak{P}\psi_{(\beta')}|_2\\
&\qquad + \|\omega\|_{H^{N-1}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big),
\end{aligned}$$

the last line stemming from Corollary 3.14. We finally deduce that

$$\begin{aligned}
|\mathfrak{P}\psi_{(\alpha)}|_2 \le M_N\Big(&|\mathfrak{P}\psi|_{H^3} + \sum_{k=x,y,\,1\le|\beta'|\le|\beta|}\|\partial_k U_{(\beta')}\|_2\\
&+ \sum_{1<|\beta'|\le|\beta|}|\mathfrak{P}\psi_{(\beta')}|_2 + \|\omega\|_{H^{N-1}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big).
\end{aligned}$$

It follows from a finite induction that

$$|\mathfrak{P}\psi_{(\alpha)}|_2 \leq M_N\Big(|\mathfrak{P}\psi|_{H^3} + \sum_{k=x,y,\,1\leq|\beta'|\leq|\beta|} \|\partial_k U_{(\beta')}\|_2 + \|\omega\|_{H^{N-1}} + |\omega_b\cdot N_b|_{H_0^{-1/2}}\Big),$$

which is the desired result. $\square$

4. Well-Posedness of the Hamiltonian Water Waves Equations (2.11)

We prove in this section the local well-posedness of the water waves equations (2.11). Let us introduce here the energy that shall be used to obtain this result:

$$\mathcal{E}^N(\zeta,\psi,\omega) := \frac{1}{2}|\zeta|^2_{H^N} + \frac{1}{2}|\mathfrak{P}\psi|^2_{H^3} + \frac{1}{2}\sum_{\alpha\in\mathbb{N}^d,\,0<|\alpha|\leq N} |\mathfrak{P}\psi_{(\alpha)}|_2^2 + \frac{1}{2}||\omega||^2_{H^{N-1}} + \frac{1}{2}|\omega_b\cdot N_b|_{H_0^{-1/2}}, \tag{4.1}$$

where we recall that $\psi_{(\alpha)} = \partial^\alpha\psi - \underline{w}\,\partial^\alpha\zeta$, that $\mathfrak{P}$ is defined in (3.31), and $H_0^{-1/2}(\mathbb{R}^d)$ in (1.8). With this energy we associate the functional space E_T^N defined for all $T\geq 0$ as

$$E_T^N = \Big\{(\zeta,\psi,\omega)\in C([0,T];H^2(\mathbb{R}^d)\times\dot{H}^2(\mathbb{R}^d)\times H^2(S)^3)\mid \mathcal{E}^N((\zeta,\psi,\omega)(\cdot))\in L^\infty([0,T])\Big\}. \tag{4.2}$$

We also denote by $\mathfrak{m}^N(\zeta,\psi,\omega)$ any constant of the form

$$\mathfrak{m}^N(\zeta,\psi,\omega) = C\left(\frac{1}{h_{\min}}, H_0, \mathcal{E}^N(\zeta,\psi,\omega)\right). \tag{4.3}$$

4.1. **A priori** *energy estimates for the vorticity.*

We recall that the equation for the vorticity $\boldsymbol{\omega}$ is given by

$$\partial_t\boldsymbol{\omega} + (\mathbf{U}\cdot\nabla_{X,z})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla_{X,z})\mathbf{U} \quad \text{in } \Omega_t,$$

with $\mathbf{U} = \mathbb{U}[\zeta](\psi,\boldsymbol{\omega})$ and $\mathbb{U}[\zeta](\psi,\boldsymbol{\omega})$ as in Definition 2.5. As explained in Remark 2.7, the vorticity $\boldsymbol{\omega}$ is studied through the *straightened vorticity* $\omega = \boldsymbol{\omega}\circ\Sigma$, where $\Sigma : S\to\Omega_t$ is the diffeomorphism introduced in Section 3.3. Written in terms of ω rather than $\boldsymbol{\omega}$, using the notation (3.24), and recalling the notation $\mathbb{U}^\sigma[\zeta](\psi,\omega) = \mathbf{U}\circ\Sigma$, the vorticity equation becomes

$$\partial_t^\sigma\omega + \mathbb{U}^\sigma[\zeta](\psi,\omega)\cdot\nabla^\sigma_{X,z}\omega = \omega\cdot\nabla^\sigma_{X,z}\mathbb{U}^\sigma[\zeta](\psi,\omega) \quad \text{in } S. \tag{4.4}$$

Solving this equation on a domain with boundaries like the flat strip S requires, in general, boundary conditions on the vorticity. We show below that such boundary conditions are not needed to derive *a priori* estimates[7] if the kinematic equation holds.

Proposition 4.1. *Let $N \in \mathbb{N}$, $N \geq 5$, $T > 0$, and $(\zeta, \psi, \omega) \in E_T^N$ be such that (2.6)$_T$ and (4.4) hold on $[0, T]$. If, moreover, $\partial_t \zeta - \mathbb{U}^\sigma[\zeta](\psi, \omega)_{|z=0} \cdot N = 0$ on $[0, T]$, then the following estimate holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}(||\omega||_{H^k}^2 + |\omega_b \cdot N_b|_{H_0^{-1/2}}) \leq \mathfrak{m}^N(\zeta, \psi, \omega),$$

with $\mathfrak{m}^N(\zeta, \psi, \omega)$ as defined in (4.3), and $H_0^{-1/2}(\mathbb{R}^d)$ given by (1.8).

Proof. The first step is to rewrite (4.4) under the form

$$\partial_t \omega + \mathbb{V}^\sigma[\zeta](\psi, \omega) \cdot \nabla \omega + \mathbb{a}[\zeta](\psi, \omega)\, \partial_z \omega = \omega \cdot \nabla_{X,z}^\sigma \mathbb{U}^\sigma[\zeta](\psi, \omega) \tag{4.5}$$

with, denoting $\tilde{N} = (-\nabla \sigma^T, 1)^T$ (so that $\tilde{N}_{|z=0} = N$),

$$\begin{aligned} \mathbb{a}[\zeta](\psi, \omega) &= \frac{1}{1 + \partial_z \sigma}(\mathbb{U}^\sigma[\zeta](\psi, \omega) \cdot \tilde{N} - \partial_t \sigma) \\ &= \frac{1}{1 + \partial_z \sigma}\left(\mathbb{U}^\sigma[\zeta](\psi, \omega) \cdot \tilde{N} - \frac{z + H_0}{H_0}\mathbb{U}^\sigma[\zeta](\psi, \omega)_{z=0} \cdot N\right), \end{aligned}$$

where we used the definition of σ (see Section 3.3) and the fact that the kinematic equation is satisfied to derive the second equality. Denoting $U = \mathbb{U}^\sigma[\zeta](\psi, \omega)$, $a = \mathbb{a}[\zeta](\psi, \omega)$, etc., the first step is to perform an L^2 *a priori* estimate on the equation

$$\partial_t \omega + V \cdot \nabla \omega + a\, \partial_z \omega = f, \tag{4.6}$$

with $f \in L^2(S)$. Taking the L^2 scalar product of this equation with ω, we get

$$\frac{1}{2}\partial_t ||\omega||_{L^2}^2 - \frac{1}{2}\int_S (\nabla \cdot V + \partial_z a)|\omega|^2 = \int_S f \cdot \omega,$$

where we used the fact that a vanishes at the bottom and at the surface. We deduce that

$$\partial_t ||\omega||_2^2 \lesssim \|U\|_{W^{1,\infty}}\, ||\omega||_2^2 + \|\omega\|_2\, \|f\|_2. \tag{4.7}$$

In order to get an L^2 *a priori* estimate for (4.4), we must take $f = \omega \cdot \nabla_{X,z}^\sigma U$, and therefore

$$\|f\|_{L^2} \leq C(|\zeta|_{W^{1,\infty}})\|\omega\|_2\, \|U\|_{W^{1,\infty}}.$$

[7] The issue of constructing a solution is a bit more complex than the derivation of a priori estimates; this aspect will be addressed in the proof of Theorem 4.7.

We can therefore deduce from (4.7) and Proposition 3.12 that

$$\frac{\mathrm{d}}{\mathrm{d}t}||\omega||_2^2 \leq \mathfrak{m}^N(\zeta,\psi,\omega).$$

We now prove higher-order *a priori* estimates. For all $j \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ with $|\beta| + j \leq N-1$, we get, after applying $\partial^\beta \partial_z^j$ to (4.5), that

$$\partial_t \partial^\beta \partial_z^j \omega + V \cdot \nabla \partial^\beta \partial_z^j \omega + a\, \partial_z \partial^\beta \partial_z^j \omega = f_{\beta,j},$$

with

$$f_{\beta,j} = -[\partial^\beta \partial_z^j, V]\,\partial_z \omega - [\partial^\beta \partial_z^j, a]\,\partial_z \omega + \partial^\beta \partial_z^j(\omega \cdot \nabla_{X,z}^\sigma U).$$

This equation is of the form (4.6), and we therefore get from (4.7) that

$$\partial_t ||\partial^\beta \partial_z^j \omega||_{L^2_\sigma}^2 \lesssim \|U\|_{W^{1,\infty}} ||\omega||_2^2 + \|\partial^\beta \partial_z^j \omega\|_2 \, \|f_{\beta,j}\|_2. \tag{4.8}$$

With product estimates similar to those of Lemma 3.15, we also get

$$\|f_{\beta,j}\|_{L^2_\sigma} \leq C(|\zeta|_{H^N}, \|U\|_{H^N}, \|\omega\|_{H^{N-1}}),$$

and therefore $\partial_t ||\partial^\beta \partial_z^j \omega||_{L^2_\sigma}^2 \leq \mathfrak{m}^N(\zeta,\psi,\omega)$, which provides the H^k-estimate on ω of the proposition.

Evaluating the vorticity equation at the bottom, we get as in Remark 2.1 that

$$\partial_t(\omega_b \cdot N_b) + \nabla \cdot (\omega_b \cdot N_b V_b) = 0, \tag{4.9}$$

and therefore

$$\partial_t\,|\,\omega_b \cdot N_b\,|_{H_0^{-1/2}}^2 \leq |(\omega_b \cdot N_b)V_b|_{H^{1/2}}\, |\omega_b \cdot N_b|_{H_0^{-1/2}};$$

as a consequence of the trace lemma and Corollary 3.14, we deduce finally

$$\partial_t\,|\,\omega_b \cdot N_b\,|_{H_0^{-1/2}}^2 \leq \mathfrak{m}^N(\zeta,\psi,\omega), \tag{4.10}$$

and the proof is complete. ◻

4.2. Quasilinearization of the equations. The following proposition gives the structure of the equations solved by the derivatives of the solutions to the water waves equations (2.11). Note that, for the equations on ψ, it is crucial to work with Alinhac's good unknown. We also recall that the condition $(2.6)_T$ is defined in Notation 3.16, while the function space E_T^N is introduced in (4.2).

Proposition 4.2. *Let $N \geq 5$, $T > 0$, and $(\zeta, \psi, \omega) \in E_T^N$ be a solution to the water waves equations (2.11) on the time interval $[0, T]$. If $(2.6)_T$ is satisfied, then for all $k = x, y$, and $\beta \in \mathbb{N}^d$, $|\beta| \leq N-1$, one has, with $\alpha \in \mathbb{N}^d$ such that $\partial^\alpha = \partial^\beta \partial_k$.*

$$\begin{aligned}(\partial_t + \underline{V} \cdot \nabla)\, \partial^\alpha \zeta - \partial_k \underline{U}_{(\beta)} \cdot N &= R_\alpha^1, \\ (\partial_t + \underline{V} \cdot \nabla)(U_{(\beta)\parallel} \cdot \mathbf{e}_k) + \mathfrak{a}\, \partial^\alpha \zeta &= R_\alpha^2, \\ (\partial_t^\sigma + U \cdot \nabla_{X,z}^\sigma)\, \partial^\beta \omega &= R_\beta^3,\end{aligned}$$

with $\mathfrak{a} = g + (\partial_t + \underline{V} \cdot \nabla)\underline{w}$, and where

$$|R_\alpha^1|_2 + |R_\alpha^2|_2 + |\mathfrak{P} R_\alpha^2|_2 + \|R_\beta^3\|_2 \leq \mathfrak{m}^N(\zeta, \psi, \omega),$$

with $\mathfrak{m}^N(\zeta, \psi, \omega)$ as in (4.3).

Proof. Let us consider the first equation of (2.11). We use the notation

$$f \sim 0 \iff |f|_2 \leq \mathfrak{m}^N(\zeta, \psi, \omega).$$

Applying ∂^α to the first equation of (2.11), we get

$$\partial_t \partial^\alpha \zeta - \partial^\alpha \underline{U} \cdot N - \underline{U} \cdot \partial^\alpha N = [\partial^\alpha, \underline{U}, N].$$

One also has

$$[\partial^\alpha, \underline{U}, N] = \sum_{0 < \beta < \alpha} *_\beta\, \partial^{\alpha - \beta} \underline{U} \cdot \partial^\beta N,$$

where $*_\beta$ are scalar coefficients of no importance. We deduce easily from the product estimate (3.36) and the assumption $N \geq 5$ that

$$\big|[\partial^\alpha, \underline{U}, N]\big|_2 \leq C(|\underline{U}|_{H^{N-1}}, |\zeta|_{H^N}),$$

and therefore, by the trace lemma and Proposition 3.12, we have $[\partial^\alpha, \underline{U}, N] \sim 0$. This yields

$$\partial_t \partial^\alpha \zeta - \partial^\alpha \underline{U} \cdot N - \underline{U} \cdot \partial^\alpha N \sim 0.$$

Since $\underline{U} \cdot \partial^\alpha N = -\underline{V} \cdot \nabla \partial^\alpha \zeta$ and $\partial^\alpha \underline{U} \cdot N \sim \partial_k \underline{U}_{(\beta)} \cdot N$, one readily deduces the first estimate of the proposition.

We now consider the second equation of (2.11). Remarking that

$$\underline{w} = \frac{\underline{U} \cdot N + \nabla \zeta \cdot U_\parallel}{1 + |\nabla \zeta|^2},$$

we can write the second equation of (2.11) under the form

$$\partial_t \psi + g\zeta + \frac{1}{2}|U_\parallel|^2 - \frac{1}{2}(1 + |\nabla\zeta|^2)\underline{w}^2 - \frac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}\cdot N\underline{V}) = 0.$$

After applying ∂_k, we obtain that

$$\partial_t\,\partial_k\psi + g\,\partial_k\zeta + \underline{V}\cdot(\partial_k U_\parallel - \underline{w}\nabla\,\partial_k\zeta) - \underline{w}\,\partial_k(\underline{U}\cdot N) - \partial_k\frac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}\cdot N\underline{V}) = 0,$$

or equivalently, after substituting $U_\parallel = \nabla\psi + \nabla^\perp\tilde{\psi}$,

$$\begin{aligned}\partial_t\,\partial_k\psi + g\,\partial_k\zeta + \underline{V}\cdot((\partial_k\nabla\psi - \underline{w}\nabla\,\partial_k\zeta) + \partial_k\nabla^\perp\tilde{\psi})\\ - \underline{w}\,\partial_k(\underline{U}\cdot N) - \partial_k\frac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}\cdot N\underline{V}) = 0.\end{aligned}$$

Before differentiating β times this equation, it is convenient to introduce the notation

$$f \approx g \iff |f-g|_2 + |\mathfrak{P}(f-g)|_2 \le \mathfrak{m}^N(\zeta,\psi,\omega)$$

and to state the following lemma.

Lemma 4.3. *The following identities hold:*

$$\begin{aligned}&[\partial^\beta,\underline{V}\,]\cdot((\nabla\,\partial_k\psi - \underline{w}\nabla\,\partial_k\zeta) + \nabla^\perp\,\partial_k\tilde{\psi}) \approx 0,\\ &-[\partial^\beta,\underline{w}\,]\,\partial_k(\underline{U}\cdot N) - \underline{V}\cdot[\partial^\beta,\underline{w}\,]\nabla\,\partial_k\zeta \approx 0.\end{aligned}$$

Proof. Let us first observe that

$$\begin{aligned}|\underline{U}\,|_{H^{N-1/2}} &\le \mathfrak{m}^N(\zeta,\psi,\omega),\\ \big|[\mathfrak{P},f]g\big|_2 &\lesssim |\nabla f|_{H^{t_0}}\,|g|_2,\\ \big|[\partial^\beta,f]g\big|_{H^{1/2}} &\lesssim |f|_{H^{N-1/2}}\,|g|_{H^{N-3/2}};\end{aligned}$$

the first estimate is a direct consequence of the trace lemma and Proposition 3.12, while the second and third ones follow from the general commutator estimates of Theorem 3 in [40]. These identities allow us to write

$$\begin{aligned}&|\mathfrak{P}[\partial^\beta,\underline{V}\,]\cdot(\nabla\,\partial_k\psi + \nabla^\perp\,\partial_k\tilde{\psi} - w\nabla\,\partial_k\zeta)|_2\\ &\qquad\le \mathfrak{m}^N(\zeta,\psi,\omega)(1 + |\nabla\tilde{\psi}|_{H^{N-1/2}}) \le \mathfrak{m}^N(\zeta,\psi,\omega),\end{aligned}$$

the last line being a consequence of Lemma 3.13; this yields the first assertion of the lemma. For the second one, we proceed along the same lines (it is important to note that the most singular terms in ζ of both terms cancel each other). $\square$

Applying ∂^β (with $\beta \neq 0$) to the equation on $\partial_k \psi$, we therefore get, using the lemma,

$$\partial_t \partial^\alpha \psi + g \partial^\alpha \zeta + \underline{V} \cdot ((\partial^\alpha \nabla \psi - \underline{w} \nabla \partial^\alpha \zeta) + \partial^\alpha \nabla^\perp \tilde{\psi})$$
$$- \underline{w} \partial^\alpha (\underline{U} \cdot N) - \partial^\alpha \frac{\nabla^\perp}{\Delta} \cdot (\underline{\omega} \cdot N \underline{V}) \approx 0.$$

Using the evolution equation on ζ, we can substitute $\partial^\alpha(\underline{U} \cdot N) = \partial_t \partial^\alpha \zeta$ to obtain

$$\partial_t \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta + \underline{V} \cdot \nabla \psi_{(\alpha)} \approx \partial^\alpha \frac{\nabla^\perp}{\Delta} \cdot (\underline{\omega} \cdot N \underline{V}) - \underline{V} \cdot \nabla^\perp \partial^\alpha \tilde{\psi}$$
$$= \left[\frac{\partial^\alpha \nabla^\perp}{\Delta}, \underline{V} \right] (\underline{\omega} \cdot N).$$

Now, a general commutator estimate (Theorem 3 in [40]) implies that

$$\left\| \left[\frac{\partial^\alpha \nabla^\perp}{\Delta}, \underline{V} \right] (\underline{\omega} \cdot N) \right\|_{H^{1/2}} \lesssim |\underline{V}|_{H^{N-1/2}} |\underline{\omega} \cdot N|_{H^{N-3/2}}$$
$$\lesssim \|V\|_{H^{N,1}} \|\omega\|_{H^{N-1,1}} \lesssim \mathfrak{m}^N(\zeta, \psi, \omega),$$

where we used the trace lemma to derive the second inequality, and Proposition 3.12 to get the third one. It follows that

$$\partial_t \psi_{(\alpha)} + \mathfrak{a} \partial^\alpha \zeta + \underline{V} \cdot \nabla \psi_{(\alpha)} \approx 0. \tag{4.11}$$

We now have to relate $\psi_{(\alpha)}$ to the quantity $U_{(\beta)\|} \cdot \mathbf{e}_k$ used in the statement of the proposition. Proceeding as for (3.50), we get

$$U_{(\beta)\|} \cdot \mathbf{e}_k = \psi_{(\alpha)} + \mathbf{e}_k \cdot \nabla^\perp \partial^\beta \tilde{\psi} - [\partial^\beta, \underline{w}, \partial_k \zeta] - \partial^\beta \zeta (\partial_z^\sigma U)_\| \cdot \mathbf{e}_\kappa. \tag{4.12}$$

We now need the following lemma.

Lemma 4.4. *Let f_1, f_2, and f_3 be defined as*

$$f_1 = \mathbf{e}_k \cdot \nabla^\perp \partial^\beta \tilde{\psi}, \quad f_2 = [\partial^\beta, \underline{w}, \partial_k \zeta], \quad f_3 = \partial^\beta \zeta (\partial_z^\sigma U)_\| \cdot \mathbf{e}_k;$$

then, one has

$$(\partial_t + \underline{V} \cdot \nabla) f_j \approx 0 \quad (j = 1, 2, 3).$$

Proof. Let us first prove the lemma for f_1. We recall that $\Delta \tilde{\psi} = \underline{\omega} \cdot N$ and that we get from the vorticity equation evaluated at the surface that

$$(\partial_t + \underline{V} \cdot \nabla)(\underline{\omega} \cdot N) + \underline{\omega} \cdot N \nabla \cdot \underline{V} = 0$$

(see Remark 2.1). Applying $\mathbf{e}_k \cdot \nabla^\perp \partial^\beta \Delta^{-1}$ to this equation, we get therefore

$$(\partial_t + \underline{V} \cdot \nabla) f_1 = -\mathbf{e}_k \cdot \nabla^\perp \partial^\beta \Delta^{-1} (\underline{\omega} \cdot N \nabla \cdot \underline{V}) \approx 0,$$

the second identity stemming from standard commutator estimates, the trace lemma, and Proposition 3.12. To treat the case of f_2, it is sufficient to prove that

(i) $(\partial_t + \underline{V} \cdot \nabla)\, \partial^{\beta'} \underline{w} \approx 0$,
(ii) $(\partial_t + \underline{V} \cdot \nabla)\, \partial^{\beta'} \partial_k \zeta \approx 0$,

for all $0 < \beta' < \beta$. For (i), the space derivative is controlled through Proposition 3.12, while the time derivative is controlled by Corollary 3.18 in terms of $\mathfrak{m}^N$ and $|\mathfrak{P}\, \partial_t \psi_{\alpha'}|_2$, with $|\alpha'| \le N-1$. Now, (4.11) gives a control of this last quantity in terms of $\mathfrak{m}^N$, since $\mathfrak{a}\, \partial^{\alpha'} \zeta \approx 0$ when $|\alpha'| \le N-1$; the identity (i) is therefore proved. For (ii), we just have to remark that $\underline{U}_{(\alpha')} \cdot N \sim 0$ when $|\alpha'| \le N-1$ (which is the case if we take $\partial^{\alpha'} = \partial^{\beta'} \partial_k$), and to use the first identity given in the statement of the proposition and proved above.

We finally turn to f_3, which is a direct consequence of Proposition 3.12, Corollary 3.18, and (ii) above. $\square$

Because of this lemma and (4.12), we can replace $\psi_{(\alpha)}$ by $U_{(\beta)\|} \cdot \mathbf{e}_k$ in (4.11), and the second assertion of the proposition is proved. The third and last assertion of the proposition is a simple byproduct of the proof of Proposition 4.1. $\square$

4.3. A* priori *estimates on the full equations. We recall that the energy $\mathcal{E}^N(\zeta, \psi, \omega)$ is defined in (4.1) as

$$\mathcal{E}^N(\zeta, \psi, \omega) := \frac{1}{2}|\zeta|_{H^N}^2 + \frac{1}{2}|\mathfrak{P}\psi|_{H^3}^2 + \frac{1}{2}\sum_{0<|\alpha|\le N} |\mathfrak{P}\psi_{(\alpha)}|_2^2 + \frac{1}{2}||\omega||_{H^{N-1}}^2 + \frac{1}{2}|\omega_b \cdot N_b|^2.$$

The following proposition gives an *a priori* estimate for this energy, provided that the *Rayleigh-Taylor* criterion is uniformly satisfied (i.e., $\mathfrak{a}$ remains strictly positive on the time interval considered) and that the water depth does not vanish.

Proposition 4.5. *Let $N \ge 5$, $T > 0$, and $(\zeta, \psi, \omega) \in E_T^N$ be a solution to the water waves equations* (2.11) *on the time interval* $[0, T]$*. If* $(2.6)_T$ *is satisfied and if*

$$\exists\, a_0 > 0,\ \forall\, t \in [0, T], \quad \mathfrak{a}(t) \ge a_0,$$

with $\mathfrak{a}$ as defined in Proposition 4.2, then

$$\forall\, 0 \le t \le T, \quad \mathcal{E}^N(\zeta, \psi, \omega)(t) \le C\left(T, \frac{1}{a_0}, \frac{1}{h_{\min}}, H_0, \mathcal{E}^N(\zeta^0, \psi^0, \omega^0)\right).$$

Proof. Energy estimates are classically obtained by controlling the time derivative of this expression. This has already partially been done in Proposition 4.1 for the vorticity. We deal here with the other components of the energy.

