J. Fluid Mech. (2014), *vol.* 759, *pp.* 642–675.
doi:10.1017/jfm.2014.593

Fully nonlinear long-wave models in the presence of vorticity

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(Received 11 June 2014; revised 8 September 2014; accepted 4 October 2014)

We study here Green–Naghdi type equations (also called fully nonlinear Boussinesq, or Serre equations) modelling the propagation of large-amplitude waves in shallow water without a smallness assumption on the amplitude of the waves. The novelty here is that we allow for a general vorticity, thereby allowing complex interactions between surface waves and currents. We show that the *a priori* $(2+1)$-dimensional dynamics of the vorticity can be reduced to a finite cascade of two-dimensional equations. With a mechanism reminiscent of turbulence theory, vorticity effects contribute to the averaged momentum equation through a Reynolds-like tensor that can be determined by a cascade of equations. Closure is obtained at the precision of the model at the second order of this cascade. We also show how to reconstruct the velocity field in the $(2+1)$-dimensional fluid domain from this set of two-dimensional equations and exhibit transfer mechanisms between the horizontal and vertical components of the vorticity, thus opening perspectives for the study of rip currents, for instance.

Key words: shallow water flows, shear waves, surface gravity waves

1. Introduction

The equations describing the motion of an inviscid and incompressible fluid of constant density ρ and delimited from a above by a free surface $\{z=\zeta(t,X)\}$ $(X\in\mathbb{R}^2)$ and below by a non-moving bottom $\{z=-H_0+b(X)\}$ are given by the so-called free-surface Euler equations. Denoting by $\boldsymbol{U}=(\boldsymbol{V}^{\mathrm{T}},\boldsymbol{w})^{\mathrm{T}}$ and P the velocity and pressure fields, these equations can be written as

$$\partial_t\boldsymbol{U}+\boldsymbol{U}\boldsymbol{\cdot}\nabla_{X,z}\boldsymbol{U}=-\frac{1}{\rho}\nabla_{X,z}P-g\boldsymbol{e}_z, \tag{1.1}$$

$$\nabla_{X,z}\boldsymbol{\cdot}\boldsymbol{U}=0, \tag{1.2}$$

in the fluid domain $\Omega_t=\{(X,z)\in\mathbb{R}^{2+1},\ -H_0+b(X)<z<\zeta(t,X)\}$; they are complemented with the boundary conditions

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$$\partial_t \zeta - \boldsymbol{U}_{|_{surf}} \cdot N = 0 \quad (\text{with } N = (-\nabla\zeta^{\mathrm{T}}, 1)^{\mathrm{T}}) \tag{1.3}$$

$$P_{|_{surf}} = \text{const.} \tag{1.4}$$

at the surface and

$$\boldsymbol{U}_{|_{bott}} \cdot N_b = 0 \quad (\text{with } N_b = (-\nabla b^{\mathrm{T}}, 1)^{\mathrm{T}}) \tag{1.5}$$

at the bottom.

It is well known that the kinetic equation (1.3) can be restated as a mass conservation equation

$$\partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \tag{1.6}$$

where h is the total depth of the fluid and $\overline{V}$ the vertical average of the horizontal component $\boldsymbol{V}$ of the velocity (equivalently, $h\overline{V}$ is the total discharge),

$$h(t, X) := H_0 + \zeta(t, X) - b(X), \quad \overline{V}(t, X) = \frac{1}{h}\int_{-H_0+b(X)}^{\zeta(t,X)} \boldsymbol{V}(t, X, z)\mathrm{d}z. \tag{1.7a,b}$$

Notation 1. We decompose any function f defined on Ω_t as an averaged part and a zero mean component, using the notation

$$f(t, X, z) = \overline{f}(t, X) + f^*(t, X, z), \quad \text{with } \overline{f} = \frac{1}{h}\int_{-H_0+b(X)}^{\zeta(t,X)} f(t, X, z)\mathrm{d}z \tag{1.8}$$

and $f^* = f - \overline{f}$.

It is therefore quite natural to look for another equation that would complement (1.6) to form a closed system of two evolution equations on ζ and $\overline{V}$. Decomposing the horizontal velocity field as

$$\boldsymbol{V}(t, X, z) = \overline{V}(t, X) + V^*(t, X, z) \tag{1.9}$$

and integrating vertically the horizontal component of (1.1), one obtains classically

$$\partial_t(h\overline{V}) + \nabla \cdot \left(\int_{-H_0+b}^{\zeta} \boldsymbol{V} \otimes \boldsymbol{V}\right) + \int_{-H_0+b}^{\zeta} \nabla P = 0. \tag{1.10}$$

Since, by construction, the vertical average of V^* vanishes, we finally obtain as in Teshukov (2007) the following set of evolution equations on ζ and $\overline{V}$:

$$\left.\begin{aligned} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t(h\overline{V}) + \nabla \cdot (h\overline{V} \otimes \overline{V}) + \nabla \cdot \left(\int_{-H_0+b}^{\zeta} V^* \otimes V^*\right) + \int_{-H_0+b}^{\zeta} \nabla P = 0. \end{aligned}\right\} \tag{1.11}$$

We shall refer to (1.11) as the averaged Euler equations. These equations are exact but too complex to work with (because V^* and ∇P are not closed expressions of ζ and $\overline{V}$) and they are replaced by simpler approximate equations for practical purposes; we shall consider here approximations of these equations in shallow water, i.e. when the depth is small compared with the typical horizontal length.

Let us first consider the case of irrotational flows for which (1.1) and (1.2) are complemented by the condition

$$\nabla_{X,z} \times \boldsymbol{U} = 0. \tag{1.12}$$

In the shallow water regime, that is, when $\mu := H_0^2/L^2 \ll 1$ (with L the typical horizontal scale), it is well known that the flow is columnar at leading order in the sense that the horizontal velocity does not depend at leading order on the vertical variable z. The 'Reynolds' tensor

$$\boldsymbol{R} := \int_{-1+b}^{\zeta} V^* \otimes V^* \tag{1.13}$$

is therefore a second-order term (it is of size $O(\mu^2)$). The terminology 'Reynolds tensor' is *sensu stricto* improper here since space derivatives do not commute with averaging (consisting in vertical integration here); we, however, use it because the analogy with Reynolds turbulence theory is instructive.

It is also classical in shallow water that the pressure is hydrostatic at leading order, $P(t, X, z) = \rho g(\zeta(t, X) - z) + O(\mu)$. At leading order, the averaged Euler equations (1.11) are therefore formally approximated by the nonlinear shallow water (or Saint-Venant) equations,

$$\left.\begin{aligned} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t(h\overline{V}) + gh\nabla\zeta + \nabla \cdot (h\overline{V} \otimes \overline{V}) = 0 \end{aligned}\right\} \tag{1.14}$$

(see Ovsjannikov 1976; Kano & Nishida 1979; Alvarez-Samaniego & Lannes 2008; Iguchi 2009 for a justification of this approximation). This model is widely used but misses, for instance, dispersive effects that can be very important in coastal oceanography. This is the reason why a more precise model taking into account the first-order terms (with respect to μ) is used for applications. As already said, the Reynolds-like tensor is a second-order term and can still be neglected at this level of approximation, but non-hydrostatic components of the pressure must be taken into account. The resulting equations are known as the Green–Naghdi (or Serre, or fully nonlinear Boussinesq equations, see Lannes & Bonneton (2009) and Bonneton *et al.* (2011*a*) for recent reviews). Under the formulation derived in Bonneton *et al.* (2011*b*), these equations can be written as

$$\left.\begin{aligned} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \left(I + h\mathcal{T}\frac{1}{h}\right)\left(\partial_t(h\overline{V}) + \nabla \cdot (h\overline{V} \otimes \overline{V})\right) + gh\nabla\zeta + h\mathcal{Q}_1(\overline{V}) = 0, \end{aligned}\right\} \tag{1.15}$$

where the non-hydrostatic effects are taken into account through the operators $\mathcal{T}$ and $\mathcal{Q}(\cdot)$, defined as

$$\mathcal{T}V = -\frac{1}{3h}\nabla(h^3\nabla \cdot V) + \frac{1}{2h}\left[\nabla(h^2\nabla b \cdot V) - h^2\nabla b\nabla \cdot V\right] + \nabla b\nabla b \cdot V, \tag{1.16}$$

and, writing $V^\perp = (-V_2, V_1)^\perp$,

$$\mathcal{Q}_1(V) = -2\mathcal{R}_1\left(\partial_1 V \cdot \partial_2 V^\perp + (\nabla \cdot V)^2\right) + \mathcal{R}_2\left(V \cdot (V \cdot \nabla)\nabla b\right), \tag{1.17}$$

with

$$\mathscr{R}_1 w = -\frac{1}{3h}\nabla(h^3 w) - \frac{h}{2} w\nabla b, \quad \mathscr{R}_2 w = \frac{1}{2h}\nabla(h^2 w) + w\nabla b. \tag{1.18a,b}$$

It is possible to derive from (1.15) an equation for the local conservation of energy, namely,

$$\partial_t \mathfrak{e} + \nabla \cdot \mathfrak{F} = 0, \tag{1.19}$$

where the energy $\mathfrak{e}$ is the sum of the potential energy $\mathfrak{e}_p$ and the kinetic energy $\mathfrak{e}_k$ given by

$$\mathfrak{e}_p = \tfrac{1}{2} g\zeta^2, \quad \mathfrak{e}_k = \tfrac{1}{2} h|\overline{V}|^2 + \tfrac{1}{2} h\left(\tfrac{1}{3}\left|h\nabla \cdot \overline{V} - \tfrac{3}{2}\nabla b \cdot \overline{V}\right|^2 + \tfrac{1}{4}|\nabla b \cdot \overline{V}|^2\right), \tag{1.20a,b}$$

and where the flux $\mathfrak{F}$ is given by the expression

$$\mathfrak{F} = (g\zeta h + \mathfrak{e}_k + \mathfrak{q})\overline{V}, \tag{1.21}$$

with

$$\mathfrak{q} = -\tfrac{1}{3} h^2(\partial_t + \overline{V} \cdot \nabla)(h\nabla \cdot \overline{V}) + \tfrac{1}{2} h^2(\partial_t + \overline{V} \cdot \nabla)(\nabla b \cdot \overline{V}). \tag{1.22}$$

The Green–Naghdi equations have been rigorously justified in Makarenko (1986), Li (2006) and Alvarez-Samaniego & Lannes (2008) (see also the monograph by Lannes (2013) and references therein). The Green–Naghdi system is now the most popular model for the numerical simulation of coastal flows, even in configurations that include vanishing depth (shoreline) and wave breaking (see, for instance, Chen *et al.* (2000), Cienfuegos, Bartélemy & Bonneton (2006), Le Métayer, Gavrilyuk & Hank (2010), Bonneton *et al.* (2011*b*), Kazolea *et al.* (2012), Dutykh *et al.* (2014), Lannes & Marche (2014) and Ricchiuto & Filippini (2014)).