Step 1. Control of the low order terms in (ζ,ψ). Computing the time derivative of these terms, one gets

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\zeta|_2^2+|\mathfrak{P}\psi|_{H^3}^2) &= (\zeta,\partial_t\zeta)+(\Lambda^3\mathfrak{P}\psi,\Lambda^3\mathfrak{P}\,\partial_t\psi)\\ &\le |\zeta|_2\,|\partial_t\zeta|_2+|\mathfrak{P}\psi|_{H^3}\,|\mathfrak{P}\,\partial_t\psi|_{H^3}.\end{aligned}$$

Replacing $\partial_t\zeta$ by $\underline{U}\cdot N$ according to the first equation of (2.11), and replacing similarly $\partial_t\psi$ using the second equation of (2.11), we get by standard product estimates and the trace lemma that

$$\begin{aligned}|\partial_t\zeta|_2+|\mathfrak{P}\,\partial_t\psi|_{H^3} &\le C(|\zeta|_{H^N},|\underline{U}|_{H^{7/2}},|\underline{\omega}|_{H^{5/2}})\\ &\le C(|\zeta|_{H^N}\,\|\underline{U}\|_{H^{4,1}},\|\omega\|_{H^3}).\end{aligned}$$

We then easily deduce with the help of Proposition 3.12 that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\zeta|_2^2+|\mathfrak{P}\psi|_{H^3}^2)\le \mathfrak{m}^N(\zeta,\psi,\omega). \tag{4.13}$$

Step 2. Control of the higher-order terms in (ζ,U). We do not directly give a control of the components of the energy that involve $\partial^\alpha\zeta$ and $\psi_{(\alpha)}$, for all $\alpha\neq 0$. The control of $\psi_{(\alpha)}$ will indeed be obtained indirectly (through Corollary 3.22) from the control of $\partial_k U_{(\beta)}$ (in the flat strip) derived here. We do not for this purpose directly use (2.11) as above, but rather the system exhibited in Proposition 4.2, namely,

$$\begin{cases}(\partial_t+\underline{V}\cdot\nabla)\,\partial^\alpha\zeta-\partial_k\underline{U}_{(\beta)}\cdot N=R^1_\alpha,\\ (\partial_t+\underline{V}\cdot\nabla)(U_{(\beta)\parallel}\cdot\mathbf{e}_k)+\mathfrak{a}\,\partial^\alpha\zeta=R^2_\alpha,\end{cases} \tag{4.14}$$

where $\partial^\alpha=\partial_k\,\partial_\beta$, with $k=x,y$ and $0\le|\beta|\le N-1$. The nondiagonal terms in (4.14) can be cancelled out in the energy estimates by multiplying the first equation by $\mathfrak{a}\,\partial^\alpha\zeta$, and the second one by $\partial_k\underline{U}_{(\beta)}\cdot N$. More precisely, multiplying the first equation of (4.14) by $\mathfrak{a}\,\partial^\alpha\zeta$, and integrating over $\mathbb{R}^d$, we get

$$\begin{aligned}&\frac{1}{2}\partial_t(\mathfrak{a}\,\partial^\alpha\zeta,\partial^\alpha\zeta)-(\partial_k\underline{U}_{(\beta)}\cdot N,\mathfrak{a}\,\partial^\alpha\zeta)\\ &\qquad=(R^1_\alpha,\mathfrak{a}\,\partial^\alpha\zeta)-\frac{1}{2}(\partial_t\mathfrak{a}\,\partial^\alpha\zeta,\partial^\alpha\zeta)-(\underline{V}\cdot\nabla\,\partial^\alpha\zeta,\mathfrak{a}\,\partial^\alpha\zeta)\le\mathfrak{m}^N(\zeta,\psi,\omega),\end{aligned}$$

the last inequality being a consequence of Proposition 4.2 and Corrollay 3.18 (and a simple integration by parts for the last term of the right-hand side). Multiplying the second equation of (4.14) by $\partial_k \underline{U}_{(\beta)} \cdot N$, and integrating over $\mathbb{R}^d$, we also get

$$\begin{aligned}(\partial_t (U_{(\beta)\|} \cdot \mathbf{e}_k), \partial_k \underline{U}_{(\beta)} \cdot N) + (\mathfrak{a}\, \partial^\alpha \zeta, \partial_k \underline{U}_{(\beta)} \cdot N) \\ + (\underline{V} \cdot \nabla (U_{(\beta)\|} \cdot \mathbf{e}_k), \partial_k \underline{U}_{(\beta)} \cdot N) = (R_\alpha^2, \partial_k \underline{U}_{(\beta)} \cdot N) \le \mathfrak{m}^N(\zeta, \psi, \omega),\end{aligned}$$

the last inequality stemming from Proposition 4.2 and Corollary 3.20. Summing up these two equations, we get

$$\frac{1}{2}\partial_t (\mathfrak{a}\, \partial^\alpha \zeta, \partial^\alpha \zeta) + ((\partial_t + \underline{V} \cdot \nabla)(U_{(\beta)\|} \cdot \mathbf{e}_k), \partial_k \underline{U}_{(\beta)} \cdot N) \le \mathfrak{m}^N(\zeta, \psi, \omega). \tag{4.15}$$

Let us denote by $U^\flat_{(\beta)\|}$ the following extension of $U_{(\beta)\|}$:

$$U^\flat_{(\beta)\|} = V_{(\beta)} + w_{(\beta)} \nabla \sigma.$$

Focusing our attention on the second term of the left-hand side of (4.15), and remarking that the kinematic equation implies

$$(\partial_t + \underline{V} \cdot \nabla)(U_{(\beta)\|} \cdot \mathbf{e}_k) = (\partial_t^\sigma + U \cdot \nabla^\sigma_{X,z})(U^\flat_{(\beta)\|} \cdot \mathbf{e}_k)_{|z=0}, \tag{4.16}$$

we can deduce from Green's identity that

$$\begin{aligned}((\partial_t + \underline{V} \cdot \nabla)&(U_{(\beta)\|} \cdot \mathbf{e}_k), \partial_k \underline{U}_{(\beta)} \cdot N) \\ &= \int_S (1 + \partial_z \sigma)(\partial_t^\sigma + U \cdot \nabla^\sigma_{X,z})(U^\flat_{(\beta)\|} \cdot \mathbf{e}_k) \nabla^\sigma_{X,z} \cdot (\partial_k U_{(\beta)}) \\ &\quad + \int_S (1 + \partial_z \sigma) \nabla^\sigma_{X,z} [(\partial_t^\sigma + U \cdot \nabla^\sigma_{X,z})(U^\flat_{(\beta)\|} \cdot \mathbf{e}_k)] \cdot \partial_k U_{(\beta)}.\end{aligned}$$

The first term of the right-hand side is controlled by $\mathfrak{m}^N(\zeta, \psi, \omega)$ by a simple application of the Cauchy-Schwarz inequality and Propositions 3.17 and 3.20. We can also remark that it is possible to commute the operators

$$\nabla^\sigma_{X,z} \text{ and } (\partial_t^\sigma + U \cdot \nabla^\sigma_{X,z})(U^\flat_{(\beta)\|} \cdot \mathbf{e}_k)$$

in the second term of the right-hand side, up to terms that can similarly be controlled by $\mathfrak{m}^N(\zeta, U, \omega)$. We therefore have

$$\begin{aligned}((\partial_t + \underline{V} \cdot \nabla)&(U_{(\beta)\|} \cdot \mathbf{e}_k), \partial_k \underline{U}_{(\beta)} \cdot N) \\ &= \int_S (1 + \partial_z \sigma)[(\partial_t^\sigma + U \cdot \nabla^\sigma_{X,z}) \nabla^\sigma_{X,z} (U^\flat_{(\beta)\|} \cdot \mathbf{e}_k)] \cdot \partial_k U_{(\beta)} + \text{l.o.t.}\end{aligned} \tag{4.17}$$

where l.o.t. stands for lower order terms that can be controlled by $\mathfrak{m}^N(\zeta, \psi, \omega)$. We shall now need the following lemma to relate the quantity $\nabla^\sigma_{X,z}(U^\flat_{(\beta)\|} \cdot \mathbf{e}_k)$ to $\partial_k U_{(\beta)}$.

Lemma 4.6. *Denoting* $\mathbf{t}_k = \mathbf{e}_k + \partial_k\sigma\,\mathbf{e}_z$, *we have*

$$\nabla^\sigma_{X,z}(U^\flat_{(\beta)\parallel}\cdot\mathbf{e}_k) = \partial_k U_{(\beta)} + \mathbf{t}_k \times \partial^\beta\omega + r_\alpha,$$

with

$$r_\alpha = ((\nabla^\sigma_{X,z}\times U_{(\beta)}) - \partial^\beta\omega) + w_{(\beta)}\nabla^\sigma_{X,z}\,\partial_k\sigma;$$

in particular,

$$\|(\partial^\sigma_t + U\cdot\nabla^\sigma_{X,z})r_\alpha\|_2 \le \mathfrak{m}^N(\zeta,\psi,\omega).$$

Proof. Noting that

$$\nabla^\sigma_{X,z}(U_{(\beta)}\cdot\mathbf{e}_j) = \partial^\sigma_j U_{(\beta)} + \mathbf{e}_j\times(\nabla^\sigma_{X,z}\times U_{(\beta)}) \quad (j = x,y,z),$$

we compute

$$\begin{aligned}\nabla^\sigma_{X,z}(U^\flat_{(\beta)\parallel}\cdot\mathbf{e}_k) &= \partial^\sigma_k U_{(\beta)} + \mathbf{e}_k\times(\nabla^\sigma_{X,z}\times U_{(\beta)}) + \partial^\sigma_z U_{(\beta)}\,\partial_k\sigma\\ &\quad + \mathbf{e}_z\times(\nabla^\sigma_{X,z}\times U_{(\beta)})\,\partial_k\sigma + w_{(\beta)}\nabla^\sigma_{X,z}\,\partial_k\sigma\\ &= \partial_k U_{(\beta)} + \mathbf{t}_\kappa\times(\nabla^\sigma_{X,z}\times U_{(\beta)}) + w_{(\beta)}\nabla^\sigma_{X,z}\,\partial_k\sigma,\end{aligned}$$

and the result follows from Proposition 3.20 and Corollary 3.18. ◻

A direct consequence of (4.17) and Lemma 4.6 is that

$$\begin{aligned}&((\partial_t + \underline{V}\cdot\nabla)(U_{(\beta)\parallel}\cdot\mathbf{e}_k), \partial_k\underline{U}_{(\beta)}\cdot N)\\ &\quad = \int_S (1+\partial_z\sigma)[(\partial^\sigma_t + U\cdot\nabla^\sigma_{X,z})\,\partial_k U_{(\beta)}]\cdot\partial_k U_{(\beta)}\\ &\qquad + \int_S (1+\partial_z\sigma)[\mathbf{t}_k\times(\partial^\sigma_t + U\cdot\nabla^\sigma_{X,z})\,\partial^\beta\omega]\cdot\partial_k U_{(\beta)} + \text{l.o.t.}\\ &\quad = A + B\end{aligned}$$

(with the same meaning as above for l.o.t.). We now turn to analyze A and B.

- *Analysis of* A. Using the integration by parts formula (3.49) together with the identity

$$\partial_t\int_S(1+\partial_z\sigma)fg = \int_S(1+\partial_z\sigma)\,\partial^\sigma_t f\,g + \int_S(1+\partial_z\sigma)f\,\partial^\sigma_t g + \int_{z=0} fg\,\partial_t\zeta,$$

we can write

$$\begin{aligned}A &= \frac12\partial_t\int_S(1+\partial_z\sigma)|\partial_k U_{(\beta)}|^2 - \frac12\int_S(\nabla^\sigma_{X,z}\cdot U)\,|\partial_k U_{(\beta)}|_2^2\\ &\quad - \frac12\int_{z=0}|\partial_k\underline{U}_{(\beta)}|^2(\partial_t\zeta - \underline{U}\cdot N)\\ &= \frac12\partial_t\int_S(1+\partial_z\sigma)|\partial_k U_{(\beta)}|^2,\end{aligned}\tag{4.18}$$

where we used that fact that $\nabla^\sigma_{X,z}\cdot U = 0$ and $\partial_t\zeta - \underline{U}\cdot N = 0$ by the first equation of (2.11).

- *Analysis of* B. By the Cauchy-Schwarz inequality, we have

$$B \leq C(|\zeta|_{W^{1,\infty}})\|(\partial_t^\sigma + U\cdot\nabla_{X,z}^\sigma)\,\partial^\beta\omega\|_2\,\|\partial_k U_{(\beta)}\|_2 \leq \mathfrak{m}^N(\zeta,\psi,\omega), \tag{4.19}$$

the last inequality stemming from the third assertion of Proposition 4.2 and Proposition 3.12.

We deduce from this analysis that

$$((\partial_t + \underline{V}\cdot\nabla)(U_{(\beta)\|}\cdot\mathbf{e}_k), \partial_k\underline{U}_{(\beta)}\cdot N) = \frac{1}{2}\partial_t\int_S (1+\partial_z\sigma)\,|\partial_k U_{(\beta)}|_2^2 + \text{l.o.t.},$$

so that (4.15) yields

$$\partial_t\Big\{(\mathfrak{a}\,\partial^\alpha\zeta,\partial^\alpha\zeta) + \int_S (1+\partial_z\sigma)\,|\partial_k U_{(\beta)}|_2^2\Big\} \leq \mathfrak{m}^N(\zeta,\psi,\omega). \tag{4.20}$$

Step 3. Energy estimate on the modified energy. Let us define the modified energy $\tilde{\mathcal{E}}^N = \tilde{\mathcal{E}}^N(\zeta,\psi,\omega)$ as

$$\tilde{\mathcal{E}}^N = |\zeta|_2^2 + |\mathfrak{P}\psi|_{H^3}^2 + \sum_{k=x,y,\,0<|\beta|\leq N-1}\Big[(\partial_k\partial^\beta\zeta, \mathfrak{a}\,\partial_k\partial^\beta\zeta) + \int_S (1+\partial_z\sigma)\,|\partial_k U_{(\beta)}|_2^2\Big] + \|\omega\|_{H^{N-1}}^2.$$

It follows directly from *Step 1*, *Step 2*, and the vorticity estimates of Proposition 4.1 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^N(\zeta,\psi,\omega) \leq \mathfrak{m}^N(\zeta,\psi,\omega).$$

Under the assumption made on $\mathfrak{a}$, we can write, for all $\alpha \neq 0$,

$$|\partial^\alpha\zeta|_2^2 \leq \frac{2}{a_0}(\mathfrak{a}\,\partial^\alpha\zeta,\partial^\alpha\zeta),$$

so that, with the help of Corollary 3.22, we have

$$\mathfrak{m}^N(\zeta,\psi,\omega) \leq C\Big(\frac{1}{a_0},\tilde{\mathcal{E}}^N(\zeta,\psi,\omega)\Big),$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^N(\zeta,\psi,\omega) \leq C\Big(\frac{1}{a_0},\tilde{\mathcal{E}}^N(\zeta,\psi,\omega)\Big). \tag{4.21}$$

We classically deduce from this differential inequality that, for all $0 \leq t \leq T$,

$$\tilde{\mathcal{E}}^N(\zeta,\psi,\omega)(t) \leq C\Big(T,\frac{1}{a_0},\frac{1}{h_{\min}},H_0,\tilde{\mathcal{E}}^N(\zeta^0,\psi^0,\omega^0)\Big). \tag{4.22}$$

Step 4. Conclusion. Corollary 3.22 implies that

$$\mathcal{E}^N(\zeta,\psi,\omega) \le C\left(\frac{1}{a_0},\frac{1}{h_{\min}},\tilde{\mathcal{E}}^N(\zeta,\psi,\omega)\right),$$

while we get from Proposition 3.12 that

$$\tilde{\mathcal{E}}^N(\zeta^0,\psi^0,\omega^0) \le C\left(\frac{1}{h_{\min}},\mathcal{E}^N(\zeta^0,\psi^0,\omega^0)\right).$$

These two inequalities, together with (4.22), imply that, for all $0 \le t \le T$, we have

$$\mathcal{E}^N(\zeta,\psi,\omega)(t) \le C\left(T,\frac{1}{a_0},\frac{1}{h_{\min}},H_0,\mathcal{E}^N(\zeta^0,\psi^0,\omega^0)\right);$$

this is exactly the result stated in the proposition. ◻

4.4. Main result. As explained in Remark 2.7, it is easier to give a functional framework for our well-posedness result if we replace the equation on the vorticity $\boldsymbol{\omega}$ by an equation on the straightened vorticity $\omega = \boldsymbol{\omega}\circ\Sigma$ in the water waves equations (2.11), where we recall that

$$\Sigma(t,X,z) = (X, z+\sigma(t,X,z)) \quad \text{and} \quad \sigma(t,X,z) = \frac{1}{H_0}(z+H_0)\zeta(t,X). \tag{4.23}$$

Recalling that, according to Notation 3.10, $\mathbb{U}^\sigma[\zeta](\psi,\omega) = \mathbb{U}[\zeta](\psi,\omega)\circ\Sigma$, with horizontal and vertical components $\mathbb{V}^\sigma[\zeta](\psi,\omega)$ and $\mathbb{w}^\sigma[\zeta](\psi,\omega)$, we also denote

$$\mathbb{U}^\sigma_\parallel[\zeta](\psi,\omega) := \underline{\mathbb{V}}^\sigma[\zeta](\psi,\omega) + \underline{\mathbb{w}}^\sigma[\zeta](\psi,\omega)\nabla\zeta = \mathbb{U}_\parallel[\zeta](\psi,\boldsymbol{\omega})$$

(as usual, quantities evaluated at the surface $z = 0$ are underlined). Because of (4.4), instead of the set of evolution equations (2.11) on $(\zeta,\psi,\boldsymbol{\omega})$, we are therefore concerned with the following set of evolution equations on (ζ,ψ,ω):

$$\begin{cases} \partial_t\zeta - \underline{\mathbb{U}}^\sigma[\zeta](\psi,\omega)\cdot N = 0, \\ \partial_t\psi + g\zeta + \dfrac{1}{2}|\mathbb{U}^\sigma_\parallel[\zeta](\psi,\omega)|^2 \\ \qquad -\dfrac{1}{2}(1+|\nabla\zeta|^2)\underline{\mathbb{w}}^\sigma[\zeta](\psi,\omega)^2 \\ \qquad\qquad -\dfrac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}\cdot N\underline{\mathbb{V}}^\sigma[\zeta](\psi,\omega)) = 0, \\ \partial_t^\sigma\omega + \mathbb{U}^\sigma[\zeta](\psi,\omega)\cdot\nabla^\sigma_{X,z}\omega = \omega\cdot\nabla^\sigma_{X,z}\mathbb{U}^\sigma[\zeta](\psi,\omega). \end{cases} \tag{4.24}$$

Note that, as for (2.11)–(2.12), these equations make sense if ω is divergence free in the sense that

$$\operatorname{div}^{\sigma} \omega = 0 \quad \text{in } \mathcal{S}, \tag{4.25}$$

but that we omit this constraint since it is propagated by the equation on ω if is initially satisfied.

We also recall the definition (4.1) of the energy

$$\begin{aligned}\mathcal{E}^N(\zeta, \psi, \omega) := \frac{1}{2}|\zeta|^2_{H^N} + \frac{1}{2}|\mathfrak{P}\psi|^2_{H^3} + \frac{1}{2}\sum_{0<|\alpha|\leq N} |\mathfrak{P}\psi_{(\alpha)}|_2^2 \\ + \frac{1}{2}\|\omega\|^2_{H^{N-1}} + \frac{1}{2}|\omega_b \cdot N_b|^2_{H_0^{-1/2}},\end{aligned}$$

and of the associated functional space E_T^N defined in (4.2) for all $T \geq 0$ as

$$\begin{aligned}E_T^N = \Big\{(\zeta, \psi, \omega) \in C([0, T]; H^2(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d) \times H^2(\mathcal{S})) \mid \\ \mathcal{E}^N((\zeta, \psi, \omega)(\cdot)) \in L^\infty([0, T])\Big\};\end{aligned}$$

we also denote by E_0^N the set of $(\zeta, \psi, \omega) \in H^2(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d) \times H^2(\mathcal{S})^3$ of finite energy. We recall finally that the Rayleigh-Taylor coefficient $\mathfrak{a}$ is defined[8] as

$$\mathfrak{a} = \mathfrak{a}(\zeta, \psi, \omega) = g + (\partial_t + \underline{\mathbb{V}}^\sigma[\zeta](\psi, \omega))\underline{\mathbb{w}}^\sigma[\zeta](\psi, \omega).$$

Theorem 4.7. *Let $N \geq 5$ and $\Theta^0 = (\zeta^0, \psi^0, \omega^0) \in E_0^N$ be such that ω^0 satisfies the divergence-free condition* (4.25)*. Assume, moreover, that*

$$\exists\, h_{\min} > 0,\ \exists\, a_0 > 0, \quad H_0 + \zeta^0 > h_{\min},\ \mathfrak{a}(\zeta^0, \psi^0, \omega^0) > a_0.$$

Then, there exists $T > 0$ and a unique solution $\Theta \in E_T^N$ to (4.24) *satisfying the divergence-free constraint* (4.25)*, and with initial condition Θ^0. Moreover,*

$$\frac{1}{T} = c^1 \quad \text{and} \quad \sup_{t\in[0,T]} \mathcal{E}^N(\Theta(t)) = c^2$$

with $c^j = C(\mathcal{E}^N(\Theta^0), 1/h_{\min}, H_0, 1/a_0)$ for $j = 1, 2$.

[8] The notation $\mathfrak{a} = \mathfrak{a}(\zeta, \psi, \omega)$ suggests that $\mathfrak{a}$ is a function of (ζ, ψ, ω) and not of their time derivatives. We actually use an alternative definition of $\mathfrak{a}$ where $\partial_t(\underline{\mathbb{w}}^\sigma[\zeta](\psi, \omega))$ is written in terms of $\partial_t\psi$, $\partial_t\zeta$, and $\partial_t^\sigma\omega$ through Proposition 3.17, and where these time derivatives are replaced by purely spatial operators using the time evolution equations provided by (4.24).

Proof. The strategy of the proof is the following. In *Step 1*, we give an equivalent formulation of the water waves equations (4.26); in view of regularizing these equations, we define and study in *Step 2* mollifiers in the horizontal and vertical variables. Since the divergence-free constraint (4.25) is a consequence of the particular structure of the vorticity equation in (4.26), it may not hold with regularized equations; this would be an obstruction to solving the div-curl problem studied in Section 2.2, which requires that the vorticity be divergence free. We therefore explain in *Step 3* how to project any vector field in $H^1(\mathcal{S})^{d+1}$ on its divergence-free component. The regularized equations are then defined in *Step 4*, and solved with standard ODEs tools. The solution thus obtained is studied in *Step 5*, where we prove *a posteriori* that its vorticity is divergence free. Energy estimates inspired by the *a priori* estimates of Proposition 4.5 are then established in *Step 6* and used in *Step 7* to prove that the solutions to the regularized equations converge to a solution of (4.26).

For the sake of simplicity, we often write $U = \mathbb{U}^\sigma[\zeta](\psi,\omega)$, $a = \mathbb{a}[\zeta](\psi,\omega)$, *etc., when no confusion is possible.*

Step 1. Modification of the vorticity equation. As explained in the proof of Proposition 4.1, we are allowed to solve the equations (4.24) with the vorticity equation replaced by

$$\partial_t\omega + \mathbb{V}^\sigma[\zeta](\psi,\omega)\cdot\nabla\omega + \mathbb{a}[\zeta](\psi,\omega)\,\partial_z\omega = \omega\cdot\nabla^\sigma_{X,z}\mathbb{U}^\sigma[\zeta](\psi,\omega)$$

with, denoting $N(z) = (-\nabla\sigma^T, 1)^T$ (so that $N(0) = N$),

$$\mathbb{a}[\zeta](\psi,\omega) = \frac{1}{1+\partial_z\sigma}\Big(\mathbb{U}^\sigma[\zeta](\psi,\omega)\cdot N(z) - \frac{z+H_0}{H_0}\mathbb{U}^\sigma[\zeta](\psi,\omega)_{|_{z=0}}\cdot N\Big).$$

The equations (4.24) are therefore equivalent to

$$\partial_t\Theta + \mathcal{F}(\Theta) = 0, \tag{4.26}$$

with

$$\Theta = \begin{pmatrix}\zeta\\ \psi\\ \omega\end{pmatrix}, \quad \mathcal{F}(\Theta)\begin{pmatrix}\mathcal{F}_1(\Theta)\\ \mathcal{F}_2(\Theta)\\ \mathcal{F}_3(\Theta)\end{pmatrix}$$

and

$$\begin{aligned}
\mathcal{F}_1(\Theta) &= -\underline{\mathbb{U}}^\sigma[\zeta](\psi,\omega)\cdot N,\\
\mathcal{F}_2(\Theta) &= g\zeta + \frac12|\mathbb{U}^\sigma_{\parallel}[\zeta](\psi,\omega)|^2 - \frac12(1+|\nabla\zeta|^2)\underline{\mathbb{w}}^\sigma[\zeta](\psi,\omega)^2,\\
\mathcal{F}_3(\Theta) &= \mathbb{V}^\sigma[\zeta](\psi,\omega)\cdot\nabla\omega + \mathbb{a}[\zeta](\psi,\omega)\partial_z\omega - \omega\cdot\nabla^\sigma_{X,z}\mathbb{U}^\sigma[\zeta](\psi,\omega).
\end{aligned}$$

Step 2. Definition of the mollifiers. For the horizontal variables, we use a standard mollifier. We define, for all $0 < \iota < 1$, the mollifier $J^\iota = \chi(\iota|\mathrm{D}|)$, where $\chi : \mathbb{R} \to \mathbb{R}$ is a smooth, even, and compactly supported function equal to 1 in a neighborhood of the origin. The mollifying properties of J^ι are classical and straightforwardly deduced from standard results on Fourier multipliers. We shall, in particular, use the fact that

$$\forall\, s,t \in \mathbb{R},\ \exists\, C^\iota_{s,t} > 0, \quad \forall\, f \in H^t(\mathbb{R}^d),\ |J^\iota f|_{H^s} \le C^\iota_{s,t}|f|_{H^t} \tag{4.27}$$

and

$$\forall\, \iota^1, \iota^2 > 0,\ \forall\, s \in \mathbb{R},\ \exists\, C > 0, \qquad \forall\, f \in H^{s+1}(\mathbb{R}^d),\ |(J^{\iota^1} - J^{\iota^2})f|_{H^s} \le C|\iota^1 - \iota^2|\,|f|_{H^{s+1}}. \tag{4.28}$$

For the vertical variable for which the Fourier transform cannot be used, we use another kind or regularization based on the lemma below; in the statement, we use the following functional spaces, where a is some smooth enough function defined on S:

$$\begin{aligned}
H(a\,\partial_z, S) &:= \{f \in L^2(S) \mid a\,\partial_z f \in L^2(S)\},\\
H^k(a\,\partial_z, S) &:= \{f \in H^k(S) \mid \forall\, \beta \in \mathbb{N}^d, \forall\, j \in \mathbb{N},\\
&\qquad\qquad |\beta| + j \le k,\ a\,\partial_z\,\partial^\beta\,\partial_z^j f \in L^2(S)\},\\
V_a(S) &:= \{f \in H^{-1}(S) \mid \exists\, \tilde{f} \in L^2(\mathbb{R}^d),\ f = a\,\partial_z \tilde{f}\}.
\end{aligned}$$

Lemma 4.8. *Let $N \ge 5$. Let $a \in W^{1,\infty}(S)$ be such that $a_{|_{z=0}} = a_{|_{z=-H_0}} = 0$ and $0 < \iota < 1$. Then, we have the following:*

(i) *The mapping*

$$(1 - \iota^2\partial_z(a^2\partial_z\cdot)) : \begin{array}{l} H(a\,\partial_z, S) \to H^{-1}(S),\\ f \mapsto f - \iota^2\,\partial_z(a^2\,\partial_z f) \end{array}$$

is well defined and one-to-one.

(ii) *We also have $L^2(S) + V_a(S) \subset \mathrm{Range}(1 - \iota^2\,\partial_z(a^2\,\partial_z\cdot))$.*

(iii) *Let $a \in H^{N-1}(S)$, and $0 \le k \le N-1$. Then, for $\iota > 0$ small enough, the mapping*

$$K^\iota[a\,\partial_z] : \begin{array}{l} H^k(S) \to H^k(a\,\partial_z, S),\\ f \mapsto (1 - \iota^2\,\partial_z(a^2\,\partial_z\cdot))^{-1}(a\,\partial_z f) \end{array}$$

is well defined and continuous; if $k \le N-2$, the result remains true if we assume only that $a \in H^{N-2}(S)$.

(iv) *Under the same assumptions, we also have*

$$\forall\, \beta \in \mathbb{N}^d,\ \forall\, j \in \mathbb{N},\ |\beta| + j \le k,$$
$$(\partial^\beta \partial_z^j K^\iota[a\,\partial_z]f, \partial^\beta \partial_z^j f) \le C(\|a\|_{H^{N-1}})\|f\|_{H^k};$$

if $k \le N-2$*, we can replace* $\|a\|_{H^{N-1}}$ *by* $\|a\|_{H^{N-2}}$ *in the right-hand side.*

(v) *For all* $0 < \iota_1 \le \iota_2 < 1$ *and all* $f \in H^{N-1}(S)$*, we have*

$$\|K^{\iota_1}[a\,\partial_z]f - K^{\iota_2}[a\,\partial_z]f\|_{H^{N-3}} \le |\iota_1 - \iota_2| C(\|a\|_{H^{N-1}})\, \|f\|_{H^{N-1}}.$$

Proof. The fact that $(1 - \iota^2 \partial_z(a^2 \partial_z \cdot))$ is one-to-one follows immediately by taking $f = g$ in the following integration-by-parts formula (recall that a is assumed to cancel at the boundaries):

$$\forall\, f, g \in H(a\,\partial_z, S),$$
$$((1 - \iota^2 \partial_z(a^2 \partial_z \cdot))f, g) = (f,g) + \iota^2(a\,\partial_z f, a\,\partial_z g) := B(f,g). \tag{4.29}$$

In order to prove the second point of the lemma, we need to prove that, for all $f_1, f_2 \in L^2(S)$, there exists $u \in H(a\,\partial_z, S)$ such that

$$(1 - \iota^2 \partial_z(a^2 \partial_z \cdot))u = f_1 + \partial_z(af_2).$$

We prove the existence of a variational solution, that is, of $u \in H(a\,\partial_z, S)$ such that

$$\forall\, g \in H(a\,\partial_z; S), \quad B(u,g) = (f_1 + \partial_z(af_2), g).$$

Since the bilinear form is obviously continuous and coercive on $H(a\,\partial_z, S)$, the existence of a variational solution is granted by Lax-Milgram's theorem if we can prove that $g \mapsto (f_1 + \partial_z(af_2), g)$ defines a continuous linear form on $H(a\,\partial_z, S)$. Since a vanishes at the boundaries, one has

$$(f_1 + \partial_z(af_2), g) = (f_1, g) - (f_2, a\,\partial_z g) \le \|f_1\|_2\, \|g\|_2 + \|f_2\|_2\, \|a\,\partial_z g\|_2,$$

which implies the desired continuity property. Note, moreover, for later use that, because of (4.29), the solution $u = (1 - \iota^2 \partial_z(a^2 \partial_z \cdot))^{-1}(f_1 + \partial_z(af_2))$ satisfies the bound

$$\|u\|_2 + \iota\|a\,\partial_z u\|_2 \le 2\|f_1\|_2 + \frac{2}{\iota}\|f_2\|_2. \tag{4.30}$$

From the second point, we can then define $K^\iota[a\,\partial_z] : L^2(S) \to H(a\,\partial_z, S)$. For $f \in L^2(S)$, let $u = K^\iota[a\,\partial_z]f$. We need to prove that if $f \in H^k(S)$, then $u \in H^k(a\,\partial_z, S)$. Applying $\partial^\beta \partial_z^j$, with $|\beta| + j \le k$, to the relation

$$(1 - \iota^2 \partial_z(a^2 \partial_z \cdot))u = a\,\partial_z f = -(\partial_z a)f + \partial_z(af),$$

we obtain

$$(4.31)\qquad (1-\iota^2\,\partial_z(a^2\,\partial_z\cdot))\,\partial^\beta\,\partial_z^j u = f_1 + \partial_z(af_2),$$

where f_1 and f_2 are of the form

$$\begin{aligned} f_1 &= \partial_z[\partial^\beta\,\partial_z^j, a]f - \partial^\beta\,\partial_z^j((\partial_z a)f) \\ &\quad + \iota^2\,\partial_z\Big[\sum_{\substack{|\beta'|+j'\geq 1\\ |\beta''|+j''\geq 1}} *(\partial^{\beta'}\,\partial_z^{j'}a)(\partial^{\beta''}\,\partial_z^{j''}a)\,\partial^{\beta-\beta'-\beta''}\,\partial_z^{j-j'-j''}\,\partial_z u\Big] \\ f_2 &= \partial^\beta\,\partial_z^j f + \iota^2\sum_{|\beta'|+j'\geq 1} *(\partial^{\beta'}\,\partial_z^{j'}a)\,\partial^{\beta-\beta'}\,\partial_z^{j-j'}\,\partial_z u, \end{aligned}$$

where $*$ denotes numerical coefficients of no importance here. From (4.30) and (4.31), and using the product estimates as in the proof of Lemma 3.15, we have

$$\begin{aligned} \|\partial^\beta\,\partial_z^j u\|_2 + \iota\|a\,\partial_z\,\partial^\beta\,\partial_z^j u\|_2 &\lesssim \|a\|_{H^{N-1}}\,\|f\|_{H^k} + \iota^2||a||_{H^{N-1}}^2\,\|u\|_{H^k} \\ &\quad + \frac{1}{\iota}(\|f\|_{H^k} + \iota^2\|a\|_{H^{N-1}}\,\|u\|_{H^k}). \end{aligned}$$

Summing over all $|\beta|+j\leq k$, and taking ι small enough to absorb the terms in $\|u\|_{H^k}$ in the right-hand side by the left-hand side of the inequality, we get

$$(4.32)\qquad \|u\|_{H^k} + \iota\sum_{|\beta|+j\leq k}\|a\,\partial_z\,\partial^\beta\,\partial_z^j u\|_2 \leq C\left(\|a\|_{H^{N-1}},\frac{1}{\iota}\right)\|f\|_{H^k},$$

which proves the third point of the lemma.

For the fourth point, let us first prove the case $k=0$. With

$$\tilde f = (1-\iota^2\,\partial_z(a^2\,\partial_z\cdot))^{-1}f,$$

we have

$$\begin{aligned} (K^\iota[a\,\partial_z]f, f) &= (a\,\partial_z(1-\iota^2\,\partial_z(a^2\,\partial_z\cdot))\tilde f, \tilde f) \\ &= -\frac{1}{2}((\partial_z a)\tilde f, \tilde f) + \iota^2(a\,\partial_z\tilde f, \partial_z(a\tilde f)), \end{aligned}$$

and therefore

$$\begin{aligned} (K^\iota[a\,\partial_z]f, f) &\leq C(\|\partial_z a\|_\infty)(||\tilde f||_2^2 + \iota^2||a\,\partial_z\tilde f||_2^2) \\ &\leq C(\|\partial_z a\|_\infty)\,\|f\|_2, \end{aligned}$$

the second inequality stemming from (4.30); the result is therefore proved when $k = 0$. When $0 < k \le N-1$, we write, for all $|\beta| + j \le k$,

$$(\partial^\beta \partial_z^j K^\iota[a\,\partial_z]f, \partial^\beta \partial_z^j f) = (\partial^\beta \partial_z^j f, K^\iota[a\,\partial_z]\,\partial^\beta \partial_z^j f) \\ + (\partial^\beta \partial_z^j f, [\partial^\beta \partial_z^j, K^\iota[a\,\partial_z]]f);$$

we can use the case $k = 0$ to control the first term of the right-hand side. The commutator $[\partial^\beta \partial_z^j, K^\iota[a\,\partial_z]]f$ can then be controlled with the same computations as in the proof of the third point, but without the term $\partial^\beta \partial_z^j f$ in f_2, which was responsible for the ι^{-1} singularity in (4.32). We therefore have

$$\|[\partial^\beta \partial_z^j, K^\iota[a\,\partial_z]]f\| \le C(\|a\|_{H^{N-1}})\,\|f\|_{H^k},$$

and the result follows.

For the last point of the lemma, let us write $u_j = K^{\iota_j}[a\,\partial_z]f$ ($j = 1, 2$). We compute that

$$u_1 - u_2 = -(\iota_2^2 - \iota_2^2)(1 - \iota_1^2\,\partial_z(a^2\,\partial_z\cdot))^{-1}\,\partial_z(a^2\,\partial_z u_2);$$

using (4.32), we deduce that

$$\begin{aligned}\|u_1 - u_2\|_{H^{N-3}} &\le |\iota_2^2 - \iota_1^2| C(\|a\|_{H^{N-1}}) \|\partial_z(a^2\,\partial_z u_2)\|_{H^{N-3}} \\ &\le |\iota_2^2 - \iota_1^2| C(\|a\|_{H^{N-1}})\, \|u_2\|_{H^{N-1}} \\ &\le \frac{|\iota_2^2 - \iota_1^2|}{\iota_2} C(\|a\|_{H^{N-1}}) \|f\|_{H^{N-1}}\end{aligned}$$

(the last line follows from the computations performed in the proof of the third point), which implies the result stated in the lemma. □

Step 3. Relaxing the divergence-free condition on the vorticity. The div-curl problem has been solved in Section 2.2 if we assume that $\nabla_{X,z} \cdot \boldsymbol{\omega} = 0$, or equivalently $\nabla_{X,z}^\sigma \cdot \omega = 0$. In a manner consistent with Definition 2.3, we introduce the notation

$$\begin{aligned} H^k(\mathrm{div}_0^\sigma, S) &= \{\omega \in H^k(S) \mid \nabla_{X,z}^\sigma \cdot \omega = 0\}, \\ H_b^k(\mathrm{div}_0^\sigma, S) &= \{\omega \in H^k(\mathrm{div}_0^\sigma\ S) \mid \omega_{|_{z=-H_0}} \cdot N_b \in H_0^{-1/2}\}. \end{aligned}$$

If $\omega \notin H_b^k(\mathrm{div}_0^\sigma, S)$, it is therefore not possible to define $\mathbb{U}^\sigma[\zeta](\psi, \omega)$. The vorticity equation in (4.24) preserves the divergence-free property, so that these equations make sense if the vorticity field is initially divergence free. However, the regularization of the vorticity equation introduced in *Step 4* below does not preserve *a priori* the divergence-free condition (we only show *a posteriori* that it does so), and we therefore have to define the projection onto divergence-free vector fields as follows, for all $1 \leq k \leq N-1$:

$$\pi[\zeta] : \begin{array}{l} H_b^k(S) \to H^k(\mathrm{div}_0^\sigma, S), \\ \omega \mapsto \omega - \nabla_{X,z}^\sigma \varphi, \end{array}$$

where $H_b^k(S)$ denotes the set of all $\omega \in H^k(S)$ such that $\omega_b \cdot N_b \in H_0^{-1/2}(\mathbb{R}^d)$, while φ solves the boundary value problem

$$\begin{cases} \nabla_{X,z} \cdot P(\Sigma)\nabla_{X,z}\varphi = (1+\partial_z\sigma)\nabla_{X,z}^\sigma \cdot \omega, \\ \varphi_{|_{z=0}} = 0, \\ \mathbf{e}_z \cdot P(\Sigma)\nabla_{X,z}\varphi_{|_{z=-H_0}} = 0, \end{cases}$$

and where we recall that $P(\Sigma) = (1+\partial_z\sigma)J_\Sigma^{-1}(J_\Sigma^{-1})^T$ and $(1+\partial_z\sigma)\nabla_{X,z}^\sigma \cdot \nabla_{X,z}^\sigma = \nabla_{X,z} \cdot P(\sigma)\nabla_{X,z}$. By simple elliptic estimates similar to those of Corollary 3.14, we get that, for all $0 \leq j \leq N-1$,

$$\|\nabla_{X,z}\varphi\|_{H^k} \leq M_N(\|\omega\|_{H^k} + |\omega_b \cdot N_b|_{H_0^{-1/2}}).$$

In particular, if $\omega \in H^1(S)$ and $\omega_b \cdot N_b \in H_0^{-1/2}(\mathbb{R}^d)$ (note that the H^1-regularity is imposed on ω so that the normal trace of ω at the bottom makes sense), then $\pi[\zeta]\omega \in H_b^1(S, \mathrm{div}_0^\sigma, S)$; when $\omega \in H^1(S) \setminus H_b^1(\mathrm{div}_0^\sigma, S)$, we can therefore replace $\mathbb{U}^\sigma[\zeta](\psi,\omega)$ by $\mathbb{U}^\sigma[\zeta](\psi, \pi[\zeta]\omega)$, which is well defined. We also have that, for all $1 \leq k \leq N-1$,

$$\forall\, \omega \in H_b^k(S), \quad \|\pi[\zeta]\omega\|_{H^k} \leq M_N(\|\omega\|_{H^k} + |\omega_b \cdot N_b|_{H_0^{-1/2}}). \tag{4.33}$$

Step 4. Existence of a local solution for a regularized system. We consider the following regularization of the water waves equations (4.26):

$$\partial_t\Theta + \mathcal{F}^{\delta,\iota}(\Theta) = 0, \tag{4.34}$$

with $0 < \delta, \iota < 1$, and

$$\begin{aligned} \mathcal{F}_1^{\delta,\iota}(\Theta) &= -J^\iota(\underline{\mathbb{U}}^\sigma[\zeta](\psi, \pi[\zeta]\omega) \cdot N), \\ \mathcal{F}_2^{\delta,\iota}(\Theta) &= gJ^\iota\zeta + \frac{1}{2}J^\iota|\mathbb{U}_\parallel^\sigma[\zeta](\psi, \pi[\zeta]\omega)|^2 \\ &\qquad - \frac{1}{2}J^\iota[(1+|\nabla\zeta|^2)\underline{\mathbb{w}}^\sigma[\zeta](\psi,\omega)^2] + \delta J^\iota\Lambda\zeta, \\ \mathcal{F}_3^{\delta,\iota}(\Theta) &= \mathbf{g}^\iota - \nabla_{X,z}^\sigma Q, \end{aligned}$$

with

$$\mathbf{g}^\iota = J^\iota(\mathbb{V}^\sigma[\zeta](\psi, \pi[\zeta]\omega) \cdot \nabla\omega) + K^\iota[\mathbb{a}[\zeta](\psi, \pi[\zeta]\omega)\,\partial_z]\omega \\ - J^\iota(\omega \cdot \nabla^\sigma_{X,z}\mathbb{U}^\sigma[\zeta](\psi, \pi[\zeta]\omega)).$$

We also consider regularized initial conditions

$$\Theta_{|t=0} = (J^\iota\zeta^0, J^\iota\psi^0, \omega^0).$$

The reasons why such a regularization is introduced are listed below:

- *Projection onto the divergence-free component of the vorticity.* As already mentioned, the mapping $\mathbb{U}^\sigma[\zeta](\psi, \omega)$ is well defined provided that $\nabla^\sigma_{X,z} \cdot \omega = 0$. Since such a condition is not necessarily propagated from the initial condition by the regularized vorticity equation, we have to replace $\mathbb{U}^\sigma[\zeta](\psi, \omega)$ by $\mathbb{U}^\sigma[\zeta](\psi, \pi[\zeta]\omega)$ in all its occurrences.
- *Mollifiers.* Horizontal derivatives are regularized with the mollifier J^ι. For the vertical derivative, the mollifier $K^\iota[a\,\partial_z]$ is used; this is made possible by the fact that a vanishes at the surface and at the bottom.
- *Dispersive regularization with parameter δ.* The purpose of the term $\delta J^\iota\Lambda\zeta$ in $\mathcal{F}_2^{\delta,\iota}(\Theta)$ is to allow for the control of an extra $\frac{1}{2}$-derivative on ζ, uniformly with respect to ι, and this will be used to control commutator terms due to the above mollifiers.
- *Pressure term Q.* If we simply regularize the equation as explained above, the equation no longer preserves the divergence-free condition. To recover this property, we choose the pressure Q so that $\partial_t \operatorname{div}^\sigma \omega = 0$. This leads to[9]

$$\begin{cases} \nabla_{X,z} \cdot P(\Sigma)\nabla_{X,z}Q \\ \qquad = (1 + \partial_z\sigma)\nabla^\sigma_{X,z} \cdot \mathbf{g}^\iota + \nabla^\sigma_{X,z}\left(\dfrac{z + H_0}{H_0}J^\iota(\underline{U} \cdot N)\right) \cdot \partial_z\omega, \\ Q_{|z=0} = 0, \\ \mathbf{e}_z \cdot P(\Sigma)\nabla_{X,z}Q = 0 \end{cases} \tag{4.35}$$

(we recall that $(1 + \partial_z\sigma)\nabla^\sigma_{X,z} \cdot \nabla^\sigma_{X,z} = \nabla_{X,z} \cdot P(\Sigma)\nabla_{X,z}$).

Before proceeding to construct a solution to the regularized equations (4.34), we need to provide some estimates on the pressure term $\nabla^\sigma_{X,z}Q$. We shall use the following lemma.

[9]We use the fact that $\partial^\sigma_t = \partial_t - (\partial_t\sigma/(1 + \partial_z\sigma\,\partial_z))\,\partial_z$, and substitute

$$\partial_t\sigma = \frac{z + H_0}{H_0}J^\iota(\underline{U} \cdot N),$$

which holds provided that the regularized kinematic equation is satisfied.

Lemma 4.9. *We let* $k = N-2, N-1$, $\mathbf{g} \in H^k(S)$, *and*

$$\mathbf{h} \in W^{k,\infty}((-H_0,0); H^k(\mathbb{R}^d))$$

be such that $\mathbf{h}_{|_{z=-H_0}} = 0$. *Then, there is a unique solution* $Q \in \dot{H}^{k+1}(S)$ *to*

$$\begin{cases} \nabla_{X,z} \cdot P(\Sigma)\nabla_{X,z} Q = (1+\partial_z\sigma)\nabla^{\sigma}_{X,z} \cdot \mathbf{g} + \mathbf{h} \cdot \partial_z \omega, \\ Q_{|_{z=0}} = 0, \\ \mathbf{e}_z \cdot P(\Sigma)\nabla^{\sigma}_{X,z} Q_{|_{z=-H_0}} = 0, \end{cases}$$

and one has

$$\|\nabla^{\sigma}_{X,z} Q\|_{H^k} \leq M_N(\|\mathbf{g}\|_{H^k} + \|\mathbf{h}\|_{W^{k,\infty}H^k}\|\omega\|_{H^k}).$$

Proof. Existence of a solution follows classically from Lax-Milgram's theorem. In order to get an L^2-estimate on $\nabla_{X,z} Q$ (and therefore on $\nabla^{\sigma}_{X,z} Q$), we multiply by Q, integrate by parts, and use the coercivity property (3.33) of $P(\Sigma)$ to get

$$\left|\left|\nabla_{X,z} Q\right|\right|_2^2 \leq M_N \|\mathbf{g}\|_2 \, \|\nabla_{X,z} Q\|_2 + \int_{\mathbb{R}^d} g_b \cdot N_b Q_b + \|\mathbf{h}\|_{W^{1,\infty}} \, \|\omega\|_2 \, \|\nabla_{X,z} Q\|,$$

where we used for the last term the fact that $\mathbf{h}$ vanishes at the bottom, and Q vanishes at the surface. From the trace lemma, we then get

$$\|\nabla_{X,z} Q\|_2 \leq M_N(\|\mathbf{g}\|_{H^{0,1}} + \|\mathbf{h}\|_{W^{1,\infty}} \, \|\omega\|_2).$$

In order to control horizontal derivatives of $\nabla_{X,z} Q$, just note that $\tilde{Q} = \Lambda^s Q$ solves

$$\begin{cases} \nabla_{X,z} \cdot P(\Sigma)\nabla_{X,z} \tilde{Q} = (1+\partial_z\sigma)\nabla^{\sigma}_{X,z} \cdot (\Lambda^s \mathbf{g} - J_\Sigma[\Lambda^s, P(\Sigma)]\nabla_{X,z} Q) \\ \qquad\qquad\qquad\qquad + \mathbf{h} \cdot \partial_z(\Lambda^s \omega) + f, \\ \tilde{Q}_{|_{z=0}} = 0, \\ \mathbf{e}_z \cdot P(\Sigma)\nabla^{\sigma}_{X,z} \tilde{Q}_{|_{z=-H_0}} = -\mathbf{e}_z \cdot [\Lambda^s, P(\Sigma)]\nabla_{X,z} Q_{|_{z=-H_0}}, \end{cases}$$

with $f = [\Lambda^s, N(z)/(1+\partial_z\sigma)]\,\partial_z \mathbf{g} + [\Lambda^s, \mathbf{h}]\,\partial_z\omega$. Proceeding as above, we get

$$\|\nabla_{X,z} \tilde{Q}\|_2 \leq M_N(\|\mathbf{g}\|_{H^{s,1}} + \|\,[\Lambda^s, P(\Sigma)]\nabla_{X,z} Q\|_2 + \|\mathbf{h}\|_{W^{1,\infty}} \, \|\omega\|_{H^{s,0}} + \|f\|_2).$$

Controlling the commutator in terms of $s-1$ derivatives of $\nabla_{X,z} Q$, we get after a finite induction (see Proposition 2.36 of [39] for details on the control of this commutator term) that, for all $0 \leq s \leq N-1$,

$$\|\Lambda^s \nabla_{X,z} Q\|_2 = \|\nabla_{X,z} \tilde{Q}\|_2 \leq M_N(\|\mathbf{g}\|_{H^{s,1}} + \|\mathbf{h}\|_{W^{1,\infty}} \, \|\omega\|_{H^{s,0}} + \|f\|_2).$$

Using commutator estimates (e.g., Corollary B.16 in [39]) and the expression of f, we also get (with $t_0 > d/2$)

$$\|f\|_2 \leq M_N(\|\mathbf{g}\|_{H^{s,1}} + \|\mathbf{h}\|_{L^\infty H^{s\vee(t_0+1)}} \|\omega\|_{H^{s,1}}),$$

and therefore

$$\|\nabla_{X,z} Q\|_{H^{s,0}} \leq M_N(\|\mathbf{g}\|_{H^{s,1}} + \|\mathbf{h}\|_{W^{1,\infty} H^{s\vee(t_0+1)}} \|\omega\|_{H^{s,1}}).$$

Using the equation to express $\partial_z^2 Q$ in terms of first- and second-order derivatives of Q involving at most one vertical derivative, we get, for all $1 \leq k \leq s$,

$$\|\nabla_{X,z} Q\|_{H^{s,k}} \leq M_N(\|\mathbf{g}\|_{H^{s,k}} + \|\mathbf{h}\|_{W^{k,\infty} H^{s\vee(t_0+1)}} \|\omega\|_{H^{s,k}}),$$

and the result of the lemma follows. $\square$

Applying the lemma to (4.35) with

$$k = N-2, \ \mathbf{g} = \mathbf{g}^\iota, \text{ and } \mathbf{h} = \nabla_{X,z}^\sigma \left(\left(\frac{z+H_0}{H_0} \right) J^\iota(\underline{U} \cdot N) \right),$$

we deduce from the product estimate $\|fg\|_{H^{N-2}} \lesssim \|f\|_{H^{N-2}} \|g\|_{H^{N-2}}$ and the third point of Lemma 4.8 that

$$\begin{aligned} \|\nabla_{X,z}^\sigma Q\|_{H^{N-2}} &\lesssim (\|U\|_{H^{N-1}} + \|a\|_{H^{N-2}})\|\omega\|_{H^{N-2}} \\ &\lesssim C(|\zeta|_{H^N}, |\nabla\psi|_{H^{N-1}}, \|\omega\|_{H^{N-2}}, |\omega_b \cdot N_b|_{H_0^{-1/2}}), \end{aligned} \tag{4.36}$$

where we used the definition of a in terms of U and Corollary 3.14 to derive the second inequality.