Despite their many advantages, the Green–Naghdi equations (1.15) can only account for configurations where rotational effects are absent (i.e. when the assumption (1.12) holds). This is unfortunately not the case when waves propagate in a zone where currents are present, or when vorticity is created by anisotropic dissipation due to wave breaking as for rip currents (see, for instance, Hammack, Scheffner & Segur (1991) and Chen, Dalrymple & Kirby (1999)). More generally, the full coupling between currents and surface waves is still largely not understood. The presence of a non-zero vorticity makes the analysis more difficult. Indeed, $(d+1)$-dimensional irrotational flows (where d is the horizontal dimension) are d-dimensional in nature: as shown by Zakharov (1968), the full Euler equations can be reduced to a Hamiltonian formulation in terms of $\zeta(t, X)$ and $\psi(t, X) = \Phi(t, X, \zeta(t, X))$, where Φ is a scalar velocity potential (i.e. $\nabla_{X,z}\Phi = \boldsymbol{U}$). Both ζ and ψ are independent of the vertical variable, and Zakharov's Hamiltonian formulation of the full water wave problem is therefore not qualitatively different in this aspect from averaged models such as the Saint-Venant or Green–Naghdi equations (1.14) and (1.15). In the rotational setting, the picture is drastically different, and $(d+1)$-dimensional flows are truly $(d+1)$-dimensional. A Hamiltonian formulation of the water wave equations in the presence of vorticity generalizing Zakharov's formulation has recently been derived in Castro & Lannes (2014). The evolution of ζ and ψ must then be coupled to the evolution equation on the vorticity $\boldsymbol{\omega} = \nabla_{X,z} \times \boldsymbol{U}$,

$$\partial_t \boldsymbol{\omega} + \boldsymbol{U} \cdot \nabla_{X,z} \boldsymbol{U} = \boldsymbol{U} \cdot \nabla_{X,z} \boldsymbol{\omega} \quad \text{in } \Omega_t, \tag{1.23}$$

which is $(d+1)$-dimensional. The reduction to a d-dimensional set of equations such as the Saint-Venant or Green–Naghdi equations is therefore a qualitative jump and is not *a priori* obvious. Technically, in the rotational setting, the 'Reynolds' tensor $\boldsymbol{R}$ is no longer a second-order term and contributes to the momentum equation through a coupling with the $(d+1)$-dimensional dynamics of the vorticity equation.

Several approaches have been proposed to get round this difficulty. Following Bowen (1969) and Longuet-Higgins (1970), many models use a time-averaging approach where radiation stresses due to the short-wave motion are considered as a forcing term in the momentum equations (see Svendsen & Putrevu (1995) for a review). A generalization of the Green–Naghdi/fully nonlinear Boussinesq/Serre equations (1.15) able to handle the presence of vorticity would be a very promising alternative because the surface wave and current motion could be handled simultaneously without requiring the computation of radiation stresses through a wave-averaged model.

It was shown in Chen *et al.* (2003) that the Green–Naghdi equations (1.15) are able to describe partially rotational flows with purely vertical vorticity. The presence of horizontal vorticity has been considered for one-dimensional surfaces ($d=1$) in Veeramony & Svendsen (2000) for the weakly nonlinear case and Musumeci, Svendsen & Veeramony (2005) for the fully nonlinear case. These authors made explicit the contribution of the 'Reynolds' tensor to the momentum equation in this case, and showed that the momentum equation in (1.15) must be modified by the addition of new terms coming from the 'Reynolds' tensor. The computation of these new terms requires the resolution of a $(1+1)$-dimensional transport equation for the vorticity (in horizontal dimension $d=1$, the stretching term disappears from (1.23)). The authors derive an approximate explicit solution to this transport equation under a small-amplitude assumption roughly equivalent to assuming that the vorticity dynamics is weakly nonlinear. In Kim, Lynett & Socolofsky (2009), a small horizontal vorticity was also allowed in a Boussinesq type model by resorting to a simplified description for the vorticity model.

Another recent approach to handle vorticity in shallow water flows was proposed in Zhang *et al.* (2013). Their strategy is reminiscent of the finite element approach: generalizing the approach of Shields & Webster (1988) and Kim *et al.* (2001), the vertical dependence of the velocity field is projected onto a finite dimensional basis of functions of z, and the 'Reynolds' tensor $\boldsymbol{R}$ is approximated by its projection onto this basis; computations are further simplified by dropping the smallest terms with respect to μ. They end up with a set of equations on the coordinates of the velocity field in this basis (these coordinates are functions of time and the horizontal variable). Let us also mention Constantin (2001) and references therein for the modelling of one-dimensional, periodic or standing rotational water waves. Many studies have dealt with the analysis of such waves, and they have exhibited several interesting phenomena that are not encountered within the framework of irrotational flows. The possibility of stagnation points beneath the surface allows, for instance, 'cats eye' patterns for the streamlines, as predicted by Kelvin and more recently rigorously established in Wahlen (2009) and Constantin & Varvaruca (2011). Such phenomena are another motivation for the derivation of asymptotic models able to handle vorticity.