We can now proceed to construct a solution to the regularized system (4.34). Let us introduce the space $\mathbb{X}$ and its open subset $\mathbb{X}_0$ as

$$\begin{aligned} \mathbb{X} &= H^N(\mathbb{R}^d) \times \dot{H}^N(\mathbb{R}^d) \times H_b^{N-2}(\mathcal{S}), \\ \mathbb{X}_0 &= \{(\zeta, \psi, \omega) \in \mathbb{X} : \inf_{\mathbb{R}^d} |H_0 + \zeta| > 0, \ \mathfrak{a}(\zeta, \psi, \omega) > 0\}. \end{aligned}$$

From (4.27) and Corollary 3.14, note that $\mathcal{F}_j^{\delta,\iota}$ ($j = 1,2$) define smooth mappings[10] on $\mathbb{X}_0$ with values in $H^\infty(\mathbb{R}^d)$. Using also the third point of Lemma 4.8, together with (4.36), $\mathcal{F}_3^{\delta,\iota}$ defines a smooth mapping on $\mathbb{X}_0$ with values in $H^{N-2}(\mathcal{S})$. We can therefore deduce that $\mathcal{F}^{\iota,\delta}$ maps $\mathbb{X}_0$ into $\mathbb{X}$ provided that, for

[10] This follows from the fact that $\mathbb{U}^\sigma[\zeta](\psi, \omega)$ has a Lipschitz dependence on ζ, ψ, and ω, as shown in Corollary 3.19.

all $\Theta \in \mathbb{X}_0$, we have $\mathcal{F}_3^{\delta,\iota}(\Theta)_{|_{z=-H_0}} \cdot N_b \in H_0^{-1/2}(\mathbb{R}^d)$. After recalling that a and w vanish at the bottom, we compute

$$\mathcal{F}_3^{\delta,\iota}(\Theta)_{|_{z=-H_0}} \cdot N_b = J^\iota(V \cdot \nabla \omega_v)_{|_{z=-H_0}} - J^\iota(\omega_v\, \partial_z^\sigma w)_{|_{z=-H_0}},$$

where ω_v stands for the vertical component of ω. Now, since by construction U is divergence free, we have $\partial_z^\sigma w_{|_{z=-H_0}} = -\nabla \cdot V_b$, and therefore

$$\mathcal{F}_3^\iota(\Theta)_{|_{z=-H_0}} \cdot N_b = \nabla \cdot J^\iota(V\omega_v)_{|_{z=-H_0}},$$

which implies the desired result. It follows that $\mathcal{F}^{\iota,\delta}$ defines a smooth mapping $\mathbb{X}_0$ with values in $\mathbb{X}$; by standard results on ODEs, there exists a maximal existence time $T^{\iota,\delta}$ such that there exists a unique solution $\Theta \in C^1([0, T^{\iota,\delta}); \mathbb{X})$ to (4.34).

Step 5. Properties of the solution to the regularized system. The solution constructed in the previous step has some extra regularity properties. We have, for instance, $(\zeta, \psi) \in C([0, T^{\iota,\delta}); H^\infty(\mathbb{R}^d) \times \dot{H}^\infty(\mathbb{R}^d))$. We deduce in particular that, for all $\alpha \in \mathbb{N}^d$, $1 < |\alpha| \leq N$, we have

$$|\partial^\alpha \psi - \underline{\mathrm{w}}^\sigma[\zeta](\psi, \pi[\zeta]\omega)\, \partial^\alpha \zeta|_{\dot{H}^{1/2}} < \infty.$$

We can therefore deduce from the third point of Lemma 4.8, Corollary 3.14, and (4.33) that, with ζ and ψ fixed, the mapping $\mathcal{F}_3^{\iota,\delta}(\zeta, \psi, \cdot)$ maps $H^{N-1}(S)$ into itself, from which we classically deduce that

$$\omega \in C([0, T^{\iota,\delta}); H^{N-1}(S)).$$

In addition to this regularity property, we now show that ω remains divergence free. After noting that

$$\begin{aligned}
\mathrm{div}^\sigma\ \partial_t \omega &= \mathrm{div}^\sigma(\partial_t^\sigma \omega + \partial_t \sigma\, \partial_z^\sigma \omega) \\
&= \partial_t^\sigma\, \mathrm{div}^\sigma\ \omega + \nabla_{X,z}^\sigma(\partial_t \sigma) \cdot \partial_z^\sigma \omega + \partial_t \sigma\, \partial_z^\sigma\, \mathrm{div}^\sigma\ \omega \\
&= \partial_t\, \mathrm{div}^\sigma\ \omega + \nabla_{X,z}^\sigma(\partial_t \sigma) \cdot \partial_z^\sigma \omega,
\end{aligned}$$

we can use the definition (4.35) of the pressure Q, and apply div^σ to the vorticity equation to get

$$\partial_t\, \mathrm{div}^\sigma\ \omega + \nabla_{X,z}^\sigma(\partial_t \sigma) \cdot \partial_z^\sigma \omega - \nabla_{X,z}^\sigma \Big(\frac{z + H_0}{H_0} J^\iota(\underline{U} \cdot N) \Big) \cdot \partial_z^\sigma \omega = 0.$$

Since we get from the equation on ζ and the definition of σ that

$$\frac{z + H_0}{H_0} J^\iota(\underline{U} \cdot N) = \partial_t \sigma,$$

we finally obtain that $\partial_t\, \mathrm{div}^\sigma\ \omega = 0$. Since the initial condition ω^0 is assumed to be divergence free, this yields that $\mathrm{div}^\sigma\ \omega = 0$ on $[0, T^{\iota,\zeta})$. A consequence of this is that $\omega = \pi[\zeta]\omega$, so that we can drop all the occurrences of $\pi[\zeta]$ in (4.34).

Step 6. Uniform energy estimates. Proceeding exactly as for Proposition 4.5, but with the regularized equations (4.34), we obtain, with the same notation,

$$
\begin{aligned}
(\partial_t + J^\iota(\underline{V}\cdot\nabla))\,\partial^\alpha\zeta - J^\iota\,\partial_k\underline{U}_{(\beta)}\cdot N &= J^\iota R^1_\alpha,\\
(\partial_t + J^\iota(\underline{V}\cdot\nabla))(U_{(\beta)\parallel}\cdot\mathbf{e}_k) + J^\iota(\mathfrak{a}\,\partial^\alpha\zeta) + \delta J^\iota\Lambda\,\partial^\alpha\zeta &= J^\iota R^2_\alpha + \tilde{R}^2_\alpha,\\
(\partial^\sigma_t + J^\iota(V\cdot\nabla) + K^\iota[a^\iota\,\partial_z])\,\partial^\beta\omega &= R^3_\beta,
\end{aligned}
$$

with the R^j_α satisfying the same estimates as in Proposition 4.2 (for the vorticity equation, we use the fact that $\|\nabla_{X,z}Q\|_{H^{N-1}} \le \mathfrak{m}^N(\zeta,\psi,\omega)$, which is a consequence of Lemma 4.9), and

$$
\tilde{R}^2_\alpha = -(1-J^\iota)(\partial_t\underline{w}\,\partial^\alpha\zeta) + [\underline{w}, J^\iota]\,\partial^\alpha(\underline{\mathbb{U}}^\sigma[\zeta](\psi,\omega)\cdot N).
$$

The control of this extra term (coming from commutators with the mollifiers J^ι) in $\dot{H}^{1/2}$ norm requires a control of ζ in $H^{N+1/2}(\mathbb{R}^d)$ instead of $H^N(\mathbb{R}^d)$, namely,

$$
|\mathfrak{P}\tilde{R}^2_\alpha|_2 \le \mathfrak{m}^N(\zeta,\psi,\omega)(1 + |\partial^\alpha\zeta|_{H^{1/2}});
$$

the dispersive regularization $\delta J^\iota\Lambda\zeta$ has been added to the second equation in order to control this extra term. Proceeding exactly as for (4.15) except for the fact that the first equation is multiplied by $\mathfrak{a}\,\partial^\alpha\zeta + \delta\Lambda\,\partial^\alpha\zeta$ instead of $\mathfrak{a}\,\partial^\alpha\zeta$, we obtain therefore

$$
\begin{aligned}
\frac{1}{2}\partial_t\big[(\mathfrak{a}\,\partial^\alpha\zeta, \partial^\alpha\zeta) + \delta\,|\partial^\alpha\zeta|^2_{H^{1/2}}\big]&\\
+ \big((\partial_t + J^\iota(\underline{V}\cdot\nabla\cdot))(U_{(\beta)\parallel}\cdot\mathbf{e}_k), \partial_k\underline{U}_{(\beta)}\cdot N\big)&\\
\le \mathfrak{m}^N(\zeta,\psi,\omega)(1 + |\partial^\alpha\zeta|_{H^{1/2}}).&
\end{aligned}
\tag{4.37}
$$

As in the proof of Proposition 4.5, the second term is handled using Green's formula; because of the presence of the mollifiers, (4.16) must be replaced by

$$
(\partial_t + J^\iota(\underline{V}\cdot\nabla\cdot))(U_{(\beta)\parallel}\cdot\mathbf{e}_k) = \big((\partial_t + J^\iota(V\cdot\nabla) + K^\iota[a^\iota\,\partial_z])(U^\flat_{(\beta)\parallel}\cdot\mathbf{e}_k)\big)_{|z=0}
$$

(we recall that a^ι vanishes at the surface). We are thus led to replace (4.17) by

$$
\begin{aligned}
&\big((\partial_t + J^\iota(\underline{V}\cdot\nabla\cdot))(U_{(\beta)\parallel}\cdot\mathbf{e}_k), \partial_k\underline{U}_{(\beta)}\cdot N\big)\\
&\quad= \int_S (1+\partial_z\sigma)\big[(\partial_t + J^\iota(V\cdot\nabla) + K^\iota[a^\iota\,\partial_z])\nabla^\sigma_{X,z}(U^\flat_{(\beta)\parallel}\cdot\mathbf{e}_k)\big]\cdot\partial_k U_{(\beta)}\\
&\qquad + \text{l.o.t.}
\end{aligned}
$$

From Lemma 4.6, we therefore get

$$\begin{aligned}
&((\partial_t + J^\iota(\underline{V}\cdot\nabla))(U_{(\beta)\parallel}\cdot\mathbf{e}_k), \partial_k \underline{U}_{(\beta)}\cdot N)\\
&\quad = \int_S (1+\partial_z\sigma)\big[(\partial_t + J^\iota(V\cdot\nabla) + K^\iota[a^\iota\partial_z])\,\partial_k U_{(\beta)}\big]\cdot\partial_k U_{(\beta)}\\
&\qquad\quad + \int_S (1+\partial_z\sigma)\big[\mathbf{t}_k\times(\partial_t + J^\iota(V\cdot\nabla) + K^\iota[a^\iota\partial_z])\,\partial^\beta\omega\big]\cdot\partial_k U_{(\beta)} + \text{l.o.t.}\\
&\quad = A + B.
\end{aligned}$$

Integrating by parts (using the fourth point of Lemma 4.8 for the term involving $K^\iota[a^\iota\partial_z]$), one readily obtains that the identity (4.18) remains true up to lower-order terms, while B is controlled as in (4.19). The energy inequality (4.21) can therefore be adapted into

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^N_\delta(\zeta,\psi,\omega) \le C\left(\frac{1}{a_0},\frac{1}{\delta},\tilde{\mathcal{E}}^N_\delta(\zeta,\psi,\omega)\right), \tag{4.38}$$

where

$$\tilde{\mathcal{E}}^N_\delta(\zeta,\psi,\omega) = \tilde{\mathcal{E}}^N(\zeta,\psi,\omega) + \delta|\zeta|^2_{H^{N+1/2}}.$$

Step 7. Conclusion. The energy estimate (4.38) is uniform with respect to ι. In particular, the solutions $(\Theta^{\iota,\delta})_{\iota,\delta}$ to (4.34) exist on a nontrivial time interval $[0,T^\delta]$ independent of ι, and one can prove that, with $\delta>0$ fixed, the sequence $(\theta^{\iota,\delta})_{\iota,\delta}$ is a Cauchy sequence in $H^2\times\dot H^2\times H^2(S)$ as $\iota\to 0$.

Lemma 4.10. *For all* $0<\delta<1$, $(\Theta^{\iota,\delta})_\iota$ *is a Cauchy sequence, as* $\iota\to 0$, *in* $C([0,T^\delta];H^2(\mathbb{R}^d)\times\dot H^2(\mathbb{R}^d)\times H^2(S))$.

Proof. We omit the proof of this result because it is very similar to what happens in the irrotational case, and the proof of Lemma 4.28 in [39] can therefore easily be adapted. Note that this result is a consequence of (4.28). The only additional property that is needed here compared to the irrotational case is that a similar property holds for the vertical mollifier $K^\iota[a\,\partial_z]$, but this is already proved in Lemma 4.8(v). ◻

The end of the proof is then quite similar to the irrotational case, and we only give the main steps (see Section 4.3.4.4 of [39] for more details). We first deduce the existence of a limit $\Theta^\delta\in C([0,T^\delta];H^2(\mathbb{R}^d)\times\dot H^2(\mathbb{R}^d)\times H^2(S))$ to the sequence $(\Theta^{\iota,\delta})_\iota$ as $\iota\to 0$; this limit solves

$$\partial_t\Theta^\delta + \mathcal{F}^{\delta,0} = 0,$$

where $\mathcal{F}^{\delta,0}$ is deduced from $\mathcal{F}^{\delta,\iota}$ by setting ι to 0; moreover, Θ^δ also satisfies the energy estimate (4.38). Because of the dependence on $1/\delta$ of this energy estimate, we cannot directly extract a converging sequence from $(\Theta^\delta)_\delta$ as $\delta \to 0$. However, this singular dependence on δ of the energy estimate comes from the term $(1 + |\partial^\alpha \zeta|_{H^{1/2}})$ in the right-hand side of (4.37), which itself comes from the control of $\tilde{R}_\alpha^2$. Since this term vanishes when $\iota = 0$, we can replace (4.38) by a nonsingular (with respect to δ) energy estimate:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_\delta^N(\zeta,\psi,\omega) \le C\left(\frac{1}{a_0}, \tilde{\mathcal{E}}_\delta^N(\zeta,\psi,\omega)\right). \tag{4.39}$$

This uniform bound can then be used to prove, as in Lemma 4.10, that $(\Theta^\delta)_{0<\delta<1}$ is a Cauchy sequence as $\delta \to 0$, and that it converges in

$$C([0,T]; H^2(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d) \times H^2(S)), \quad \text{with } T > 0 \text{ independent of } \delta,$$

to a solution Θ of the nonregularized equations (4.26). Moreover, the solution satisfies (4.39) with δ set to 0, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}^N(\zeta,\psi,\omega) \le C\left(\frac{1}{a_0}, \tilde{\mathcal{E}}^N(\zeta,\psi,\omega)\right),$$

and the bound on $\mathcal{E}^N(\Theta)$ given in the statement of the theorem follows as for the *a priori* estimates of Proposition 4.5. Uniqueness of the solution is then obtained by estimating the difference of two solutions in $C([0,T]; H^2(\mathbb{R}^d) \times \dot{H}^2(\mathbb{R}^d) \times H^2(S))$ using the uniform bound provided by the energy estimates, along lines quite similar to the proof of Lemma 4.10. □

5. ASYMPTOTIC REGIMES

5.1. The dimensionless free surface Euler equations. The fluid motion depends qualitatively on several physical parameters: the typical amplitude a of the waves, the depth at rest H_0, and the typical horizontal scale L. Using these quantities, it is possible to form two dimensionless parameters:

$$\varepsilon = \frac{a}{H_0}, \quad \mu = \frac{H_0^2}{L^2};$$

the parameter ε is often called the nonlinearity (or amplitude) parameter, and the parameter μ is the shallowness parameter.

We also use a, H_0, and L to define dimensionless variables and unknowns (written with a tilde):

$$\tilde{z} = \frac{z}{H_0}, \quad \tilde{X} = \frac{X}{L}, \quad \tilde{\zeta} = \frac{\zeta}{a};$$

the nondimensionalization of the time variable, velocity, and pressure fields is less obvious, and is based on linear analysis of the equations (see, e.g., [39, Chapter 1])

$$\tilde{\mathbf{V}} = \frac{\mathbf{V}}{V_0}, \quad \tilde{\mathbf{w}} = \frac{\mathbf{w}}{w_0}, \quad \tilde{t} = \frac{t}{t_0}, \quad \tilde{P} = \frac{P}{P_0},$$

with

$$V_0 = a\sqrt{\frac{g}{H_0}}, \quad w_0 = \frac{aL}{H_0}\sqrt{\frac{g}{H_0}}, \quad t_0 = \frac{L}{\sqrt{gH_0}}, \quad P_0 = \rho g H_0.$$

With these variables and unknowns, and with the notation

$$\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu}\,\mathbf{V} \\ \mathbf{w} \end{pmatrix}, \quad \nabla^\mu = \begin{pmatrix} \sqrt{\mu}\,\nabla \\ \partial_z \end{pmatrix}, \quad N^\mu = \begin{pmatrix} -\varepsilon\sqrt{\mu}\,\nabla\zeta \\ 1 \end{pmatrix},$$

and

$$\mathrm{curl}^\mu = \nabla^\mu \times, \quad \mathrm{div}^\mu = (\nabla^\mu)^T, \quad \underline{U}^\mu = (\sqrt{\mu}\,\underline{V}^T, \underline{w})^T := \mathbf{U}^\mu_{|_{z=\varepsilon\zeta}},$$

the incompressible Euler equations take the form (omitting the tildes)

$$\begin{aligned} &\partial_t \mathbf{U}^\mu + \frac{\varepsilon}{\mu}\mathbf{U}^\mu \cdot \nabla^\mu \mathbf{U}^\mu = -\frac{1}{\varepsilon}(\nabla^\mu P + \mathbf{e}_z) && \text{in } \Omega, \\ &\mathrm{div}^\mu\, \mathbf{U}^\mu = 0 && \text{in } \Omega, \end{aligned}$$

where Ω now stands for the dimensionless fluid domain,

$$\Omega = \{(X, z) \in \mathbb{R}^{d+1} \mid -1 < z < \varepsilon\zeta(t, X)\},$$

and with the nonvanishing depth condition now reading

$$\exists\, h_{\min} > 0,\ \forall\, X \in \mathbb{R}^d, \quad 1 + \varepsilon\zeta \geq h_{\min}. \tag{5.1}$$

Finally, the boundary conditions on the velocity read in dimensionless form

$$\begin{aligned} &\partial_t \zeta - \frac{1}{\mu}\underline{U}^\mu \cdot N^\mu = 0 && \text{at the surface,} \\ &\mathbf{U}^\mu_{|_{z=-1}} \cdot N^\mu_b = 0 && \text{at the bottom,} \end{aligned}$$

where $N^\mu_b = \mathbf{e}_z$, while for the pressure, we still have $P = 0$ at the surface.

5.2. Notation. We give here a list of notation specific to the study of the shallow-water regime. Most of them are the dimensionless version of notation already used in the dimensional case; for the sake of clarity, we write them in the same way. Below is a list of adaptations we need to make to handle the dimensionless case:

$$\begin{aligned}
\Omega = \{(X,z) \mid -H_0 < z < \zeta(X)\} &\rightsquigarrow \Omega = \{(X,z) \mid -1 < z < \varepsilon\zeta(X)\},\\
S = \mathbb{R}^d \times (-H_0, 0) &\rightsquigarrow S = \mathbb{R}^d \times (-1, 0),\\
\sigma(X,z) = \zeta\frac{z+H_0}{H_0} &\rightsquigarrow \sigma(X,z) = \varepsilon\zeta(z+1),\\
\mathfrak{P} = \frac{|\mathrm{D}|}{(1+|\mathrm{D}|)^{1/2}} &\rightsquigarrow \mathfrak{P} = \frac{|\mathrm{D}|}{(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}},\\
|u|_{H_0^{-1/2}} = \left|\frac{1}{|\mathrm{D}|}u\right|_{H^{1/2}} &\rightsquigarrow |u|_{H_0^{-1/2}} = \left|\frac{(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}}{|\mathrm{D}|}u\right|_2.
\end{aligned}$$

We also adapt the notation (1.6) as follows:

$$A_\parallel = \frac{1}{\sqrt{\mu}}\underline{A}_h + \varepsilon\underline{A}_v\nabla\zeta \tag{5.2}$$

so that

$$\underline{A}\times^\mu = \sqrt{\mu}\begin{pmatrix} -A_\parallel^\perp \\ -\varepsilon\sqrt{\mu}A_\parallel^\perp\cdot\nabla\zeta\end{pmatrix}.$$

Finally, we also write

$$\nabla^{\sigma,\mu} = \begin{pmatrix}\sqrt{\mu}\,\nabla\\ 0\end{pmatrix} + \begin{pmatrix}-\sqrt{\mu}\,\nabla\sigma\\ 1\end{pmatrix}\partial_z^\sigma. \tag{5.3}$$

5.3. The dimensionless generalized ZCS formulation. According to the notation (5.2), we have

$$U_\parallel^\mu = \underline{V} + \varepsilon\underline{w}\nabla\zeta = \frac{1}{\sqrt{\mu}}(\underline{U}^\mu \times N^\mu)_h^T,$$

and proceeding as in Section 2.3, we deduce the following dimensionless version of (2.1),

$$\partial_t U_\parallel^\mu + \nabla\zeta + \frac{\varepsilon}{2}\nabla|U_\parallel^\mu|^2 - \frac{\varepsilon}{2\mu}\nabla((1+\varepsilon^2\mu|\nabla\zeta|^2)\underline{w}^2) + \varepsilon\underline{\omega}_\mu\cdot N^\mu\underline{V}^\perp = 0, \tag{5.4}$$

where $\underline{\omega}_\mu = \boldsymbol{\omega}_{\mu|_{z=\varepsilon\zeta}}$, and $\boldsymbol{\omega}_\mu$ is given by

$$\boldsymbol{\omega}_\mu = \begin{pmatrix}\frac{1}{\sqrt{\mu}}(\partial_z\mathbf{V}^\perp - \nabla^\perp\mathbf{w})\\ -\nabla\cdot\mathbf{V}^\perp\end{pmatrix} = \frac{1}{\mu}\operatorname{curl}^\mu\mathbf{U}^\mu.$$

The dimensionless version of the orthogonal decomposition of $U_\parallel$ performed in Section 2.3 is then given by $U_\parallel^\mu = \nabla\psi + \nabla^\perp\tilde{\psi}$, with $\Delta\tilde{\psi} = \underline{\omega}_\mu \cdot N^\mu$. The equation on ψ corresponding to the dimensionless version of (2.2) is therefore

$$(5.5)\qquad \partial_t\psi + \zeta + \frac{\varepsilon}{2}|U_\parallel^\mu|^2 - \frac{\varepsilon}{2\mu}((1+|\nabla\zeta|^2)\underline{w}^2) + \varepsilon\frac{\nabla}{\Delta}\cdot(\underline{\omega}_\mu \cdot N^\mu \underline{V}^\perp) = 0.$$

Finally, the dimensionless vorticity equation is obtained by applying ∇^μ to the dimensionless Euler equation:

$$(5.6)\qquad \partial_t\boldsymbol{\omega}_\mu + \frac{\varepsilon}{\mu}\mathbf{U}^\mu\cdot\nabla^\mu\boldsymbol{\omega}_\mu = \frac{\varepsilon}{\mu}\boldsymbol{\omega}_\mu\cdot\nabla^\mu\mathbf{U}^\mu.$$

In order to write $\mathbf{U}^\mu$ as a function of ζ, ψ, and $\boldsymbol{\omega}_\mu$, we need to solve the following dimensionless version of the div-curl problem (2.7):

$$(5.7)\qquad \begin{cases} \operatorname{curl}^\mu \mathbf{U}^\mu = \mu\boldsymbol{\omega}_\mu & \text{in } \Omega,\\ \operatorname{div}^\mu \mathbf{U}^\mu = 0 & \text{in } \Omega,\\ U_\parallel^\mu = \nabla\psi + \nabla^\perp\Delta^{-1}(\underline{\omega}_\mu\cdot N^\mu) & \text{at the surface},\\ U_b^\mu\cdot N_b^\mu = 0 & \text{at the bottom}. \end{cases}$$

We write the solution

$$\mathbf{U}^\mu = \mathbb{U}^\mu[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu) = \begin{pmatrix} \sqrt{\mu}\,\mathbb{V}[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)\\ \mathbb{w}[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)\end{pmatrix};$$

the generalized Zakharov-Craig-Sulem formulation takes therefore the following form in dimensionless form:

$$(5.8)\qquad \begin{cases} \partial_t\zeta - \dfrac{1}{\mu}\underline{\mathbb{U}}^\mu[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)\cdot N^\mu = 0,\\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\mathbb{U}_\parallel^\mu[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)|^2\\ \qquad - \dfrac{\varepsilon}{2\mu}(1+\varepsilon^2\mu|\nabla\zeta|^2)\underline{\mathbb{w}}[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)^2\\ \qquad - \varepsilon\dfrac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}_\mu\cdot N^\mu\underline{\mathbb{V}}[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)) = 0,\\ \partial_t\boldsymbol{\omega}_\mu + \dfrac{\varepsilon}{\mu}\mathbb{U}^\mu[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)\cdot\nabla^\mu\boldsymbol{\omega}_\mu = \dfrac{\varepsilon}{\mu}\boldsymbol{\omega}_\mu\cdot\nabla^\mu\mathbb{U}^\mu[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu). \end{cases}$$

5.4. Statement of the main result. As in the dimensional case, the statement of the well-posedness result requires us to work with a straightened vorticity. We therefore use a diffeomorphism Σ to straighten the fluid domain; it now takes the form $\Sigma(X,z) = (z, z+\sigma(X,z))$, where $\sigma(X,z) = (1+z)\varepsilon\zeta(X)$, and maps the strip $S = \mathbb{R}^d \times (-1,0)$ to Ω. We denote $U^\mu := \mathbf{U}^\mu \circ \Sigma$, $\omega_\mu := \boldsymbol{\omega}_\mu \circ \Sigma$, and so on, and also

$$\mathbb{U}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)\begin{pmatrix}\sqrt{\mu}\,\mathbb{V}^\sigma[\varepsilon\zeta](\psi,\omega_\mu)\\ \mathbb{w}^\sigma[\varepsilon\zeta](\psi,\omega_\mu)\end{pmatrix} := \mathbb{U}^\mu[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)\circ\Sigma.$$

The well-posedness result deals therefore with the following straightened version of (5.8):

$$\text{(5.9)}\qquad \begin{cases} \partial_t\zeta - \dfrac{1}{\mu}\underline{\mathbb{U}}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)\cdot N^\mu = 0,\\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\mathbb{U}_\parallel^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)|^2\\ \qquad -\dfrac{\varepsilon}{2\mu}(1+\varepsilon^2\mu|\nabla\zeta|^2)\underline{\mathbb{w}}^\sigma[\varepsilon\zeta](\psi,\boldsymbol{\omega}_\mu)^2,\\ \qquad -\varepsilon\dfrac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}_\mu\cdot N^\mu\underline{\mathbb{V}}^\sigma[\varepsilon\zeta](\psi,\omega_\mu)) = 0,\\ \partial_t^\sigma\omega_\mu + \dfrac{\varepsilon}{\mu}\mathbb{U}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)\cdot\nabla^{\sigma,\mu}\omega_\mu\\ \qquad = \dfrac{\varepsilon}{\mu}\omega_\mu\cdot\nabla^{\sigma,\mu}\mathbb{U}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu), \end{cases}$$

together with the divergence-free condition on ω_μ, which is propagated from the initial condition

$$\text{(5.10)}\qquad \nabla^{\sigma,\mu}\cdot\omega_\mu = 0 \quad \text{in } S.$$

The statement of the theorem also requires the introduction of the dimensionless energy

$$\text{(5.11)}\qquad \begin{aligned} \mathcal{E}^N(\zeta,\psi,\omega_\mu) :={}& \frac{1}{2}|\zeta|_{H^N}^2 + \frac{1}{2}|\mathfrak{P}\psi|_{H^3}^2 + \frac{1}{2}\sum_{0<|\alpha|\le N}|\mathfrak{P}\psi_{(\alpha)}|_2^2\\ &+\frac{1}{2}\|\omega_\mu\|_{H^{N-1}}^2 + \frac{1}{2}|\omega_{\mu,b}\cdot N_b^\mu|_{H_0^{-1/2}}^2, \end{aligned}$$

with $\psi_{(\alpha)} = \partial^\alpha\psi - \varepsilon\underline{w}\,\partial^\alpha\zeta$ (and $\underline{w} = \underline{\mathbb{w}}^\sigma[\varepsilon\zeta](\psi,\omega_\mu)$), and we still denote by $\mathfrak{m}^N(\zeta,\psi,\omega_\mu)$ any constant of the form

$$\mathfrak{m}^N(\zeta,\psi,\omega_\mu) = C\left(\frac{1}{h_{\min}},\mathcal{E}^N(\zeta,\psi,\omega_\mu)\right),$$

and by E_T^N the associated functional space defined in (4.2). Note also that the dimensionless version of the Rayleigh-Taylor coefficient is

$$\mathfrak{a} = \mathfrak{a}(\zeta,\psi,\omega_\mu) = 1 + \varepsilon(\partial_t + \underline{\mathbb{V}}^\sigma[\varepsilon\zeta](\psi,\omega_\mu))\underline{\mathbb{w}}^\sigma[\varepsilon\zeta](\psi,\omega_\mu).$$

The theorem states that the solution furnished by Theorem 4.7 exists on a time interval $[0,T/\varepsilon]$, with T independent of $\varepsilon\in(0,\varepsilon_0)$ and $\mu\in(0,\mu_0)$, and that it is uniformly bounded on this time interval.

Theorem 5.1. *Let $\varepsilon_0, \mu_0 > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\mu \in (0, \mu_0)$, and $N \geq 5$. Let also $\Theta^0 = (\zeta^0, \psi^0, \omega_\mu^0) \in E_0^N$ be such that ω_μ^0 satisfies the divergence-free condition* (5.10). *Assume, moreover, that*

$$\exists\, h_{\min} > 0,\ \exists\, a_0 > 0, \quad 1 + \varepsilon\zeta^0 > h_{\min},\ \mathfrak{a}(\zeta^0, \psi^0, \omega^0) > a_0.$$

Then, there exists $T > 0$ (independent of ε and μ), and a unique solution $\Theta \in E_{T/\varepsilon}^N$ to (5.9) *satisfying the divergence-free constraint* (5.10), *and with initial condition Θ^0. Moreover,*

$$\frac{1}{T} = c^1 \quad \text{and} \quad \sup_{t\in[0,T]} \mathcal{E}^N(\Theta(t)) = c^2$$

with $c^j = C(\mathcal{E}^N(\Theta^0), 1/h_{\min}, 1/a_0, \varepsilon_0, \mu_0)$ for $j = 1, 2$.

Remark 5.2. Note that no smallness assumption is made on ε_0 and μ_0. The theorem furnishes in particular an existence time and bounds on the solutions that are relevant to study many asymptotic regimes of interest in oceanography:

- *The* (*large amplitude*) *shallow-water regime.* Here, $\varepsilon \sim 1$ and $\mu \ll 1$. We get an existence on a time interval of order 1, uniformly with respect to μ.
- *The long wave* (*also called Boussinesq, or KdV in dimension $d = 1$*) *regime.* Here, $\varepsilon \sim \mu \ll 1$. The existence is then on a larger time interval of order $O(1/\varepsilon)$ with uniform bounds on this time scale.
- *The deep water regime.* Here, $\mu \sim 1$ and $\varepsilon \ll 1$, and asymptotics can be studied in terms of ε, on a time interval of order $O(1/\varepsilon)$.

5.5. Proof of Theorem 5.1. Theorem 4.7 furnishes the existence of a solution. We just need to prove the necessary bounds on the solution with respect to ε and μ. This is done by deriving uniform *a priori* estimates on the solution. The derivation of these estimates follows sometimes the same steps as for the dimensional case already treated, but sometimes requires specific attention. We only focus on these latter aspects, and omit (or only sketch) the proof of the former ones.

Dependence on ε, μ of the div-curl problem is investigated in Section 5.5.1, the vorticity energy estimates are addressed in Section 5.5.2, and the *a priori* estimates on the full equations are finally derived in Section 5.6.

We always assume throughout this section that $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in (0, \mu_0)$ for some $\varepsilon_0, \mu_0 > 0$. For the sake of clarity, we never make explicit the dependence on ε_0 and μ_0.

5.5.1. The div-curl problem with parameters. We can still invoke Theorem 2.4 to insure the existence and uniqueness of a solution to (5.7); however, the dependence on the parameter μ is not obvious in the estimates on the solution provided in Theorem 2.4, and special attention must be paid to avoid singular terms as $\mu \to 0$ (in the estimates involving the bottom vorticity, for instance). This dependence is made precise in the following proposition.

Proposition 5.3. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (5.1) is satisfied. Then, for all $\boldsymbol{\omega}_\mu \in L^2(\Omega)^3$ such that $\operatorname{div}^\mu \boldsymbol{\omega}_\mu = 0$, and all $\psi \in \dot{H}^{3/2}(\mathbb{R}^d)$, there exists a unique solution $\mathbf{U} \in H^1(\Omega)^3$ to (5.7), and one can decompose it as $\mathbf{U}^\mu = \operatorname{curl}^\mu \mathbf{A} + \nabla^\mu \Phi$, where $\Phi \in \dot{H}^2(\Omega)$ solves*

$$\begin{cases} (\partial_z^2 + \mu\Delta)\Phi = 0 & \text{in } \Omega, \\ \Phi_{|_{\mathrm{surf}}} = \psi, \\ \partial_z \Phi_{|_{\mathrm{bott}}} = 0, \end{cases}$$

while $\mathbf{A} \in \dot{H}^2(\Omega)^3$ solves

$$\begin{cases} \operatorname{curl}^\mu \operatorname{curl}^\mu \mathbf{A} = \mu \boldsymbol{\omega}_\mu & \text{in } \Omega, \\ \operatorname{div}^\mu \mathbf{A} = 0 & \text{in } \Omega, \\ N_b^\mu \times A_b = 0, \\ N^\mu \cdot \underline{A} = 0, \\ (\operatorname{curl}^\mu \mathbf{A})_{\parallel} = \nabla^\perp \Delta^{-1} \underline{\omega}_\mu \cdot N^\mu, \\ N_b^\mu \cdot (\operatorname{curl}^\mu \mathbf{A})_{|_{\mathrm{bott}}} = 0. \end{cases}$$

Moreover, one has

$$\|\mathbf{U}^\mu\|_2 \le \sqrt{\mu} C\left(|\zeta|_{W^{2,\infty}}, \frac{1}{h_{\min}}\right)(\sqrt{\mu}\,\|\boldsymbol{\omega}_\mu\|_2 + |\boldsymbol{\omega}_{\mu,b} \cdot N_\mu|_{H_0^{-1/2}} + |\mathfrak{P}\psi|_2),$$
$$\|\nabla^\mu \mathbf{U}^\mu\|_2 \le \mu C\left(|\zeta|_{W^{2,\infty}}, \frac{1}{h_{\min}}\right)(\|\boldsymbol{\omega}_\mu\|_2 + |\boldsymbol{\omega}_{\mu,b} \cdot N_b^\mu|_{H_0^{-1/2}} + |\mathfrak{P}\psi|_{H^1}).$$

Proof. The existence/uniqueness of $\mathbf{U}^\mu$ follows directly (up to a rescaling) from Theorem 2.4. The fact that $\nabla^\mu \Phi$ satisfies the estimates of the proposition is known from the irrotational case (Corollary 2.40 in [39]); by linearity, we can therefore assume that $\psi = 0$ and therefore $\mathbf{U}^\mu = \operatorname{curl}^\mu \mathbf{A}$ with $\mathbf{A}$ as in the statement of the proposition; we also know from the proof of Theorem 2.4 that $\mathbf{A}$ is the unique solution to the dimensionless version of the variational equation (3.12):

$$\int_\Omega \operatorname{curl}^\mu \mathbf{A} \cdot \operatorname{curl}^\mu \mathbf{C} = \mu \int_\Omega \boldsymbol{\omega}_\mu \cdot \mathbf{C} + \sqrt{\mu} \int_{\mathbb{R}^d} (\nabla \tilde{\psi}, \sqrt{\mu}\varepsilon \nabla \tilde{\psi} \cdot \nabla \zeta) \cdot \underline{C}, \tag{5.12}$$

for all $\mathbf{C} \in \mathfrak{X}^\mu$, and where the space $\mathfrak{X}^\mu$ is given by

$$\mathfrak{X}^\mu = \{\mathbf{C} \in H^1(\Omega)^{d+1} \mid \nabla^\mu \cdot \mathbf{C} = 0,\ N^\mu \times C_b = 0,\ N_b^\mu \cdot \underline{C} = 0\}.$$

We shall use the following lemma instead of the standard trace lemma that does not provide a sharp dependence on μ.

Lemma 5.4.

(i) *For all* $\mathbf{C} \in \mathfrak{X}^\mu$, *one has*

$$\|\mathbf{C}\|_2 \leq C(|\zeta|_{W^{1,\infty}})\|\partial_z \mathbf{C}\|_2 \quad \text{and} \quad \|\nabla^\mu \mathbf{C}\|_2 \leq C(|\zeta|_{W^{2,\infty}})\|\operatorname{curl}^\mu \mathbf{C}\|_2$$

(*it is not necessary that* $\mathbf{C}$ *be divergence free for the first inequality*).

(ii) *For all* $\mathbf{C} \in H^1(\Omega)$ *such that* $\mathbf{C}_{|_{z=-1}} = 0$, *one has*

$$|(1+\sqrt{\mu}|\mathrm{D}|)^{1/2}\underline{C}|_2 \leq C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right)\|\nabla^\mu \mathbf{C}\|_2.$$

(iii) *For all* $\mathbf{C} \in H^1(\Omega)$ *such that* $\mathbf{C}_{|_{z=-1}} = 0$, *one has*

$$|\mathfrak{P}\underline{C}|_2 \leq \frac{1}{\sqrt{\mu}} C(|\zeta|_{W^{1,\infty}})\|\nabla^\mu \mathbf{C}\|_2;$$

if $\mathbf{C}$ *does not vanish at the bottom, then we still have*

$$|\mathfrak{P}\underline{C}|_2 \leq \frac{1}{\sqrt{\mu}} C(|\zeta|_{W^{1,\infty}})(\|\nabla^\mu \mathbf{C}\|_2 + \|\mathbf{C}\|_2).$$

Proof. For the first point, one must simply track the dependance on μ in the proofs of Lemmas 3.1, 3.2, and 3.3). For the second point, denoting by $\Sigma : S \to \Omega$ the diffeomorphism defined by $\Sigma(X,z) = (X, z + (1+z)\varepsilon\zeta)$, and writing $C = \mathbf{C} \circ \Sigma$, we have

$$\begin{aligned}
\big|(1+\sqrt{\mu}|\mathrm{D}|)^{1/2}\underline{C}\big|_2^2 &= 2\,\mathrm{Re}\int_{\mathbb{R}^d}\int_{-1}^{\varepsilon\zeta}(1+\sqrt{\mu}|\xi|)\hat{C}\,\partial_z\hat{C} \\
&\leq 2(\|C\|_2 + \sqrt{\mu}\|\nabla C\|_2)\|\partial_z C\|_2 \\
&\leq C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{1,\infty}}\right)(\|\mathbf{C}\|_2 + \|\nabla^\mu \mathbf{C}\|_2)\|\partial_z \mathbf{C}\|_2,
\end{aligned}$$

and the result follows from the first part of the lemma.

For the third point, we have, with $C = \mathbf{C} \circ \Sigma$,

$$\begin{aligned}
\big|\mathfrak{P}\underline{C}\big|_2^2 = \big|\mathfrak{P}C(\cdot,0)\big|_2^2 &= \int_{\mathbb{R}^d}\frac{|\xi|^2}{1+\sqrt{\mu}|\xi|}\big|\hat{C}(\xi,0)\big|_{L^2}^2 \\
&\leq 2\int_{\mathbb{R}^d}\int_{-1}^0 \frac{|\xi|^2}{1+\sqrt{\mu}|\xi|}|\hat{C}(\xi,z)|\,|\widehat{\partial_z C}(\xi,z)|\,\mathrm{d}\xi\,\mathrm{d}z \\
&= \frac{2}{\mu}\int \frac{\sqrt{\mu}|\xi|}{1+\sqrt{\mu}|\xi|}|\widehat{\sqrt{\mu}\nabla C}(\xi,z)|\,|\widehat{\partial_z C}(\xi,z)|\,\mathrm{d}\xi\,\mathrm{d}z.
\end{aligned}$$

From the Cauchy-Schwarz inequality and Plancherel's identity, we then get

$$|\mathfrak{P}\underline{C}|_2^2 \leq \frac{C}{\mu}\|\sqrt{\mu}\,\nabla\tilde{C}\|_{L^2(S)}\,\|\partial_z\tilde{C}\|_{L^2(S)} \leq \frac{1}{\mu}C(|\zeta|_{W^{1,\infty}})\|\nabla^\mu\mathbf{C}\|_{L^2(\Omega)}^2,$$

which implies the result.

Finally, if $\mathbf{C}$ does not vanish at the bottom, then one can apply the result to $\tilde{\mathbf{C}} := \varphi(X,z)\mathbf{C}$, where $\varphi \in W^{1,\infty}(\Omega)$ is equal to one in a neighborhood of the surface, and vanishes at the bottom. This yields the result since $\tilde{\mathbf{C}}_{|_{\mathrm{surf}}} = \underline{C}$ and $\|\nabla^\mu\tilde{\mathbf{C}}\|_2 \lesssim \|\nabla^\mu\mathbf{C}\|_2 + \|\mathbf{C}\|_2$. $\square$

We can use (5.12) and the lemma to get a control on the L^2-norm of $\mathbf{U}^\mu$:

$$\begin{aligned}\|\mathbf{U}^\mu\|_2^2 &\leq C(|\zeta|_{W^{2,\infty}})\Big(\mu\|\boldsymbol{\omega}_\mu\|_2\,\|\mathbf{A}\|_2 \\ &\qquad + \sqrt{\mu}\,|\mathfrak{P}\tilde{\psi}|_2\,|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}(\underline{A}_h + \varepsilon\sqrt{\mu}\,\nabla\zeta\underline{A}_v)|_2\Big) \\ &\leq C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)(\mu\|\boldsymbol{\omega}_\mu\|_2 + \sqrt{\mu}\,|\mathfrak{P}\tilde{\psi}|_2)\|\mathbf{U}^\mu\|_2,\end{aligned}$$

and therefore,

$$\begin{aligned}\|\mathbf{U}^\mu\|_2 &\leq C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)(\mu\|\boldsymbol{\omega}_\mu\|_2 + \sqrt{\mu}\,|\mathfrak{P}\tilde{\psi}|_2) \\ &\leq \mu C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)\left(\|\boldsymbol{\omega}_\mu\|_2 + \frac{1}{\sqrt{\mu}}|\omega_{\mu,b}\cdot N_b^\mu|_{H_0^{-1/2}}\right),\end{aligned}\tag{5.13}$$

where we used the fact that $|\mathfrak{P}\tilde{\psi}|_2 \leq |\nabla\tilde{\psi}|_2$, and the following lemma that makes explicit the dependence on μ of the estimate given in Lemma 3.7.

Lemma 5.5. *The solution $\tilde{\psi}$ to the equation $\Delta\tilde{\psi} = \underline{\omega}_\mu\cdot N^\mu$ satisfies*

$$|\nabla\tilde{\psi}|_2 \leq \sqrt{\mu}C\left(|\zeta|_{W^{1,\infty}},\frac{1}{h_{\min}}\right)\left(\|\boldsymbol{\omega}_\mu\|_2 + \frac{1}{\sqrt{\mu}}|\omega_{\mu,b}\cdot N_b^\mu|_{H_0^{-1/2}}\right)$$

and

$$\begin{aligned}&|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}\nabla\tilde{\psi}|_2 \\ &\qquad \leq \sqrt{\mu}C\left(|\zeta|_{W^{1,\infty}},\frac{1}{h_{\min}}\right)\left(\|\boldsymbol{\omega}_\mu\|_2 + \frac{1}{\sqrt{\mu}}|\omega_{\mu,b}\cdot N_b^\mu|_{H_0^{-1/2}}\right).\end{aligned}$$

Proof. Multiplying the equation $\Delta\tilde{\psi} = \underline{\omega}_\mu\cdot N^\mu$ by $\tilde{\psi}$, we get, as in the proof of Lemma 3.7,

$$\begin{aligned}|\nabla\tilde{\psi}|_2^2 &= -\int_{\mathbb{R}^d}\tilde{\psi}\underline{\omega}_\mu\cdot N^\mu \\ &= -\int_{\mathbb{R}^d}\tilde{\psi}_b^{\mathrm{ext}}\omega_{\mu,b}\cdot N_b^\mu - \int_S(1+\partial_z\sigma)\nabla^{\sigma,\mu}\tilde{\psi}^{\mathrm{ext}}\cdot\omega_\mu,\end{aligned}$$

where $\omega_\mu = \boldsymbol{\omega}_\mu \circ \Sigma$ (and Σ as in the proof of Lemma 5.4) and $\nabla^{\sigma,\mu}$ as in (5.3), while $\tilde{\psi}^{\text{ext}}$ is now given by

$$\tilde{\psi}^{\text{ext}} = \frac{\cosh(\sqrt{\mu}(z+1)|\mathrm{D}|)}{\cosh(\sqrt{\mu}\,|\mathrm{D}|)}\tilde{\psi}.$$

We deduce that

$$\begin{aligned}
|\nabla\tilde{\psi}|_2^2 &\leq \sqrt{\mu}\left|\frac{|\mathrm{D}|}{(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}}\tilde{\psi}^{\text{ext}}_{|z=-1}\right|_2 \left|\frac{(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}}{\sqrt{\mu}\,|\mathrm{D}|}(\boldsymbol{\omega}_{\mu,b}\cdot N_b^\mu)\right|_2 \\
&\quad + C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right)\|\boldsymbol{\omega}_\mu\|_2\,\|\nabla^\mu\tilde{\psi}^{\text{ext}}\|_2 \\
&\leq \sqrt{\mu}C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right)\left(\|\boldsymbol{\omega}_\mu\|_2 + \frac{1}{\sqrt{\mu}}|\omega_{\mu,b}\cdot N_b^\mu|_{H_0^{-1/2}}\right)|\nabla\tilde{\psi}|_2,
\end{aligned}$$

which gives the first estimate of the lemma. For the second one, we multiply the equation by $(1+\sqrt{\mu}\,|\mathrm{D}|)\psi$ and proceed as above to get

$$\begin{aligned}
&|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}\nabla\tilde{\psi}|_2^2 \\
&\quad\leq C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right)\|\boldsymbol{\omega}_\mu\|_2\,\|(1+\sqrt{\mu}\,|\mathrm{D}|)\nabla^\mu\tilde{\psi}^{\text{ext}}\|_2 \\
&\qquad + \sqrt{\mu}\,|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}|\mathrm{D}|\tilde{\psi}^{\text{ext}}_b|_2 \left|\frac{(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}}{\sqrt{\mu}\,|\mathrm{D}|}(\boldsymbol{\omega}_{\mu,b}\cdot N_b^\mu)\right|_2 \\
&\quad\leq \sqrt{\mu}C\left(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\right)(\|\boldsymbol{\omega}_\mu\|_2 + |\omega_{\mu,b}\cdot N_b^\mu|_{H_0^{-1/2}})|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}\nabla\tilde{\psi}|_2
\end{aligned}$$

(in both terms of the right-hand side in the first inequality, a smoothing argument must be used to gain half a derivative; the dependence of this smoothing on μ is crucial here, and is of the form $(1+\sqrt{\mu}\,|\mathrm{D}|)^{-1/2}$; see, e.g., Lemma 2.20 in [39]). The result follows directly. ◻

Similarly, the dependence on μ of the H^1-estimate of Lemma 3.8 must be made precise. For the energy estimates on the vorticity, it shall be crucial that no $1/\sqrt{\mu}$ singularity appear in front of the bottom vorticity term of this H^1-estimate. For the L^2-estimate (5.13), this singularity comes from the control of $|\mathfrak{P}\tilde{\psi}|_2$; for the H^1-estimate, this term is expected to be replaced by $|\mathfrak{P}\tilde{\psi}|_{H^1}$, which is also singular. However, it turns out that a control in terms of $|\mathfrak{P}\nabla\tilde{\psi}|_2$ is enough, and that, for this term, low frequencies are damped, and the $1/\sqrt{\mu}$ singularity can be removed. This is done in the following lemma.