In this paper, we derive a set of equations, all of them d-dimensional, that generalize the Green–Naghdi equations (1.15) in the presence of vorticity. No assumption is made other than dropping order $O(\mu^2)$ terms as in the irrotational theory. Our strategy to get rid of the $(d+1)$-dimensional dynamics of the vorticity is inspired by standard turbulence theory. Solving the vorticity equation is indeed sufficient to compute the 'Reynolds' tensor $\boldsymbol{R}$, but it is not necessary to do so. One can rather look for an equation solved by $\boldsymbol{R}$. This approach leads to a cascade of equations (the equation on $\boldsymbol{R}$ involves a third-order tensor, which satisfies itself an equation involving a fourth-order tensor, etc.). The closure of this cascade of equations is one of the main challenges in turbulence theory; in the present context, we show that at the order $O(\mu^2)$ of the Green–Naghdi approximation, this cascade of equations is actually finite. To be more precise, we need to introduce first the shear velocity representing the contribution to the horizontal velocity of the horizontal vorticity,

$$V_{sh} = \int_z^\zeta \boldsymbol{\omega}_h^\perp. \tag{1.24}$$

The 'Reynolds' tensor can then be decomposed into a component (denoted $\boldsymbol{E}$) due to the self-interaction of the shear velocity V_{sh}, and another one containing the interaction of the shear velocity with the standard dispersive vertical dependence of the horizontal velocity due to non-hydrostatic terms,

$$\boldsymbol{R} = \boldsymbol{E} + \frac{1}{2}\int_{-1+b}^{\zeta} [(V^* - V_{sh}^*) \otimes (V^* + V_{sh}^*) + (V^* + V_{sh}^*) \otimes (V^* - V_{sh}^*)], \tag{1.25}$$

with

$$\boldsymbol{E} = \int_{-1+b}^{\zeta} V_{sh}^* \otimes V_{sh}^*. \tag{1.26}$$

These two components are handled separately, and the finite cascade of equations is derived for $\boldsymbol{E}$; we also show that the other component has a behaviour qualitatively similar to the contribution of the vorticity to non-hydrostatic corrections to the pressure. Three different configurations of increasing complexity are considered.

(*a*) The one-dimensional case with constant vorticity $\boldsymbol{\omega} = (0, \omega, 0)$. In this case, the 'Reynolds' tensor can be explicitly computed and the vorticity equation is trivial. Writing $\overline{V} = (\overline{v}, 0)^{\mathrm{T}}$, the resulting equations are then

$$\left.\begin{aligned} &\partial_t \zeta + \partial_x (h\overline{v}) = 0,\\ &\left(1 + h\mathscr{T}\frac{1}{h}\right)\left(\partial_t (h\overline{v}) + \partial_x (h\overline{v}^2)\right) + gh\partial_x \zeta + h\mathscr{Q}_1(\overline{v}) + \partial_x\left(\frac{1}{12}h^3\omega^2\right)\\ &\qquad + h\mathscr{C}(\omega h, \overline{v}) + h\mathscr{C}_b(\omega h, \overline{v}) = 0, \end{aligned}\right\} \tag{1.27}$$

where $\mathscr{T}$ and $\mathscr{Q}_1$ are the one-dimensional versions of the operators defined in (1.16) and (1.17).

The last term in the second line corresponds to $\partial_x \boldsymbol{E}$, and the cascade of equations for $\boldsymbol{E}$ is therefore trivial (it is equivalent to the mass conservation equation); the two terms of the third line gather the non-hydrostatic correction due to the vorticity and the interaction between the shear velocity and the dispersive vertical variations (see (3.10) for the definition of $\mathscr{C}$ and $\mathscr{C}_b$).

(*b*) The one-dimensional case with general vorticity. In this case, the 'Reynolds' tensor cannot be computed explicitly and the cascade of equations for $\boldsymbol{E}$ is no longer trivial. For bottom variations of medium amplitude (see (4.2)), the

Green–Naghdi equations then become

$$\left.\begin{array}{c}\partial_t\zeta+\partial_x(h\overline{v})=0,\\ \left(1+h\mathscr{T}\frac{1}{h}\right)\left(\partial_t(h\overline{v})+\partial_x(h\overline{v}^2)\right)+gh\partial_x\zeta+h\mathscr{Q}_1(\overline{v})+\partial_x\boldsymbol{E}+h\mathscr{C}(v^\sharp,\overline{v})=0,\\ \partial_t v^\sharp+\overline{v}\partial_x v^\sharp+v^\sharp\partial_x\overline{v}=0,\\ \partial_t\boldsymbol{E}+\overline{v}\partial_x\boldsymbol{E}+3\boldsymbol{E}\partial_x\overline{v}+\partial_x\boldsymbol{F}=0,\\ \partial_t\boldsymbol{F}+\overline{v}\partial_x\boldsymbol{F}+4\boldsymbol{F}\partial_x\overline{v}=0.\end{array}\right\} \tag{1.28}$$

In these equations, the quantity $\boldsymbol{F}$ that appears in the cascade of equations for $\boldsymbol{E}$ is the third-order self-interaction tensor, while $v^\sharp$ is introduced to capture corrections due to the non-hydrostatic effects of the vorticity and to the interaction of the shear velocity with the dispersive vertical variations of the horizontal velocity,

$$\boldsymbol{F}=\int_{-1+b}^{\zeta}(v_{sh}^*)^3 \quad \text{and}$$

$$v^\sharp=-\frac{24}{h^3}\int_{-1+b}^{\zeta}\int_z^{\zeta}\int_{-1+b}^{z}v_{sh}^*=\frac{12}{h^3}\int_{-1+b}^{\zeta}(z+1-b)^2v_{sh}^*. \tag{1.29a,b}$$

(c) The two-dimensional case with general vorticity. This case is technically more involved because $\boldsymbol{E}$ is now a 2×2 tensor and $\boldsymbol{F}=\int_{-1+b}^{\zeta}V_{sh}^*\otimes V_{sh}^*\otimes V_{sh}^*$ is a $2\times 2\times 2$ tensor (with coordinates $\boldsymbol{F}_{ijk}$). The main qualitative difference is the presence of a source term in the equation for $\boldsymbol{E}$ that takes into account the interaction between the horizontal and vertical components of the vorticity. The equations are

$$\left.\begin{array}{l}\partial_t\zeta+\nabla\cdot(h\overline{V})=0,\\ \left(1+h\mathscr{T}\frac{1}{h}\right)(\partial_t(h\overline{V})+\nabla\cdot(h\overline{V}\otimes\overline{V}))+gh\nabla\zeta+h\mathscr{Q}_1(\overline{V})+\nabla\cdot\boldsymbol{E}+h\mathscr{C}(V^\sharp,\overline{V})=0,\\ \partial_t V^\sharp+(\overline{V}\cdot\nabla)V^\sharp+(V^\sharp\cdot\nabla)\overline{V}=0,\\ \partial_t\boldsymbol{E}+\overline{V}\cdot\nabla\boldsymbol{E}+\nabla\cdot\overline{V}\boldsymbol{E}+\nabla\overline{V}^{\mathrm{T}}\boldsymbol{E}+\boldsymbol{E}\nabla\overline{V}+\nabla\cdot\boldsymbol{F}=\mathscr{D}(V^\sharp,\overline{V}),\\ \partial_t\boldsymbol{F}_{ijk}+\overline{V}\cdot\nabla\boldsymbol{F}_{ijk}+\boldsymbol{F}_{lkj}\partial_l\overline{V}_i+\boldsymbol{F}_{ilk}\partial_l\overline{V}_j+\boldsymbol{F}_{ijl}\partial_l\overline{V}_k+\nabla\cdot\overline{V}\boldsymbol{F}_{ijk}=0,\end{array}\right\} \tag{1.30}$$

where the interaction between the horizontal and vertical components of the vorticity operates through the operator $\mathscr{D}$ (see (5.31) for its definition).

A local conservation of energy also holds for these new systems of equations; we show that (1.19) can be generalized into an equation of the form

$$\partial_t\left(\mathfrak{e}+\mathfrak{e}_{rot}\right)+\nabla\cdot\left(\mathfrak{F}+\mathfrak{F}_{rot}\right)=0, \tag{1.31}$$

(see Remarks 9, 11 and 15 for more details).

The above Green–Naghdi equations with vorticity allow one to determine the surface elevation ζ and the averaged velocity $\overline{V}$ from the knowledge of their initial values and of the initial value of the horizontal vorticity (more precisely, of $V^\sharp$, $\boldsymbol{E}$ and $\boldsymbol{F}$). To investigate sediment transport, for instance, one must be able to reconstruct the velocity field in the fluid domain. In the irrotational framework, this structure can be explicitly recovered from ζ and $\overline{V}$. At the level of precision of the Saint-Venant equations (1.14) (i.e. up to $O(\mu)$ terms in the approximation for V), one obtains

$$\boldsymbol{V}(t, X, z) = \overline{V}(t, X), \quad w(t, X, z) = 0. \tag{1.32a,b}$$

We show that this reconstruction is no longer true in the presence of vorticity and that at the precision of the model, one has for each level line θ ($\theta \in [0, 1]$),

$$\boldsymbol{V}(t, X, -H_0 + b(X) + \theta h(t, X)) = \overline{V} + V_\theta^*, \tag{1.33}$$

where $V_\theta^* = V_{sh}^*(t, X, -H_0 + b(X) + \theta h(t, X))$ can be simply determined from its initial value by solving

$$\partial_t V_\theta^* + \overline{V} \cdot \nabla V_\theta^* + V_\theta^* \cdot \nabla \overline{V} = 0. \tag{1.34}$$

At the level of precision of the Green–Naghdi equations (i.e. keeping the $O(\mu)$ terms neglected in the Saint-Venant equations), the formula for the horizontal velocity becomes

$$\boldsymbol{V}(t, X, -H_0 + b(X) + \theta h(t, X)) = \overline{V} + V_\theta^* + T_\theta^* \overline{V}, \tag{1.35}$$

where the new term T_θ^* accounts for dispersive corrections. These corrections are the same as in the irrotational theory; the main difference from the first-order approximation is that the equation for V_θ^* now contains quadratic nonlinear terms and a source term $\mathscr{S}$, namely,

$$\partial_t V_\theta^* + \overline{V} \cdot \nabla V_\theta^* + V_\theta^* \cdot \nabla \overline{V} + V_\theta^* \cdot \nabla V_\theta^* = \mathscr{S}. \tag{1.36}$$

The presence of this source term induces an important phenomenon: the creation of a horizontal shear from vertical vorticity, even if the initial horizontal vorticity is equal to zero.