Lemma 5.6. *The solution $\tilde{\psi}$ to the equation $\Delta\tilde{\psi} = \underline{\omega}_\mu \cdot N^\mu$ satisfies*

$$|\mathfrak{P}\nabla\tilde{\psi}|_2 \leq C\left(\frac{1}{h_{\min}}, |\zeta|_{W^{1,\infty}}\right)\|\boldsymbol{\omega}_\mu\|_2.$$

Proof. Proceed as in the proof of Lemma 5.5 to obtain

$$\begin{aligned}|\mathfrak{P}\nabla\tilde{\psi}|_2^2 &= -\int_{\mathbb{R}^d}\left(\frac{\mathrm{D}^2}{1+\sqrt{\mu}\,|\mathrm{D}|}\tilde{\psi}\right)\underline{\omega}_\mu\cdot N\\ &= -\int_S (1+\partial_z\sigma)\nabla^{\sigma,\mu}\left(\frac{\mathrm{D}^2}{1+\sqrt{\mu}\,|\mathrm{D}|}\tilde{\psi}_0^{\mathrm{ext}}\right)\cdot\omega_\mu,\end{aligned}$$

where we have chosen a different extension of $\tilde{\psi}$ than in the proof of Lemma 5.5, namely,

$$\tilde{\psi}_0^{\mathrm{ext}} = \frac{\sinh(\sqrt{\mu}(z+1)|\mathrm{D}|)}{\sinh(\sqrt{\mu}\,|\mathrm{D}|)}\tilde{\psi};$$

in particular, $\tilde{\psi}_0^{\mathrm{ext}}$ vanishes at the bottom, and this is the reason why no bottom boundary term appears in the expression above. One readily deduces that

$$(5.14)\qquad |\mathfrak{P}\nabla\tilde{\psi}|_2^2 \le C\left(|\zeta|_{W^{1,\infty}},\frac{1}{h_{\min}}\right)\left\|\nabla^{\sigma,\mu}\left(\frac{\mathrm{D}^2}{1+\sqrt{\mu}\,|\mathrm{D}|}\tilde{\psi}_0^{\mathrm{ext}}\right)\right\|_2\|\boldsymbol{\omega}_\mu\|_2,$$

and we therefore need to control

$$\|\nabla^{\sigma,\mu}((\mathrm{D}^2/(1+\sqrt{\mu}\,|\mathrm{D}|))\tilde{\psi}_0^{\mathrm{ext}})\|_2.$$

We distinguish the horizontal and vertical derivatives involved in $\nabla^{\sigma,\mu}$; we start with the vertical derivative, which is the the most delicate:

Control of $(\|\mathrm{D}^2/(1+\sqrt{\mu}\,|\mathrm{D}|))\,\partial_z\tilde{\psi}_0^{\mathrm{ext}}\|_2$. We have

$$\begin{aligned}\left\|\frac{\mathrm{D}^2}{1+\sqrt{\mu}\,|\mathrm{D}|}\,\partial_z\tilde{\psi}_0^{\mathrm{ext}}\right\|_2^2 &= \left\|\mathrm{D}^2\frac{\sqrt{\mu}\,|\mathrm{D}|}{1+\sqrt{\mu}\,|\mathrm{D}|}\frac{\cosh(\sqrt{\mu}(z+1)|\mathrm{D}|)}{\sinh(\sqrt{\mu}\,|\mathrm{D}|)}\tilde{\psi}\right\|_2^2\\ &= \int_{\mathbb{R}^d}\int_{-1}^0\left|\sqrt{\mu}\,|\xi|\frac{\cosh(\sqrt{\mu}(z+1)|\xi|)}{\sinh(\sqrt{\mu}\,|\xi|)}\right|^2|\widehat{\mathfrak{P}\nabla\tilde{\psi}}|^2\,\mathrm{d}z\,\mathrm{d}\boldsymbol{\xi}.\end{aligned}$$

Let $F(r) = \frac{1}{2}r + \frac{1}{4}\sinh(2r)$ (F is the primitive of $\cosh(r)^2$ vanishing at zero); integrating with respect to z in the above expression, we get

$$\left\|\frac{\mathrm{D}^2}{1+\sqrt{\mu}\,|\mathrm{D}|}\,\partial_z\tilde{\psi}_0^{\mathrm{ext}}\right\|_2 = \int_{\mathbb{R}^d}\frac{F(\sqrt{\mu}\,|\xi|)}{\sinh(\sqrt{\mu}\,|\xi|)^2}|\widehat{\mathfrak{P}\nabla\tilde{\psi}}|^2\,\mathrm{d}z\,\mathrm{d}\boldsymbol{\xi}.$$

Since $F(\sqrt{\mu}\,|\xi|)/(\sinh(\sqrt{\mu}\,|\xi|)^2)$ is uniformly bounded from above (with respect to ξ and μ), we deduce finally from Plancherel's identity that

$$\left\|\frac{\mathrm{D}^2}{1+\sqrt{\mu}\,|\mathrm{D}|}\,\partial_z\tilde{\psi}_0^{\mathrm{ext}}\right\|_2 \lesssim |\mathfrak{P}\nabla\tilde{\psi}|_2.$$

Control of $(\|\mathrm{D}^2/(1+\sqrt{\mu}\,|\mathrm{D}|))\sqrt{\mu}\,\nabla\tilde{\psi}_0^{\mathrm{ext}}\|_2$. We have

$$\begin{aligned}
&\left\|\frac{\mathrm{D}^2}{1+\sqrt{\mu}\,|\mathrm{D}|}\sqrt{\mu}\,\nabla\tilde{\psi}_0^{\mathrm{ext}}\right\|_2\\
&\qquad\leq \|\mathrm{D}^2\tilde{\psi}_0^{\mathrm{ext}}\|_2 \leq \left|\frac{\mathrm{D}^2}{(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}}\tilde{\psi}\right|_2 = |\mathfrak{P}\nabla\tilde{\psi}|_2,
\end{aligned}$$

the second inequality stemming from a smoothing argument similar to the one used to control the vertical derivative.

Together with (5.14), these two controls give the result of the lemma. ◻

We can therefore provide a control of $\nabla^\mu\mathbf{U}^\mu$ without the $1/\sqrt{\mu}$ singularity in front of the bottom vorticity component.

Lemma 5.7. *The following estimate holds:*

$$\|\nabla^\mu\mathbf{U}^\mu\|_2 \leq \mu C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)(\|\boldsymbol{\omega}_\mu\|_2 + |\omega_{\mu,b}\cdot N_b^\mu|_{H_0^{-1/2}}).$$

Proof. The proof is similar to that of Lemma 3.8. Here, we present the main differences. The dimensionless version of (3.19) is

$$\begin{aligned}
\int_\Omega |\nabla^\mu\mathbf{U}^\mu|^2 &= \mu^2\int_\Omega|\omega_\mu|^2 + 2\mu\int_{\mathbb{R}^d}\underline{V}\cdot\nabla\underline{w} - \mu^2\varepsilon\int_{\mathbb{R}^d}(\nabla^\perp\zeta\cdot\nabla)\underline{V}^\perp\cdot\underline{V}\\
&:= \mu^2\int_\Omega|\omega_\mu|^2 + \mu I_1 + \mu^2 I_2.
\end{aligned}\tag{5.15}$$

Now we proceed to bound the integral I_1 and I_2. We recall that $\underline{V} = \nabla^\perp\tilde{\psi} - \varepsilon\underline{w}\nabla\zeta$, and we can write

$$I_1 \leq 2\varepsilon\int_{\mathbb{R}^d}|\nabla\zeta\cdot\nabla\underline{w}\underline{w}| \leq C(|\zeta|_{W^{2,\infty}})\|\nabla^\mu\mathbf{U}^\mu\|_2\,\|\mathbf{U}^\mu\|_2,$$

where we used the fact that, since $w_b = 0$, one has $|\underline{w}|_2^2 \leq \|\mathbf{w}\|_2\,\|\partial_z\mathbf{w}\|_2$. Together with (5.13), this yields

$$\begin{aligned}
I_1 &\leq C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)(\mu\|\boldsymbol{\omega}_\mu\|_2 + \sqrt{\mu}\,|\mathfrak{P}\tilde{\psi}|_2)\|\nabla^\mu\mathbf{U}^\mu\|_2\\
&\leq \sqrt{\mu}C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)(\sqrt{\mu}\,\|\boldsymbol{\omega}_\mu\|_2 + |\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}})\|\nabla^\mu\mathbf{U}^\mu\|_2,
\end{aligned}$$

with the last inequality stemming from Lemma 5.5 and the observation that $|\mathfrak{P}\tilde{\psi}|_2 \leq |\nabla\tilde{\psi}|_2$.

For I_2, after substituting $\underline{V} = \nabla^{\perp}\tilde{\psi} - \varepsilon \underline{w}\nabla\zeta$, the integrand can be written as a sum of terms of the form

$$C(\zeta)\,\partial^2\tilde{\psi}\partial\tilde{\psi},\ C(\zeta)w\,\partial^2\tilde{\psi},\ C(\zeta)\,\partial\underline{w}\,\partial\tilde{\psi},\ C(\zeta)\underline{w}\,\partial\tilde{\psi},\ C(\zeta)\underline{w}^2,$$

where $C(\zeta)$ stands for any polynomial expression in the first- and second-order derivatives of ζ, while ∂ and ∂^2 stand here for any first- and second-order partial derivative, respectively. We consequently get

$$I_2 \le C(|\zeta|_{W^{2,\infty}})\big[\,\big|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}\nabla\tilde{\psi}\big|_2(|\mathfrak{P}\nabla\tilde{\psi}|_2 + |\mathfrak{P}\underline{w}|_2 + |\underline{w}|_2) + |\underline{w}|_2^2\,\big].$$

Noting that $|\mathfrak{P}f|_2 \le \mu^{-1/2}|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}f|_2$, recalling the inequality $|\underline{\mathbf{w}}|_2^2 \le \|\mathbf{w}\|_2\,\|\partial_z\mathbf{w}\|_2$, and with the help of (5.13) and Lemmas 5.4, 5.5, and 5.6, this yields

$$\begin{aligned} I_2 \le{}& \sqrt{\mu}C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)\left(\sqrt{\mu}\,\|\boldsymbol{\omega}_\mu\|_2\frac{1}{\sqrt{\mu}}|\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}}\right)\|\boldsymbol{\omega}_\mu\|_2 \\ &+ C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)\left(\|\boldsymbol{\omega}_\mu\|_2 + \frac{1}{\sqrt{\mu}}|\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}}\right)\|\nabla^\mu\mathbf{U}^\mu\|_2. \end{aligned}$$

Gathering the estimates on I_1 and I_2, one readily gets

$$\begin{aligned} I_1+\mu I_2 \le{}& \sqrt{\mu}C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)(\sqrt{\mu}\,\|\boldsymbol{\omega}_\mu\|_2 + |\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}})\|\nabla^\mu\mathbf{U}^\mu\|_2 \\ &+ \mu C\left(|\zeta|_{W^{2,\infty}},\frac{1}{h_{\min}}\right)(\sqrt{\mu}\,\|\boldsymbol{\omega}_\mu\|_2 + |\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}})\|\boldsymbol{\omega}_\mu\|_2; \end{aligned}$$

with (5.15), this yields the result. □

The result of the proposition directly follows from (5.13) and Lemma 5.7 (and the aforementioned estimates on the irrotational part). □

For higher-order regularity estimates, we use as in Section 3.3 the straightened version of the velocity and vorticity introduced in Section 5.4. The proposition below shows how the higher-order estimate of Corollary 3.14 depends on μ.

Proposition 5.8. *Let $N \in \mathbb{N}$, $N \ge 5$. Then, for all $0 \le \ell \le k \le N-1$, the straightened velocity $U^\mu = \mathbb{U}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)$ satisfies the estimate*

$$\begin{aligned} &\|\nabla^\mu U^\mu\|_{H^{k,\ell}} \\ &\quad\le \mu M_N\Big(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha|\le k+1}|\mathfrak{P}\psi_{(\alpha)}|_2 + \|\omega_\mu\|_{H^{k,\ell}} + |\Lambda^k\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}}\Big), \end{aligned}$$

where $\psi_{(\alpha)} := \partial^\alpha\psi - \varepsilon\underline{w}\,\partial^\alpha\zeta$ (and $\underline{w} = \mathrm{w}^\sigma[\varepsilon\zeta](\psi,\omega_\mu)_{|z=0}$).

Proof. The proof is based on the dimensionless version of (3.29), which reads

$$\int_S \nabla^\mu U^\mu \cdot P^\mu(\Sigma)\nabla^\mu U^\mu = \mu\int_S (1+\sigma_z)\omega_\mu \cdot \nabla^{\sigma,\mu}\times C + \int_{\mathbb{R}^d} F^\mu \cdot \underline{C}, \tag{5.16}$$

for all $C \in H^1(S)$, and with $\nabla^{\sigma,\mu}$ as in (5.3) while

$$P^\mu(\Sigma) = (1+\sigma_z)\left(J_\Sigma^\mu\right)^{-1}\left(J_\Sigma^\mu\right)^{-1\,T},$$

$$\text{with } \left(J_\Sigma^\mu\right)^{-1\,T} = \begin{pmatrix} 1 & 0 & \dfrac{-\sqrt{\mu}\sigma_x}{1+\sigma_z} \\ 0 & 1 & \dfrac{-\sqrt{\mu}\sigma_y}{1+\sigma_z} \\ 0 & 0 & \dfrac{1}{1+\sigma_z}, \end{pmatrix}$$

and

$$\begin{aligned} F^\mu &= (N^\mu \times \nabla^\mu \times \mathbf{U}^\mu + (N^\mu \cdot \nabla^\mu)\mathbf{U}^\mu)_{|\text{surf}} \\ &= (\sqrt{\mu}\,\nabla \underline{w} - \mu^{3/2}(\nabla\zeta^\perp \cdot \nabla)\underline{V}^\perp, -\mu\nabla\cdot \underline{V}). \end{aligned}$$

The key point is that the matrix $P^\mu(\Sigma)$ is uniformly coercive with respect to μ so that the same structure as for Proposition 3.12 can be used for the proof. In particular, (3.34) is replaced by $\|\nabla^\mu \partial^\beta U^\mu\|_2^2 \le M_N(I_1+I_2+I_3)$, with the following definition and upper bounds on I_i, $i = 1,2,3$:

Upper bound for I_1. Proceeding exactly as for (3.35), we get

$$I_1 = \int_S \nabla^\mu \partial^\beta U \cdot [\partial^\beta, P^\mu(\Sigma)]\nabla^\mu U \le M_N \|\nabla^\mu \partial^\beta U^\mu\|_2 \, \|\Lambda^{k-1}\nabla^\mu U^\mu\|_2.$$

Upper bound for I_2. With straightforward adaptations, we get as for Proposition 3.12 that

$$I_2 = \int_S \Lambda^k \omega_\mu \cdot \Lambda^{-k}\nabla^{\sigma,\mu} \times \partial^{2\beta} U^\mu \le \mu M_N \|\Lambda^k \omega_\mu\|_2 \, \|\partial^\beta \nabla^\mu U^\mu\|_2.$$

Upper bound for I_3. We split I_3 into three terms:

$$\begin{aligned} I_3 &= 2\mu\int_{\mathbb{R}^d} \partial^\beta \nabla \underline{w} \cdot \partial^\beta \underline{V} - \mu^2\varepsilon \int_{\mathbb{R}^d} (\nabla^\perp\zeta \cdot \nabla)\,\partial^\beta \underline{V}^\perp \cdot \partial^\beta \underline{V} \\ &\quad - \mu^2\varepsilon \int_{\mathbb{R}^d} [\partial^\beta, \nabla^\perp \zeta] \cdot \nabla \underline{V}^\perp \cdot \partial^\beta \underline{V} \\ &= I_{31} + I_{32} + I_{33}. \end{aligned}$$

Replacing the product estimate (3.38) by

$$\forall f,g \in H^{1/2}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} f\,\partial_j g \le |\mathfrak{P} f|_2\,|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2} g|_2 \ (1 \le j \le d),$$

and using the μ-dependent version of the trace lemma furnished by the second point of Lemma 5.4, the upper bound on I_{31} given in the proof of Proposition 3.12 can be adapted into

$$I_{31} \le M_N\big(\mu|\mathfrak{P}(\nabla\partial^\beta\psi - \varepsilon\underline{w}\nabla\partial^\beta\zeta)|_2 + \sqrt{\mu}\,\|\Lambda^{k-1}\nabla^\mu U^\mu\|_2\big)\|\Lambda^k\nabla^\mu U^\mu\|_2.$$

For I_{32}, we proceed as for I_{31} to get

$$I_{32} \le \mu^2 M_N\Big[\big(|\mathfrak{P}(\nabla\partial^\beta\psi - \varepsilon\underline{w}\nabla\partial^\beta\zeta)|_2 + |\mathfrak{P}\,\partial^\beta\tilde{\psi}|_2 + |\mathfrak{P}\Lambda^{k-1}\underline{w}|_2\big) \times \frac{1}{\sqrt{\mu}}\,|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}\,\partial^\beta\underline{U}^\mu|_2 + |\partial^\beta\underline{w}|_2^2\Big].$$

With Lemma 5.4 and the inequality $|\partial^\beta\underline{w}|_2^2 \le \|\partial^\beta w\|_2\,\|\partial^\beta\,\partial_z w\|_2$, we thus get

$$I_{32} \le M_N\Big(\mu|\mathfrak{P}(\nabla\partial^\beta\psi - \varepsilon\underline{w}\nabla\partial^\beta\zeta)|_2 + \mu|\mathfrak{P}\,\partial^\beta\tilde{\psi}|_2 + \sqrt{\mu}\,\|\Lambda^{k-1}\nabla^\mu w\|_2\Big)\|\Lambda^k\nabla^\mu U^\mu\|_2.$$

Finally, we readily get that $I_{33} \le M_N\|\Lambda^{k-1}\nabla^\mu U^\mu\|_2\,\|\Lambda^k\nabla^\mu U^\mu\|_2$. Summing up the upper bounds on I_{31}, I_{32}, and I_{33}, we finally get

$$I_3 \le M_N\Big(\mu|\mathfrak{P}(\nabla\partial^\beta\psi - \varepsilon\underline{w}\nabla\partial^\beta\zeta)|_2 + \mu|\mathfrak{P}\,\partial^\beta\tilde{\psi}|_2 + \|\Lambda^{k-1}\nabla^\mu U^\mu\|_2\Big)\|\Lambda^k\nabla^\mu U^\mu\|_2.$$

As a result of these upper bounds on I_1, I_2, and I_3, we obtain

$$\|\nabla^\mu\,\partial^\beta U^\mu\|_2^2 \le M_N\Big(\mu|\mathfrak{P}(\nabla\partial^\beta\psi - \varepsilon\underline{w}\nabla\partial^\beta\zeta)|_2 + \mu|\mathfrak{P}\,\partial^\beta\tilde{\psi}|_2 + \|\Lambda^{k-1}\nabla^\mu U^\mu\|_2 + \mu\|\Lambda^k\omega_\mu\|_2\Big) \times \|\Lambda^k\nabla^\mu U^\mu\|_2,$$

with $k = |\beta|$. With the same induction method as in the proof of Proposition 3.12, we then deduce

$$\|\Lambda^k\nabla^\mu U^\mu\|_2 \le M_N\Big(\mu\sum_{1<|\alpha|\le k+1}|\mathfrak{P}\psi_{(\alpha)}|_2 + \mu|\mathfrak{P}\,\partial^\beta\tilde{\psi}|_2 + \mu\|\Lambda^k\omega_\mu\|_2 + \|\nabla^\mu U^\mu\|_2\Big).$$

With the estimate on $\nabla^\mu U^\mu$ provided by Proposition 5.3, and the estimate that generalizes Lemma 5.5 in the spirit of Lemma 3.13, namely,

$$|\mathfrak{P}\,\partial^\beta\tilde{\psi}| \le |\Lambda^k\nabla\tilde{\psi}|_2 \le \sqrt{\mu}C\Big(|\zeta|_{W^{1,\infty}}, \frac{1}{h_{\min}}\Big)\Big(\|\Lambda^k\omega_\mu\|_2 + \frac{1}{\sqrt{\mu}}|\omega_{\mu,b}\cdot N_\mu|_{H_0^{-1/2}}\Big),$$

we finally get a dimensionless version of the upper bound of Proposition 3.12:

$$\|\Lambda^k \nabla^\mu U^\mu\|_2 \le \mu M_N\Big(|\mathfrak{P}\psi|_{H^1} + \sum_{1<|\alpha|\le k+1} |\mathfrak{P}\psi_{(\alpha)}|_2 + \|\Lambda^k \omega_\mu\|_2 + |\Lambda^k \omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}}\Big).$$

Following the same steps as in the proof of Corollary 3.14, we deduce an $H^{k,\ell}$-estimate of $\nabla^\mu U^\mu$ from this L^2-estimate of $\Lambda^k\nabla^\mu U^\mu$. □

The adaptation to the dimensionless case of the other results presented in Section 3.5 and Section 3.6 is then straightforward, and we therefore omit it.