We finally use these reconstruction formulae for the velocity to determine the dynamics of the evolution of the vorticity. We show in particular that the averaged vertical vorticity $\overline{\boldsymbol{\omega}}_{\mu,v} = 1/h \int_{-H_0+b}^{\zeta} \boldsymbol{\omega}_{\mu,v}$ can be created during the evolution of the flow through a mechanism of transfer from horizontal to vertical vorticity, which is likely to play an important role for the study of rip currents, for instance.

Remark 1. All the models are derived in dimensionless variables in this article, and with an evolution equation for $\overline{V}$ rather than $h\overline{V}$ as above; we chose for the sake of clarity to give the dimensional version of these systems in this introduction.

The paper is organized as follows. Section 2 is devoted to an asymptotic analysis of the averaged Euler equations (1.11). The first step is to introduce, in § 2.1, a dimensionless version of the equations. An asymptotic expansion is then derived in § 2.2 for the velocity field; this expansion involves a 'shear velocity' for which an equation is derived in § 2.3, while an expression for the pressure contribution to (1.11) is derived in § 2.4.

We then turn to the derivation of Green–Naghdi type equations in the presence of vorticity; the simplest case of constant vorticity in one dimension ($d = 1$) is first addressed in § 3. The component $\boldsymbol{E}$ of the 'Reynolds' tensor $\boldsymbol{R}$ and the pressure contribution can then be explicitly computed (see §§ 3.1 and 3.2 respectively). The Green–Naghdi equations with constant vorticity are then derived in § 3.3. Section 4 deals with the case of a general vorticity in dimension $d = 1$; there is now a coupling of the momentum equation with other equations describing the effects of the vorticity (see §§ 4.3–4.5). The corresponding $1d$ Green–Naghdi equations with vorticity are derived in § 4.6. The two-dimensional case is then handled in § 5. We explain in § 6 how to reconstruct the velocity field in the fluid domain and comment on the dynamics of the vorticity. A first-order reconstruction (Saint-Venant) is performed in § 6.1 and a second-order (Green–Naghdi) one in § 6.2. These reconstructions allow us to describe the dynamics of the vertical vorticity in § 6.3. Finally, a conclusion and perspectives are given in § 7.

2. Asymptotic analysis of the averaged Euler equations (1.11)

2.1. *The dimensionless free-surface Euler equations*

We non-dimensionalize the equations by using several lengths: the typical amplitude a_{surf} of the waves, the typical amplitude a_{bott} of the bottom variations, the typical depth H_0 and the typical horizontal scale L. Using these quantities, it is possible to form three dimensionless parameters,

$$\varepsilon = \frac{a_{surf}}{H_0}, \quad \beta = \frac{a_{bott}}{H_0}, \quad \mu = \frac{H_0^2}{L^2}; \tag{2.1a–c}$$

the parameters ε and β are often called nonlinearity (or amplitude) and topography parameters, and the parameter μ is the shallowness parameter.

Remark 2. We are interested here in shallow water flows and therefore assume that $\mu \ll 1$; on the contrary, we allow for large-amplitude waves and we do not make any smallness assumption on ε; it is therefore possible to set $\varepsilon = 1$ throughout this article. We, however, chose to keep track of this parameter because the simplifications obtained for small-amplitude (weakly nonlinear) or medium-amplitude waves can then be performed straightforwardly (see Remark 12, for instance).

We also use a_{surf}, a_{bott}, H_0 and L to define dimensionless variables and unknowns (written with a tilde),

$$\tilde{z} = \frac{z}{H_0}, \quad \tilde{X} = \frac{X}{L}, \quad \tilde{\zeta} = \frac{\zeta}{a_{surf}}, \quad \tilde{b} = \frac{b}{a_{bott}}; \tag{2.2a–d}$$

the non-dimensionalization of the time variable and of the velocity and pressure fields is based on the linear analysis of the equations (see for instance Lannes (2013, Chap. 1))

$$\tilde{\boldsymbol{V}} = \frac{\boldsymbol{V}}{V_0}, \quad \tilde{\boldsymbol{w}} = \frac{\boldsymbol{w}}{w_0}, \quad \tilde{t} = \frac{t}{t_0}, \quad \tilde{P} = \frac{P}{P_0}, \tag{2.3a–d}$$

with

$$V_0 = a\sqrt{\frac{g}{H_0}}, \quad w_0 = \frac{aL}{H_0}\sqrt{\frac{g}{H_0}}, \quad t_0 = \frac{L}{\sqrt{gH_0}}, \quad P_0 = \rho g H_0. \tag{2.4a–d}$$

With these variables and unknowns, and with the notations

$$\boldsymbol{U}^\mu = \begin{pmatrix} \sqrt{\mu}\boldsymbol{V} \\ \boldsymbol{w} \end{pmatrix}, \quad \boldsymbol{\nabla}^\mu = \begin{pmatrix} \sqrt{\mu}\boldsymbol{\nabla} \\ \partial_z \end{pmatrix}, \quad N^\mu = \begin{pmatrix} -\varepsilon\sqrt{\mu}\boldsymbol{\nabla}\zeta \\ 1 \end{pmatrix}, \tag{2.5a–c}$$

and

$$\mathrm{curl}^\mu = \boldsymbol{\nabla}^\mu \times, \quad \mathrm{div}^\mu = (\boldsymbol{\nabla}^\mu)^{\mathrm{T}}, \quad \underline{U}^\mu = (\sqrt{\mu}\underline{V}^{\mathrm{T}}, \underline{w})^{\mathrm{T}} := \boldsymbol{U}^\mu_{|_{z=\varepsilon\zeta}}, \tag{2.6a–c}$$

the incompressible Euler equations take the form (omitting the tildes)

$$\partial_t \boldsymbol{U}^\mu + \frac{\varepsilon}{\mu}\boldsymbol{U}^\mu \cdot \nabla^\mu \boldsymbol{U}^\mu = -\frac{1}{\varepsilon}\left(\nabla^\mu P + \boldsymbol{e}_z\right) \quad \text{in } \Omega_t, \tag{2.7}$$

$$\mathrm{div}^\mu \boldsymbol{U}^\mu = 0 \quad \text{in } \Omega_t, \tag{2.8}$$

where Ω_t now stands for the dimensionless fluid domain,

$$\Omega_t = \{(X, z) \in \mathbb{R}^{d+1}, -1 + \beta b(X) < z < \varepsilon\zeta(t, X)\}. \tag{2.9}$$

Finally, the boundary conditions on the velocity read in dimensionless form

$$\partial_t \zeta + \nabla \cdot (h\overline{V}) = 0 \quad \text{at the surface}, \tag{2.10}$$

$$P = 0 \quad \text{at the surface}, \tag{2.11}$$

$$\boldsymbol{U}^\mu_{|z=-1+\beta b} \cdot N^\mu_b = 0 \quad \text{at the bottom}, \tag{2.12}$$

where $N^\mu_b = (-\varepsilon\sqrt{\mu}\nabla b^{\mathrm{T}}, 1)^{\mathrm{T}}$ and the dimensionless versions of h and $\overline{V}$ are given by

$$h(t, X) := 1 + \varepsilon\zeta(t, X) - \beta b(X), \quad \overline{V}(t, X) = \frac{1}{h}\int_{-1+\beta b(X)}^{\varepsilon\zeta(t,X)} \boldsymbol{V}(t, X, z)\mathrm{d}z. \tag{2.13a,b}$$

Decomposing the horizontal velocity into

$$\boldsymbol{V}(t, X, z) = \overline{V}(t, X) + \sqrt{\mu}V^*(t, X, z), \tag{2.14}$$

the dimensionless version of (1.11) is then given by

$$\left.\begin{aligned} &\partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ &\partial_t(h\overline{V}) + \varepsilon\nabla \cdot (h\overline{V} \otimes \overline{V}) + \varepsilon\mu\nabla \cdot \left(\int_{-1+\beta b}^{\varepsilon\zeta} V^* \otimes V^*\right) + \frac{1}{\varepsilon}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla P = 0. \end{aligned}\right\} \tag{2.15}$$

We therefore need to express the 'rotational Reynolds tensor' $\int_{-1+\beta b}^{\varepsilon\zeta} V^* \otimes V^*$ and the pressure contribution $(1/\varepsilon)\int_{-1+\beta b}^{\varepsilon\zeta} \nabla P$ in terms of ζ and $\overline{V}$ in order to obtain a closed set of equations. This requires a good understanding of the behaviour of the vorticity $\boldsymbol{\omega}_\mu$, defined as

$$\boldsymbol{\omega}_\mu = \begin{pmatrix} \dfrac{1}{\sqrt{\mu}}(\partial_z \boldsymbol{V}^\perp - \nabla^\perp \boldsymbol{w}) \\ -\nabla \cdot \boldsymbol{V}^\perp \end{pmatrix} = \frac{1}{\mu}\,\mathrm{curl}^\mu \boldsymbol{U}^\mu. \tag{2.16}$$

We shall assume here that $\boldsymbol{\omega}_\mu$ is of order $O(1)$ with respect to μ (in the terminology of Teshukov (2007) and Richard & Gavrilyuk (2012), we consider therefore weakly sheared flows). It is rigorously shown in Castro & Lannes (2014) that in this regime, ζ, $\overline{V}$ and $\boldsymbol{\omega}_\mu$ remain $O(1)$ quantities during the time evolution of the flow. All the formal asymptotic descriptions made throughout this article therefore have a firm basis, and the models derived here could be rigorously justified using the procedure hinted at in Castro & Lannes (2014) (and carried through for the Saint-Venant equations with vorticity).