5.5.2. A priori estimates for the vorticity. We prove here that the estimates of Proposition 4.1 can be replaced in the dimensionless case by

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|\omega_\mu\|^2_{H^k} + |\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}}\big) \le \varepsilon \mathfrak{m}^N(\zeta,\psi,\omega). \tag{5.17}$$

We proceed as in the proof of Proposition 4.1, to obtain the following dimensionless version of the vorticity equation (4.5):

$$\partial_t\omega_\mu + \varepsilon \mathbb{V}^\sigma[\varepsilon\zeta](\psi,\omega_\mu)\cdot\nabla\omega + \frac{\varepsilon}{\mu}\mathbb{a}[\varepsilon\zeta](\psi,\omega_\mu)\,\partial_z\omega_\mu = \frac{\varepsilon}{\mu}\omega_\mu\cdot\nabla^{\sigma,\mu}\mathbb{U}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)$$

with, denoting $\tilde N^\mu = (-\sqrt{\mu}\,\nabla\sigma^T, 1)^T$ (so that $\tilde N^\mu_{|_{z=0}} = N^\mu$),

$$\begin{aligned}\mathbb{a}[\varepsilon\zeta](\psi,\omega_\mu) &= \frac{1}{1+\partial_z\sigma}\big(\mathbb{U}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)\cdot\tilde N^\mu - \mu\,\partial_t\sigma\big)\\ &= \frac{1}{1+\partial_z\sigma}\Big(\mathbb{U}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)\cdot\tilde N^\mu - \frac{z+H_0}{H_0}\underline{\mathbb{U}}^{\sigma,\mu}[\varepsilon\zeta](\psi,\omega_\mu)\cdot N^\mu\Big).\end{aligned}$$

We are thus led to study the following equation instead of (4.6):

$$\partial_t\omega_\mu + \varepsilon V\cdot\nabla\omega + \frac{\varepsilon}{\mu}a\,\partial_z\omega_\mu = \varepsilon f$$

(with $V = \mathbb{V}^\sigma[\varepsilon\zeta](\psi,\omega_\mu)$ and $a = \mathbb{a}[\varepsilon\zeta](\psi,\omega_\mu)$). The L^2-estimate (4.7) must be refined to get a good dependence on μ. Taking the L^2 scalar product of this equation with ω_μ, we get

$$\frac{1}{2}\partial_t\|\omega_\mu\|^2_{L^2} - \frac{\varepsilon}{2}\int_S\Big(\nabla\cdot V + \frac{1}{\mu}\partial_z a\Big)|\omega_\mu|^2 = \varepsilon\int_S f\cdot\omega_\mu,$$

and therefore

$$\partial_t ||\omega_\mu||_2^2 \lesssim \frac{\varepsilon}{\mu}(\|\nabla^\mu U^\mu\|_\infty + \sqrt{\mu}\,\|U^\mu\|_\infty)||\omega_\mu||_2^2 + \varepsilon\|\omega_\mu\|_2\,\|f\|_2.$$

From the continuous embedding $H^{N-1}(S) \subset L^\infty(S)$ and the fact that

$$\|U^\mu\|_{H^{N-1}} \le \|U^\mu\|_2 + \frac{1}{\sqrt{\mu}}\|\nabla^\mu U^\mu\|_{H^{N-2}},$$

we deduce that

$$\begin{aligned}\partial_t ||\omega_\mu||_2^2 &\lesssim \frac{\varepsilon}{\mu}(\|\nabla^\mu U^\mu\|_{H^{N-1}} + \sqrt{\mu}\,\|U^\mu\|_2)||\omega_\mu||_2^2 + \|\omega_\mu\|_2\,\|f\|_2 \\ &\lesssim \varepsilon\mathfrak{m}^N(\zeta,\psi,\boldsymbol{\omega})||\omega_\mu||_2^2 + \|\omega_\mu\|_2\,\|f\|_2,\end{aligned}$$

the second inequality stemming from Propositions 5.3 and 5.8. Using this generalization of (4.7), we obtain the estimate on the H^k-norm of ω_μ of (5.17). For the bottom vorticity, we have to replace (4.9) by

$$\partial_t(\omega_{\mu,b}\cdot N_b) + \varepsilon\nabla\cdot(\omega_{\mu,b}\cdot N_b V_b) = 0,$$

and therefore

$$\partial_t|\omega_{\mu,b}\cdot N_b|_{H_0^{-1/2}} \le \varepsilon|(1+\sqrt{\mu}\,|\mathrm{D}|)^{1/2}((\omega_{\mu,b}\cdot N_b)V_b)|_2 \le \varepsilon\mathfrak{m}^N(\zeta,\psi,\omega_\mu),$$

the last inequality stemming from the trace lemma, standard product estimates, and Proposition 5.8. This completes the proof of (5.17).

5.6. A priori estimates on the full equations. In dimensionless variables, the good unknown becomes

$$\forall\,\alpha\in\mathbb{N}^d\setminus\{0\},\quad U^\mu_{(\alpha)} = \partial^\alpha U^\mu - \partial^\alpha\sigma\,\partial_z^\sigma U^\mu.$$

With a straightforward adaptation, and using the div-curl estimates derived in Section 5.5.1, the quasilinear structure exhibited in Proposition 4.2 takes the following form in dimensionless variables:

$$\begin{aligned}(\partial_t + \varepsilon\underline{V}\cdot\nabla)\,\partial^\alpha\zeta - \frac{1}{\mu}\,\partial_k\underline{U}^\mu_{(\beta)}\cdot N^\mu &= \varepsilon R^1_\alpha,\\ (\partial_t + \varepsilon\underline{V}\cdot\nabla)(U^\mu_{(\beta)\parallel}\cdot\mathbf{e}_k) + \mathfrak{a}\,\partial^\alpha\zeta &= \varepsilon R^2_\alpha,\\ \left(\partial_t^\sigma + \frac{\varepsilon}{\mu}U^\mu\cdot\nabla^{\sigma,\mu}\right)\partial^\beta\omega_\mu &= \varepsilon R^3_\beta,\end{aligned}$$

with $\mathfrak{a} = 1 + (\partial_t + \varepsilon \underline{V} \cdot \nabla)\underline{w}$, and where

$$|R^1_\alpha|_2 + |\mathfrak{P}R^2_\alpha|_2 + \|R^3_\beta\|_2 \le \mathfrak{m}^N(\zeta, \psi, \omega_\mu).$$

The following dimensionless version of (4.15) can then be derived along the same lines as in the dimensional case:

$$\frac{1}{2}\partial_t(\mathfrak{a}\,\partial^\alpha\zeta, \partial^\alpha\zeta) + \left((\partial_t + \varepsilon\underline{V}\cdot\nabla)(U^\mu_{(\beta)\|}\cdot \mathbf{e}_k), \frac{1}{\mu}\partial_k \underline{U}^\mu_{(\beta)}\cdot N^\mu\right)$$
$$\le \varepsilon\mathfrak{m}^N(\zeta, \psi, \omega_\mu),$$

from which, proceeding as for (4.20), we get

$$\partial_t\left\{(\mathfrak{a}\,\partial^\alpha\zeta, \partial^\alpha\zeta) + \int_S (1 + \partial_z\sigma)\,|\partial_k U^\mu_{(\beta)}|_2^2\right\} \le \varepsilon\mathfrak{m}^N(\zeta, \psi, \omega_\mu).$$

Together with the vorticity estimate (5.17), and mimicking *Step 3* and *Step 4* of the proof of Proposition 4.5, we get that, for all $0 \le t \le T/\varepsilon$,

$$\mathcal{E}^N(\zeta, \psi, \boldsymbol{\omega}_\mu)(t) \le C\left(T, \frac{1}{a_0}, \frac{1}{h_{\min}}, \mathcal{E}^N(\zeta^0, \psi^0, \omega^0_\mu)\right).$$

By a classical prolongation argument, this allows us to extend the solution provided by Theorem 4.7 on a time interval $[0, T/\varepsilon]$, with T independent of ε and μ, thus completing the proof of Theorem 5.1.

5.7. Justification of the shallow-water equations with vorticity. Shallow-water models provide simplified models for the propagation of water waves when $\mu \ll 1$. In the irrotational case, they are typically stated as a set of equations coupling the evolution of the surface elevation ζ to the vertically averaged horizontal velocity $\bar{V}$:

$$\bar{V}(t, X, z) = \frac{1}{h(t, X)}\int_{-1}^{\varepsilon\zeta(t,X)} \mathbf{V}(t, X, z)\,\mathrm{d}z \quad (h = 1 + \varepsilon\zeta).$$

Various models exist, depending on the precision of the approximation, and possible smallness assumptions on ε. The derivation and justification of these shallow-water models is now well understood; we refer to [39] for references and a detailed description of the many shallow-water models and for their rigorous justification. In the so-called *shallow-water, large amplitude* regime corresponding to

$$\varepsilon = 1, \ \mu \ll 1,$$

we obtain, for instance, at first-order (i.e., up to $O(\mu)$ terms) the well-known Nonlinear Shallow-Water (or Saint-Venant) equations

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\bar{V}) = 0, \\ \partial_t\bar{V} + \bar{V}\cdot\nabla\bar{V} + \nabla\zeta = 0. \end{cases} \tag{5.18}$$

A byproduct of the derivation and justification of this model is that the horizontal velocity $\mathbf{V}$ is, at the precision of the model, independent of the vertical variable z (in the physics literature, this is is often an assumption, called "columnar motion" assumption). A consequence is that at the precision of the model, the velocity at the surface $\underline{V}$ is equal to the averaged velocity $\bar{V}$ for which (5.18) is derived. This is of importance since direct experimental data are more accessible for $\underline{V}$ (by using buoys, for instance) than for $\bar{V}$.

In the rotational case, the picture is less clear, and there exists no fully justified shallow-water model. Even at the formal level, there is no real consensus in the physics literature. The assumption of "columnar motion" is often made to derive asymptotic models, but, as shown below, it is in general wrong at the precision of the model. As shown in [15] by the authors, a consequence of this fact is that even though the Nonlinear Shallow-Water model (5.18) remains the same[11] in the presence of vorticity when written in $(\zeta, \bar{V})$ variables, the recovery of the velocity $\underline{V}$ at the surface requires the resolution of one more equation:

$$\underline{V} = \bar{V} - \sqrt{\mu} Q, \quad \text{with } \partial_t Q + \bar{V} \cdot \nabla Q + Q \cdot \nabla \bar{V} = 0. \tag{5.19}$$

The goal of this section is to show that Theorem 5.1 provides all the necessary bounds on the solution to justify the formal computations of [15], and therefore to bring a full justification of the Nonlinear Shallow-Water model[12] (5.18)–(5.19) as a model for the description of shallow-water waves in the presence of vorticity.

5.7.1. Derivation of the model. For the sake of completeness, we sketch here the derivation of the NSW model (5.18)–(5.19). We refer to [15] for details. For clarity, we also use the notation

$$f = O(\mu^\alpha) \iff \exists\, k \geq 0,\ \forall\, n \geq 0, \quad |f|_{H^n} \leq \mu^\alpha C(\mathcal{E}^{n+k}(\zeta, \psi, \omega_\mu))$$

for a function defined on $\mathbb{R}^d$, and with obvious adaptation for functions defined on Ω. We recall that the energy $\mathcal{E}^{n+k}(\zeta, \psi, \omega_\mu)$ is controlled uniformly with respect to μ by Theorem 5.1.

Let us consider, therefore, $(\zeta, \psi, \omega_\mu)$, the solution provided by Theorem 5.1. We denote $\boldsymbol{\omega}_\mu = \omega_\mu \circ \Sigma^{-1}$ the corresponding vorticity in the fluid domain. One can show that the velocity field $\mathbf{U}^\mu$ has the following structure:

[11] This statement is not obvious and deserves a proof!

[12] Other, more precise, models are derived in [15], showing for instance the creation of horizontal vorticity from an initially purely vertical vorticity. A good local well-posedness theory for these models is the only thing to prove in order to deduce a full justification along the procedure described here for the NSW model (5.18)–(5.19).

$$\mathbf{U}^{\mu} = \begin{pmatrix} \sqrt{\mu}\,\mathbf{V} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \sqrt{\mu}\bar{V} + \mu\Big(\int_z^{\zeta} (\boldsymbol{\omega}_\mu)_h^{\perp} - Q\Big) + O(\mu^{3/2}) \\ -\mu(1+z)\Delta\psi - \mu^{3/2}\nabla\cdot\int_{-1}^{z}\int_{z'}^{\zeta}(\boldsymbol{\omega}_\mu)_h^{\perp} - \mu^2\int_{-1}^{z}\nabla\cdot V^{(1)} \end{pmatrix}, \tag{5.20}$$

with

$$Q := \frac{1}{h}\int_{-1}^{\zeta}\int_{z'}^{\zeta}(\boldsymbol{\omega}_\mu)_h^{\perp}.$$

Plugging this expression into the vorticity equation (5.6), we obtain, for the horizontal components,

$$\begin{aligned} \partial_t(\boldsymbol{\omega}_\mu)_h + \bar{V}\cdot\nabla(\boldsymbol{\omega}_\mu)_h - (1+z)\nabla\cdot\bar{V}\,\partial_z(\boldsymbol{\omega}_\mu)_h \\ = (\boldsymbol{\omega}_\mu)_h\cdot\nabla\bar{V} - (\nabla^{\perp}\cdot\bar{V})(\boldsymbol{\omega}_\mu)_h^{\perp} + O(\sqrt{\mu}); \end{aligned}$$

integrating this equation then gives the following equation for Q:

$$\partial_t Q + Q\cdot\nabla\bar{V} + \bar{V}\cdot\nabla Q = O(\sqrt{\mu}). \tag{5.21}$$

Using (5.20) again, we can relate $U_{\parallel}^{\mu}$ to the averaged velocity $\bar{V}$ through the approximation

$$U_{\parallel}^{\mu} = \underline{V} + \underline{w}\nabla\zeta = \bar{V} - \sqrt{\mu}Q + O(\mu), \quad \text{where } Q := \frac{1}{h}\int_{-1}^{\zeta}\int_{z'}^{\zeta}(\boldsymbol{\omega}_\mu)_h^{\perp}.$$

This approximation is then plugged into (5.4) to obtain

$$\partial_t\bar{V} + \bar{V}\cdot\nabla\bar{V} + \nabla\zeta = \sqrt{\mu}\,[\partial_t Q + Q\cdot\nabla\bar{V} + \bar{V}\cdot\nabla Q] + O(\mu) = O(\mu), \tag{5.22}$$

the second identity stemming from (5.21).

Using the exact relation $\underline{U}^{\mu}\cdot N^{\mu} = -\mu\nabla\cdot(h\bar{V})$ in the equation for the surface elevation, we get, moreover,

$$\partial_t\zeta + \nabla\cdot(h\bar{V}) = 0. \tag{5.23}$$

The Nonlinear Shallow-Water model with vorticity (5.18)–(5.19) corresponds therefore to (5.21), (5.22), and (5.23) without all the terms of order $O(\mu)$.

5.7.2. Justification of the model. Let us denote by $(\zeta_{\mathrm{SW}}, \bar{V}_{\mathrm{SW}}, Q_{\mathrm{SW}})$ the exact solution to (5.18)–(5.19) with initial conditions

$$\begin{aligned} \zeta_{\mathrm{SW}}^0 = \zeta^0, \quad \bar{V}_{\mathrm{SW}}^0 = \frac{1}{1+\zeta^0}\int_{-1}^{\zeta^0}\mathbf{V}^0, \\ Q_{\mathrm{SW}}^0 = \frac{1}{1+\zeta^0}\int_{-1}^{\zeta^0}\int_{z'}^{\zeta^0}(\boldsymbol{\omega}_\mu^0)_h^{\perp}. \end{aligned} \tag{5.24}$$

The following proposition shows that $(\zeta_{\mathrm{SW}}, \bar{V}_{\mathrm{SW}}, Q_{\mathrm{SW}})$ is a good approximation[13] at order $O(\mu)$ to the full water waves equations (5.9).

Proposition 5.9. *Let $N \in \mathbb{N}$ be large enough, $\varepsilon = 1$, and $\mu \in (0,1)$. Let $(\zeta^0, \psi^0, \omega^0_\mu)$ be such that the assumptions of Theorem 5.1 are satisfied.*

Then, there exists $T > 0$ (independent of μ) such that the following hold:

(i) *There exists a unique solution*

$$(\zeta_{\mathrm{SW}}, \bar{V}_{\mathrm{SW}}, Q_{\mathrm{SW}}) \in C([0,T]; H^N(\mathbb{R}^d) \times H^N(\mathbb{R}^d)^2 \times H^{N-1}(\mathbb{R}^d)^d)$$

to (5.18)–(5.19) *with initial condition* (5.24).

(ii) *There exists a unique solution $(\zeta, \psi, \omega_\mu) \in E^N_T$ to* (5.9) *with initial data $(\zeta^0, \psi^0, \omega^0_\mu)$.*

(iii) *The following error estimates hold:*

$$\begin{aligned}|\zeta - \zeta_{\mathrm{SW}}|_{L^\infty([0,T]\times\mathbb{R}^d)} + |\bar{V} - \bar{V}_{\mathrm{SW}}|_{L^\infty([0,T]\times\mathbb{R}^d)} \\ + \sqrt{\mu}\,|Q - Q_{\mathrm{SW}}|_{L^\infty([0,T]\times\mathbb{R}^d)} \le \mu c,\end{aligned}$$

with $c = C(\mathcal{E}^N(\zeta^0, \psi^0, \omega^0_\mu), 1/h_{\min}, 1/a_0)$.

Proof. The first point of the proposition is classical and stems directly from the hyperbolic structure of (5.18), and the second point is a direct consequence of Theorem 5.1. For the third point, we have shown in Section 5.7.1 that $(\zeta, \bar{V}, Q)$ solves (5.18)–(5.19) up to $O(\mu)$ terms. Standard hyperbolic estimates then give a $O(\mu)$ control of the error in H^2-norm (provided than N is chosen large enough), from which the L^∞-estimate given in the statement of the lemma follows from Sobolev embeddings. ◻

6. A Hamiltonian Formulation of the Water Waves Equations with Vorticity (2.11)

The total energy is given by the sum of the potential energy E and the kinetic energy K,

$$H = E + K = \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 + \frac{1}{2}\int_{\Omega_\zeta} |\mathbf{U}|^2,$$

where we always assume that ζ satisfies the nonvanishing depth condition (2.6), and where Ω_ζ is the fluid domain delimited above by the graph of ζ and below by the flat bottom $z = -H_0$ (when no confusion is possible, we simply write

[13] In the statement of the theorem, the quantities $\bar{V}$ and Q are given by

$$\bar{V} = \frac{1}{1+\zeta}\int_{-1}^{\zeta} \mathbb{V}[\zeta](\psi, \boldsymbol{\omega}_\mu), \quad Q = \frac{1}{1+\zeta}\int_{-1}^{\zeta}\int_{z'}^{\zeta} (\boldsymbol{\omega}_\mu)_h^\perp,$$

with $\boldsymbol{\omega}_\mu = \omega_\mu \circ \Sigma^{-1}$ and $\mathbb{V}[\zeta](\psi, \boldsymbol{\omega}_\mu)$ the horizontal component of the $\mathbb{U}[\zeta](\psi, \boldsymbol{\omega}_\mu)$, as given in Definition 2.5.

$\Omega = \Omega_\zeta$ as everywhere else in this paper). In [59], Zakharov showed that, *in the irrotational case*, the water waves equations could be formally written under a canonical Hamiltonian formulation, namely,

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = J \operatorname{grad}_{\zeta,\psi} H, \quad \text{with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.1}$$

Several authors have proposed formulations of the water waves equations in the presence of vorticity that also have a Hamiltonian structure (but without addressing the well-posedness of these formulations). Let us mention, for instance, [17, 46] in a Lagrangian framework, [54] in an Eulerian framework for one-dimensional flows with constant vorticity, and the general approach of [43] (see also [37], and [36] for comments on the validity of these formulations). We investigate in this section if our new well-posed formulation (2.11) has a structure similar to (6.1) when the vorticity is non zero.

We first define admissible functionals in Section 6.1, show how to compute the gradients of such functionals in Section 6.2, and finally show in Section 6.3 that the formulation (2.11) has a formal Hamiltonian structure.

6.1. The set of admissible functionals. With the notation introduced in Definition 2.5, we can write the energy as a function of $(\zeta, \psi, \boldsymbol{\omega})$:

$$H = H(\zeta, \psi, \boldsymbol{\omega}) = \frac{1}{2}\int_{\mathbb{R}^d} g\zeta^2 + \frac{1}{2}\int_\Omega |\mathbb{U}(\zeta, \psi, \boldsymbol{\omega})|^2 =: \mathcal{H}(\zeta, \mathbb{U}(\zeta, \psi, \boldsymbol{\omega})), \tag{6.2}$$

where H and $\mathcal{H}$ are, respectively, functionals on $\mathcal{M}$ and $\mathcal{N}$, defined as follows.

Definition 6.1.

(i) We denote

$$\mathcal{M} = \Big\{(\zeta, \psi, \boldsymbol{\omega}) \mid (\zeta, \psi) \in H^\infty(\mathbb{R}^d)^2,\ \zeta \text{ satisfies (2.6)},\ \boldsymbol{\omega} \in H^\infty(\Omega_\zeta)^3,$$
$$\operatorname{div} \boldsymbol{\omega} = 0,\ \omega_b \cdot N_b \in H_0^\infty(\mathbb{R}^d)\Big\}.$$

(ii) We denote

$$\mathcal{N} = \Big\{(\zeta, \mathbf{U}) \mid \zeta \in H^\infty(\mathbb{R}^d) \text{ satisfies (2.6)},\ \mathbf{U} \in H^\infty(\Omega_\zeta)^3,$$
$$\operatorname{div} \mathbf{U} = 0,\ U_b \cdot N_b = 0\Big\}.$$

Denoting $\sigma_\zeta = (1/H_0)(z + H_0)\zeta$, and recalling that $\operatorname{div}^{\sigma_\zeta}$ is defined in (3.22), we also define the straightened version of $\mathcal{N}$ by

$$\mathcal{N}^\sigma = \Big\{(\zeta, U) \mid \zeta \in H^\infty(\mathbb{R}^d) \text{ satisfies (2.6)},\ U \in H^\infty(S)^3,$$
$$\operatorname{div}^{\sigma_\zeta} U = 0,\ U_b \cdot N_b = 0\Big\}.$$

Any functional $\mathcal{F}$ on $\mathcal{N}$ can be equivalently defined as a functional $\mathcal{F}^\sigma$ on $\mathcal{N}^\sigma$ through the relation

$$\mathcal{F}^\sigma(\zeta U) = \mathcal{F}(\zeta, U \circ \Sigma_\zeta^{-1}) \quad \text{with } \Sigma_\zeta(X,z) = (X, z + \sigma_\zeta(X,z));$$

we use this observation to define the class $C^\infty(\mathcal{N})$ of smooth functionals on $\mathcal{N}$.

Definition 6.2. A functional $\mathcal{F}$ belongs to $C^\infty(\mathcal{N})$ if and only if $\mathcal{F}^\sigma$ belongs to $C^\infty(\mathcal{N}^\sigma)$.

We can also use this observation to define Gâteaux-derivatives of functional $C^\infty(\mathcal{N})$; the following assumption is made on these derivatives:

$$\text{(6.3)} \quad \begin{cases} \exists\, \dfrac{\delta \mathcal{F}}{\delta \zeta} \in H^\infty(\mathbb{R}^d),\ \forall\, \delta\zeta \in H^\infty(\mathbb{R}^d), \quad \mathrm{d}_\zeta \mathcal{F} \cdot \delta\zeta = \displaystyle\int_{\mathbb{R}^d} \frac{\delta \mathcal{F}}{\delta \zeta}\delta\zeta, \\ \exists\, \dfrac{\delta \mathcal{F}}{\delta \mathbf{U}} \in H^\infty(\Omega_\zeta)^3,\ \operatorname{div} \dfrac{\delta \mathcal{F}}{\delta \mathbf{U}} = 0,\ \dfrac{\delta \mathcal{F}}{\delta \mathbf{U}}\Big|_{z=-H_0} \cdot N_b = 0, \\ \qquad \forall\, \delta\mathbf{U} \in H^\infty(\Omega_\zeta)^3 \text{ such that } \operatorname{div} \delta\mathbf{U} = 0 \text{ and} \\ \qquad (\delta\mathbf{U})_{|z=-H_0} \cdot N_b = 0, \text{ one has } \mathrm{d}_\mathbf{U}\mathcal{F} \cdot \delta\mathbf{U} = \displaystyle\int_{\Omega_\zeta} \frac{\delta \mathcal{F}}{\delta \mathbf{U}} \cdot \delta\mathbf{U}. \end{cases}$$

We can finally define the class $\mathcal{A}$ of admissible functionals to which the Hamiltonian H belongs.

Definition 6.3. A functional F on $\mathcal{M}$ belongs to the set $\mathcal{A}$ of *admissible* functionals if and only if there exists $\mathcal{F} \in C^\infty(\mathcal{N})$ satisfying (6.3), and such that

$$\forall\, (\zeta, \psi, \boldsymbol{\omega}) \in \mathcal{M}, \quad F(\zeta, \psi, \boldsymbol{\omega}) = \mathcal{F}(\zeta, \mathbb{U}[\zeta](\psi, \boldsymbol{\omega})).$$

6.2. Gradients of admissible functionals. We give in the following proposition an expression for the gradient of admissible functionals, as well as an expression for the cotangent bundle $T^*\mathcal{M}$. Note that the tangent space $T_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M}$, in which we take the variations $(\delta\zeta, \delta\psi, \delta\boldsymbol{\omega})$, is defined[14] as

$$T_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M} = \Big\{(\delta\zeta, \delta\psi, \delta\boldsymbol{\omega}) \in H_0^\infty(\mathbb{R}^d) \times \dot{H}^\infty(\mathbb{R}^d) \times H^\infty(\Omega_\zeta)^3 \mid \\ \operatorname{div} \delta\boldsymbol{\omega} = 0,\ (\delta\omega)_b \cdot N_b \in H_0^\infty(\mathbb{R}^d)\Big\};$$

we also recall that the operator curl^{-1} is defined in Corollary 3.9.

[14] The fact that the variations $\delta\zeta$ are taken in $H_0^\infty(\mathbb{R}^d)$ is to ensure the conservation of the volume of the fluid domain; consequently, the variations $\delta\psi$ are taken in $\dot{H}^\infty(\mathbb{R}^d)$.

Proposition 6.4. *Let F be an admissible functional and $\mathcal{F}$ be the associated functional on $\mathcal{N}$. One can write, for all variations $(\delta\zeta, \delta\psi, \delta\boldsymbol{\omega}) \in T_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M}$, that*

$$\begin{aligned}\left\langle d_{\zeta,\psi,\boldsymbol{\omega}}F, \begin{pmatrix}\delta\zeta\\ \delta\psi\\ \delta\boldsymbol{\omega}\end{pmatrix}\right\rangle_{T^*_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M}-T_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M}} &= \int_{\mathbb{R}^d}\frac{\delta F}{\delta\zeta}\delta\zeta + \int_{\mathbb{R}^d}\frac{\delta F}{\delta\psi}\delta\psi + \int_{\Omega_\zeta}\frac{\delta F}{\delta\boldsymbol{\omega}}\cdot\delta\boldsymbol{\omega}\\ &= \left(\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}} F, \begin{pmatrix}\delta\zeta\\ \delta\psi\\ \delta\boldsymbol{\omega}\end{pmatrix}\right)_{L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\times L^2(\Omega_\zeta)},\end{aligned}$$

with the L^2-gradient $\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}} F$ given by

$$\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}} F := \begin{pmatrix}\dfrac{\delta F}{\delta\zeta}\\ \dfrac{\delta F}{\delta\psi}\\ \dfrac{\delta F}{\delta\boldsymbol{\omega}}\end{pmatrix} = \begin{pmatrix}\dfrac{\delta\mathcal{F}}{\delta\zeta} - \underline{w}\dfrac{\delta\mathcal{F}}{\delta\mathbf{U}}\cdot N - \underline{\omega}_h^\perp\cdot\nabla\Delta^{-1}\dfrac{\delta\mathcal{F}}{\delta\mathbf{U}}\cdot N\\ \dfrac{\delta\mathcal{F}}{\delta\mathbf{U}}\Big|_{z=\zeta}\cdot N\\ \operatorname{curl}^{-1}\dfrac{\delta\mathcal{F}}{\delta\mathbf{U}}.\end{pmatrix}.$$

Identifying the cotangent space with the set of the L^2-gradients of all the admissible functionals, one has, moreover,

$$T^*_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M} = \Big\{(a,b,\mathbf{C}) \in H^\infty(\mathbb{R}^d)\times H^\infty(\mathbb{R}^d)\times H^\infty(\Omega_\zeta)^3 \mid \operatorname{div}\mathbf{C} = 0,\ \nabla^\perp\cdot C_\parallel = b,\ C_b = 0\Big\}.$$

Proof. Let us first consider the derivative with respect to ψ. Since F is admissible, we have

$$F(\zeta,\psi,\boldsymbol{\omega}) = \mathcal{F}(\zeta,\mathbb{U}[\zeta](\psi,\boldsymbol{\omega}))$$

with $\mathcal{F}\in C^\infty(\mathcal{N})$ satisfying (6.3), and therefore

$$\begin{aligned}\mathrm{d}_\psi F\cdot\delta\psi &= \mathrm{d}_U\mathcal{F}\cdot(\mathrm{d}_\psi\mathbb{U}[\zeta](\cdot,\boldsymbol{\omega})\cdot\delta\psi)\\ &= \int_\Omega\frac{\delta\mathcal{F}}{\delta\mathbf{U}}\cdot\mathbb{U}_I[\zeta]\delta\psi = \int_{\mathbb{R}^d}\frac{\delta\mathcal{F}}{\delta\mathbf{U}}\Big|_{z=\zeta}\cdot N\delta\psi,\end{aligned}$$

where we used the definition of $\mathbb{U}_I[\zeta]\delta\psi$, the fact that $\delta\mathcal{F}/\delta\mathbf{U}$ is divergence free, and the fact that its normal component vanishes at the bottom.

For the derivative with respect to $\boldsymbol{\omega}$, we proceed along the same lines as above to get

$$\mathrm{d}_{\boldsymbol{\omega}}F\cdot\delta\boldsymbol{\omega} = \int_\Omega\frac{\delta\mathcal{F}}{\delta\mathbf{U}}\cdot\mathbb{U}_{II}[\zeta]\delta\boldsymbol{\omega}.$$

Since $\delta\mathcal{F}/\delta\mathbf{U}$ is divergence free and has zero normal component at the bottom, we can use Corollary 3.9 and Green's identity, and write

$$\begin{aligned}
\mathrm{d}_{\boldsymbol{\omega}} F \cdot \delta\boldsymbol{\omega} &= \int_\Omega \operatorname{curl}^{-1} \frac{\delta\mathcal{F}}{\delta\mathbf{U}} \cdot \delta\boldsymbol{\omega} + \int_{\mathbb{R}^d} N \times \Big(\operatorname{curl}^{-1} \frac{\delta\mathcal{F}}{\delta\mathbf{U}}\Big) \cdot (\mathbb{U}_{II}[\zeta]\delta\boldsymbol{\omega})_{|\mathrm{surf}} \\
&= \int_\Omega \operatorname{curl}^{-1} \frac{\delta\mathcal{F}}{\delta\mathbf{U}} \cdot \delta\boldsymbol{\omega} + \int_{\mathbb{R}^d} \Big(\operatorname{curl}^{-1} \frac{\delta\mathcal{F}}{\delta\mathbf{U}}\Big)_{\parallel}^{\perp} \cdot (\mathbb{U}_{II}[\zeta]\delta\boldsymbol{\omega})_{\parallel}.
\end{aligned}$$

The second term in the above expression vanishes because

$$\Big(\operatorname{curl}^{-1} \frac{\delta\mathcal{F}}{\delta\mathbf{U}}\Big)_{\parallel}^{\perp} = -\nabla\Delta^{-1}(\underline{C} \cdot N),$$

which is L^2-orthogonal to $(\mathbb{U}_{II}[\zeta]\delta\boldsymbol{\omega})_{\parallel} = \nabla^{\perp}\Delta^{-1}(\delta\omega_{|_{\mathrm{surf}}} \cdot N)$, and we therefore have

$$\mathrm{d}_{\boldsymbol{\omega}} F \cdot \delta\boldsymbol{\omega} = \int_\Omega \operatorname{curl}^{-1} \frac{\delta\mathcal{F}}{\delta\mathbf{U}} \cdot \delta\boldsymbol{\omega}.$$

Finally, for the derivative with respect to ζ, we get

$$\begin{aligned}
\mathrm{d}_\psi F \cdot \delta\psi &= \mathrm{d}_\zeta \mathcal{F} \cdot \delta\zeta + d_U\mathcal{F} \cdot (\mathrm{d}_\zeta \mathbb{U}[\cdot](\psi, \boldsymbol{\omega}) \cdot \delta\zeta) \\
&= \int_{\mathbb{R}^d} \frac{\delta\mathcal{F}}{\delta\zeta}\delta\zeta + \int_\Omega \frac{\delta\mathcal{F}}{\delta\mathbf{U}} \cdot \mathbb{U}_I[\zeta]\Big(-\underline{w}\delta\zeta + \frac{\nabla}{\Delta} \cdot (\underline{\omega}_h^{\perp}\delta\zeta)\Big),
\end{aligned}$$

where we used[15] Proposition 3.17 to substitute

$$\mathrm{d}_\zeta \mathbb{U}[\cdot](\psi, \boldsymbol{\omega}) \cdot \delta\zeta = \mathbb{U}_I[\zeta]\Big(-\underline{w}\delta\zeta + \frac{\nabla}{\Delta} \cdot (\underline{\omega}_h^{\perp}\delta\zeta)\Big)$$

in the second term. Since $\delta\mathcal{F}/\delta\mathbf{U}$ is divergence free and has zero normal component at the bottom, we can use Green's identity to get

$$\begin{aligned}
\mathrm{d}_\psi F \cdot \delta\psi &= \int_{\mathbb{R}^d} \frac{\delta\mathcal{F}}{\delta\zeta}\delta\zeta + \int_{\mathbb{R}^d} \frac{\delta\mathcal{F}}{\delta\mathbf{U}} \cdot N\Big(-\underline{w}\delta\zeta + \frac{\nabla}{\Delta} \cdot (\underline{\omega}_h^{\perp}\delta\zeta)\Big) \\
&= \int_{\mathbb{R}^d} \Big(\frac{\delta\mathcal{F}}{\delta\zeta} - \underline{w}\frac{\delta\mathcal{F}}{\delta\mathbf{U}} \cdot N - \underline{\omega}_h^{\perp} \cdot \nabla\Delta^{-1}\frac{\delta\mathcal{F}}{\delta\mathbf{U}} \cdot N\Big)\delta\zeta,
\end{aligned}$$

and the expression for $\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}} F$ given in the statement of the proposition follows easily.

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[15]Proposition 3.17 actually gives a formula for the time derivative of $\mathbb{U}[\zeta](\psi, \boldsymbol{\omega})$; the shape derivative formula used here is obtained exactly in the same way.

Recalling the identity $\nabla^\perp\cdot C_\parallel=(\operatorname{curl}\mathbf{C})_{|_{\rm surf}}\cdot N$, we immediatly deduce that

$$T^*_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M}\subset\Big\{(a,b,\mathbf{C})\in H^\infty(\mathbb{R}^d)\times H^\infty(\mathbb{R}^d)\times H^\infty(\Omega_\zeta)\mid \nabla^\perp\cdot C_\parallel=b,\ C_b=0\Big\}.$$

To prove the reverse inclusion, we need to construct, for all $(a,b,\mathbf{C})$ satisfying the condition defining the space in the right part of the identity, a functional $\mathcal{F}\in C^\infty(\mathcal{N})$ satisfying (6.3), and such that

$$\frac{\delta\mathcal{F}}{\delta\mathbf{U}}(\zeta,\mathbf{U})=\operatorname{curl}\mathbf{C},\quad \frac{\delta\mathcal{F}}{\delta\zeta}(\zeta,\mathbf{U})=a+\underline{w}b-\omega_h^\perp\cdot\nabla\Delta^{-1}b.$$

This is achieved by taking $\mathcal{F}$ defined as

$$\mathcal{F}(\zeta',\mathbf{U}')=\int_{\Omega_{\zeta'}}\operatorname{curl}\mathbf{C}\cdot\mathbf{U}'+\int_{\mathbb{R}^d}(a+\underline{w}b-\omega_h^\perp\cdot\nabla\Delta^{-1}b-(\operatorname{curl}\mathbf{C})_{|_{z=\zeta}}\cdot\underline{U})\zeta',$$

for all $(\zeta',\mathbf{U}')\in\mathcal{N}$. ◻

This proposition can be used to compute the gradient of the total energy (6.2).

Corollary 6.5. *Let H be the functional on $\mathcal{M}$ associated with the total energy* (6.2)*. Then,*

$$\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}}H=\begin{pmatrix} g\zeta+\frac{1}{2}|U_\parallel|^2-\frac{1}{2}(1+|\nabla\zeta|^2)\underline{w}^2-\underline{\omega}_h^\perp\cdot\nabla\Delta^{-1}\underline{U}\cdot N\\ \underline{U}\cdot N\\ \operatorname{curl}^{-1}\mathbf{U}\end{pmatrix}.$$

Proof. The functional H is admissible, with associated function $\mathcal{H}$ on $\mathcal{N}$ given by

$$\mathcal{H}(\zeta,\mathbf{U})=\frac{1}{2}\int_{\mathbb{R}^d}g\zeta^2+\frac{1}{2}\int_{\Omega_\zeta}|\mathbf{U}|^2.$$

The result follows from Proposition 6.4 and the observation that

$$\frac{\delta\mathcal{H}}{\delta\zeta}=g\zeta+\frac{1}{2}|\underline{U}|^2,\quad\frac{\delta\mathcal{H}}{\delta\mathbf{U}}=\mathbf{U}.$$ ◻

6.3. The Poisson bracket and the Hamiltonian formulation. As said above, Zakharov showed that the irrotational water waves equations (2.13) can be written under the form

$$\partial_t\begin{pmatrix}\zeta\\ \psi\end{pmatrix}=J\operatorname{grad}_{\zeta,\psi}H,\quad\text{with } J=\begin{pmatrix}0&1\\-1&0\end{pmatrix};$$

in particular, J is an antisymmetric operator on $H_0^\infty(\mathbb{R}^d)\times\dot H^\infty(\mathbb{R}^d)$. In order to generalize this result to the rotational case, we need to define the notion of antisymmetric operator on the cotangent bundle $T^*\mathcal{M}$.

Definition 6.6. A mapping $J : T^*\mathcal{M}\to T\mathcal{M}$ is *antisymmetric* if, on each fiber of the cotangent bundle, the bilinear mapping

$$\begin{aligned}&T^*_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M}\times T^*_{\zeta,\psi,\boldsymbol{\omega}}\mathcal{M}\to\mathbb{R}\\ &(d_{\zeta,\psi,\boldsymbol{\omega}}F, d_{\zeta,\psi,\boldsymbol{\omega}}G)\mapsto(\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}}F, J_{\zeta,\psi,\boldsymbol{\omega}}\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}}G)_{L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\times L^2(\Omega_\zeta)}\end{aligned}$$

is antisymmetric.

We can now state the following generalization of Zakharov's result.

Theorem 6.7. *The water waves equations* (2.11) *can be written*

$$\partial_t\begin{pmatrix}\zeta\\ \psi\\ \boldsymbol{\omega}\end{pmatrix}=J_{\zeta,\psi,\boldsymbol{\omega}}\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}}H,$$

with

$$\begin{aligned}&J_{\zeta,\psi,\boldsymbol{\omega}}\\ &=\begin{pmatrix}0 & 1 & 0\\ -1 & \left(\underline{\omega}_h\cdot\frac{\nabla^\perp}{\Delta}\bullet+\frac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}_h\bullet)\right) & \frac{\nabla^\perp}{\Delta}\cdot\left(\underline{\omega}_h(\operatorname{curl}\bullet)\cdot N-\underline{\omega}\cdot N(\operatorname{curl}\bullet)_h\right)\\ 0 & 0 & \operatorname{curl}(\boldsymbol{\omega}\times\operatorname{curl}\bullet)\end{pmatrix};\end{aligned}$$

the field of linear mappings $J=(J_{\zeta,\psi,\boldsymbol{\omega}})_{(\zeta,\psi,\boldsymbol{\omega})\in\mathcal{M}} : T^*\mathcal{M}\to T\mathcal{M}$ is antisymmetric.

Proof. Let us define $J^0_{\boldsymbol{\omega}}$ by

$$J^0_{\boldsymbol{\omega}}=\begin{pmatrix}0 & 1\\ -1 & \left(\underline{\omega}_h\cdot\frac{\nabla^\perp}{\Delta}\bullet+\frac{\nabla^\perp}{\Delta}\cdot(\underline{\omega}_h\bullet)\right)\end{pmatrix}.$$

We have, therefore, for all admissible functionals $F,G\in\mathcal{A}$ (and writing $\delta_\zeta F=\delta F/(\delta\zeta)$ etc.) the following:

(6.4)

$$\begin{aligned}&(\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}}F, J_{\zeta,\psi,\boldsymbol{\omega}}\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}}G)_{L^2\times L^2\times L^2}\\ &\quad=\left(\begin{pmatrix}\delta_\zeta F\\ \delta_\psi F\end{pmatrix}, J^0_{\zeta,\boldsymbol{\omega}}\begin{pmatrix}\delta_\zeta G\\ \delta_\psi G\end{pmatrix}\right)_{L^2\times L^2}\\ &\qquad+\left(\delta_\psi F,\frac{\nabla^\perp}{\Delta}\cdot\left(\underline{\omega}_h(\operatorname{curl}\delta_{\boldsymbol{\omega}}G)_{|\mathrm{surf}}\cdot N-\underline{\omega}\cdot N(\operatorname{curl}\delta_{\boldsymbol{\omega}}G)_{|\mathrm{surf},h}\right)\right)_{L^2(\mathbb{R}^d)}\\ &\qquad+(\delta_{\boldsymbol{\omega}}F,\operatorname{curl}(\boldsymbol{\omega}\times\operatorname{curl}\delta_{\boldsymbol{\omega}}G))_{L^2(\Omega)}.\end{aligned}$$

Focusing our attention on the last two terms of the right-hand side, we note first that

$$\begin{aligned}
&\Big(\delta_\psi F, \frac{\nabla^\perp}{\Delta}\cdot\big(\underline{\omega}_h(\operatorname{curl}\delta_{\boldsymbol\omega}G)_{|\mathrm{surf}}\cdot N - \underline{\omega}\cdot N(\operatorname{curl}\delta_{\boldsymbol\omega}G)_{|\mathrm{surf},h}\big)\Big)\\
&\quad = -\Big(\frac{\nabla^\perp}{\Delta}\delta_\psi F, \underline{\omega}_h(\operatorname{curl}\delta_{\boldsymbol\omega}G)_{|\mathrm{surf}}\cdot N - \underline{\omega}\cdot N(\operatorname{curl}\delta_{\boldsymbol\omega}G)_{|\mathrm{surf},h}\Big)\\
&\quad = \left(\operatorname{curl}\delta_{\boldsymbol\omega}G_{|\mathrm{surf}}, \underline{\omega}\times\begin{pmatrix}\frac{\nabla}{\Delta}\delta_\psi F\\ \frac{\nabla}{\Delta}\delta_\psi F\cdot\nabla\zeta\end{pmatrix}\right).
\end{aligned}$$

For the last term of (6.4), we use Green's identity to get

$$\begin{aligned}
&(\delta_{\boldsymbol\omega}F, \operatorname{curl}(\boldsymbol\omega\times\operatorname{curl}\delta_{\boldsymbol\omega}G))_{L^2(\Omega)}\\
&\quad = (\operatorname{curl}\delta_{\boldsymbol\omega}F, \boldsymbol\omega\times\operatorname{curl}\delta_{\boldsymbol\omega}G)_{L^2(\Omega)}\\
&\qquad + ((\operatorname{curl}\delta_{\boldsymbol\omega}G)_{|\mathrm{surf}}, \underline{\omega}\times(N\times(\delta_{\boldsymbol\omega}F))_{|\mathrm{surf}})_{L^2(\mathbb{R}^d)}\\
&\quad = (\operatorname{curl}\delta_{\boldsymbol\omega}F, \boldsymbol\omega\times\operatorname{curl}\delta_{\boldsymbol\omega}G)_{L^2(\Omega)}\\
&\qquad + \left((\operatorname{curl}\delta_{\boldsymbol\omega}G)_{|\mathrm{surf}}, \underline{\omega}\times\begin{pmatrix}(\delta_{\boldsymbol\omega}F)_\parallel^\perp\\ (\delta_{\boldsymbol\omega}F)_\parallel^\perp\cdot\nabla\zeta\end{pmatrix}\right)_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

We therefore get from (6.4) that

$$\begin{aligned}
&\left(\begin{pmatrix}\delta_\zeta F\\ \delta_\psi F\\ \delta_{\boldsymbol\omega}F\end{pmatrix}, J_{\zeta,\psi,\boldsymbol\omega}\begin{pmatrix}\delta_\zeta G\\ \delta_\psi G\\ \delta_{\boldsymbol\omega}G\end{pmatrix}\right)_{L^2\times L^2\times L^2}\\
&\quad = \left(\begin{pmatrix}\delta_\zeta F\\ \delta_\psi F\end{pmatrix}, J^0_{\zeta,\boldsymbol\omega}\begin{pmatrix}\delta_\zeta G\\ \delta_\psi G\end{pmatrix}\right)_{L^2\times L^2} + (\operatorname{curl}\delta_{\boldsymbol\omega}F, \boldsymbol\omega\times\operatorname{curl}\delta_{\boldsymbol\omega}G)_{L^2(\Omega)}\\
&\qquad + \left((\operatorname{curl}\delta_{\boldsymbol\omega}G)_{|\mathrm{surf}}, \underline{\omega}\times\begin{pmatrix}(\delta_{\boldsymbol\omega}F)_\parallel^\perp + \frac{\nabla}{\Delta}\delta_\psi F\\ \Big((\delta_{\boldsymbol\omega}F)_\parallel^\perp + \frac{\nabla}{\Delta}\delta_\psi F\Big)\cdot\nabla\zeta\end{pmatrix}\right)_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Now, the assumption that $F \in A$ implies by Proposition 6.4 that the last term vanishes, so that

$$\begin{aligned}
(6.5)\quad &\left(\begin{pmatrix}\delta_\zeta F\\ \delta_\psi F\\ \delta_{\boldsymbol\omega}F\end{pmatrix}, J_{\zeta,\psi,\boldsymbol\omega}\begin{pmatrix}\delta_\zeta G\\ \delta_\psi G\\ \delta_{\boldsymbol\omega}G\end{pmatrix}\right)_{L^2\times L^2\times L^2}\\
&\quad = \left(\begin{pmatrix}\delta_\zeta F\\ \delta_\psi F\end{pmatrix}, J^0_{\zeta,\boldsymbol\omega}\begin{pmatrix}\delta_\zeta G\\ \delta_\psi G\end{pmatrix}\right)_{L^2\times L^2} + (\operatorname{curl}\delta_{\boldsymbol\omega}F, \boldsymbol\omega\times\operatorname{curl}\delta_{\boldsymbol\omega}G)_{L^2(\Omega)}.
\end{aligned}$$

Since, moreover, $J^0_{\zeta,\boldsymbol{\omega}}$ is obviously skew-symmetric for the $L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)$ scalar product, the result follows directly. □

We can now deduce the following corollary that shows that the water waves equations with vorticity can be formally written in Hamiltonian form.

Corollary 6.8. *The water waves equations* (2.11) *are equivalent to the Hamiltonian equation*

$$\forall\, F \in \mathcal{A}, \qquad \dot{F} = \{F, H\},$$

where H is the Hamiltonian (6.2)*, while the* Poisson bracket $\{\cdot,\cdot\}$ *is defined as*

$$\{F,G\} = \int_{\mathbb{R}^d} \frac{\delta F}{\delta \zeta}\frac{\delta G}{\delta \psi} - \frac{\delta F}{\delta \psi}\frac{\delta G}{\delta \zeta} - \int_{\mathbb{R}^d} \underline{\omega}_h \cdot \Big[\frac{\delta F}{\delta \psi}\frac{\nabla^\perp}{\Delta}\frac{\delta G}{\delta \psi} - \frac{\delta G}{\delta \psi}\frac{\nabla^\perp}{\Delta}\frac{\delta F}{\delta \psi}\Big] + \int_\Omega \Big(\operatorname{curl}\frac{\delta F}{\delta \boldsymbol{\omega}}\Big)\cdot\Big(\boldsymbol{\omega}\times \operatorname{curl}\frac{\delta G}{\delta \boldsymbol{\omega}}\Big),$$

for all $F, G \in \mathcal{A}$.

Remark 6.9. As said above, the Hamiltonian formulation derived above is only formal. In order to obtain a *valid* Hamiltonian structure [36], one must also prove that the Poisson bracket satisfies Jacobi's identity, and that it is closed (i.e., that for all $F, G \in \mathcal{A}$, the functional F, G is also an admissible one). Checking these points is left for future work. Note that it is proved in [36] that the Poisson brackets derived in [43] are not valid; actually, even in the irrotational case, it does not seem to be known whether Zakharov's formulation (6.1) provides a *valid* Hamiltonian structure.

Proof. Note first that (6.5) corresponds exactly to the Poisson bracket. Therefore, as a simple consequence of Theorem 6.7, if $(\zeta, \psi, \boldsymbol{\omega})$ solves (2.11), then the Hamiltonian equation is satisfied.

Conversely, if the Hamiltonian equation is satisfied for all admissible functionals F, we deduce from Theorem 6.7 that

$$\forall\, F \in \mathcal{A}, \qquad \left(\operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}} F, \begin{pmatrix}\partial_t \zeta\\ \partial_t \psi\\ \partial_t \boldsymbol{\omega}\end{pmatrix} - J_{\zeta,\psi,\boldsymbol{\omega}} \operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}} H\right)_{L^2\times L^2\times L^2} = 0.$$

Using the last point of Proposition 6.4, we readily deduce that

$$\begin{pmatrix}\partial_t \zeta\\ \partial_t \psi\\ \partial_t \boldsymbol{\omega}\end{pmatrix} - J_{\zeta,\psi,\boldsymbol{\omega}} \operatorname{grad}_{\zeta,\psi,\boldsymbol{\omega}} H = 0,$$

and the result is proved. □

Acknowledgements. The authors warmly thank B. Kolev for his help on the various Hamiltonian structures encountered in fluid mechanics. The first author is supported by the grant MTM2011-266696 (Spain), ICMAT Severo Ochoa project SEV-2011-0087, and ERC grant 307179-GFTIPFD. The second author acknowledges support from the ANR-13-BS01-0003-01 DYFICOLTI, the ANR-13-BS01-0009-01 BOND, and the INSU-CNRS LEFE-MANU project Soli.

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KEY WORDS AND PHRASES: Water waves, shallow water models, vorticity.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 35Q35, 76B15.

Received: February 2, 2014.