2.2. *Asymptotic expansion of the velocity field*

As explained above, the quantities ζ, $\overline{V}$ and $\boldsymbol{\omega}_\mu$ are all of order $O(1)$ with respect to μ. An asymptotic description of $\boldsymbol{U}^\mu$ can then be found by considering the boundary value problem

$$\left.\begin{array}{ll} \mathrm{curl}^\mu \boldsymbol{U}^\mu = \mu\boldsymbol{\omega}_\mu & \text{in } \Omega \\ \mathrm{div}^\mu \boldsymbol{U}^\mu = 0 & \text{in } \Omega \\ U^\mu_b \cdot N^\mu_b = 0 & \text{at the bottom} \end{array}\right\} \tag{2.17}$$

(the subscript b is used to denote quantities evaluated at the bottom).

Remark 3. Only the 'rotational' part of $\boldsymbol{U}^\mu$ is fully determined from this boundary value problem. The 'irrotational part' of $\boldsymbol{U}^\mu$ is determined from the tangential component of the velocity at the interface; more precisely, one can show that $\boldsymbol{U}^\mu_\parallel := \boldsymbol{V}_{|_{z=\varepsilon\zeta}} + \varepsilon \boldsymbol{w}_{|_{z=\varepsilon\zeta}} \nabla\zeta$ can be written under the form

$$\boldsymbol{U}^\mu_\parallel = \nabla\psi + \nabla^\perp \Delta^{-1}(\boldsymbol{\omega}_{\mu_{|_{z=\varepsilon\zeta}}} \cdot N^\mu), \tag{2.18}$$

for some scalar function ψ defined over $\mathbb{R}^2$. This function fully determines the 'irrotational' part of $\boldsymbol{U}^\mu$, which is given by $\nabla^\mu \Phi$, with

$$\left.\begin{array}{l} (\partial_z^2 + \mu\Delta)\Phi = 0 \quad \text{in } \Omega \\ \Phi_{|_{surf}} = \psi, \quad \partial_z \Phi_{|_{bott}} = 0 \end{array}\right\} \tag{2.19}$$

(and which obviously leaves (2.17) unchanged). We refer to Castro & Lannes (2014) for more details; it is, in particular, shown in this reference that ζ, ψ and $\boldsymbol{\omega}_\mu$ remain uniformly bounded with respect to μ during the time evolution of the flow. The analysis of the full boundary value problem then shows that $\boldsymbol{V}$, $\boldsymbol{w}$, $\overline{V}$ etc. also remain bounded, which justifies the formal asymptotics made here.

We show here how to construct an approximate solution to this system of equations. Replacing

$$\boldsymbol{U}^\mu = \begin{pmatrix} \sqrt{\mu}\boldsymbol{V} \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} \sqrt{\mu}\overline{V} + \mu V^* \\ \mu\tilde{\boldsymbol{w}} \end{pmatrix} \tag{2.20}$$

in (2.17) and using the fact that $\partial_z \overline{V} = 0$, we obtain

$$\left.\begin{array}{ll} \partial_z V^* - \sqrt{\mu}\nabla\tilde{\boldsymbol{w}} = -\boldsymbol{\omega}^\perp_{\mu,h} & \text{in } \Omega \\ \nabla^\perp \cdot \overline{V} + \sqrt{\mu}\nabla^\perp \cdot V^* = \boldsymbol{\omega}_{\mu,v} & \text{in } \Omega \\ \nabla \cdot \overline{V} + \sqrt{\mu}\nabla \cdot V^* + \partial_z \tilde{\boldsymbol{w}} = 0 & \text{in } \Omega \\ \tilde{\boldsymbol{w}}_b - \beta\nabla b \cdot (\overline{V} + \sqrt{\mu} V^*_b) = 0 & \text{at the bottom,} \end{array}\right\} \tag{2.21}$$

where $\boldsymbol{\omega}_{\mu,h}$ and $\boldsymbol{\omega}_{\mu,v}$ denote respectively the horizontal and vertical components of the vorticity $\boldsymbol{\omega}_\mu$. From the third and last equations of (2.21), we first get an expression for $\tilde{\boldsymbol{w}}$ in terms of $\overline{V}$ and V^*,

$$\tilde{\boldsymbol{w}} = -\nabla \cdot \left[(1 + z - \beta b)\overline{V}\right] - \sqrt{\mu}\nabla \cdot \int_{-1+\beta b}^{z} V^*. \tag{2.22}$$

Replacing this expression in the first equation of (2.21) then gives

$$\partial_z V^* = \sqrt{\mu}\nabla\tilde{\boldsymbol{w}} - \boldsymbol{\omega}^\perp_{\mu,h} \tag{2.23}$$

and therefore

$$V^* = \sqrt{\mu}\left(\int_z^{\varepsilon\zeta} \nabla\nabla\cdot[(1+z'-\beta b)\overline{V}]\mathrm{d}z'\right)^* + \mu\left(\int_z^{\varepsilon\zeta} \nabla\nabla\cdot\int_{-1+\beta b}^{z} V^*\right)^* + \left(\int_z^{\varepsilon\zeta} \boldsymbol{\omega}_{\mu,h}^{\perp}\right)^*. \tag{2.24}$$

Defining the operators $T = T[\beta b, \varepsilon\zeta]$ and $T^* = T^*[\beta b, \varepsilon\zeta]$ by

$$T[\beta b, \varepsilon\zeta]W = \int_z^{\varepsilon\zeta} \nabla\nabla\cdot\int_{-1+\beta b}^{z'} W \quad \text{and} \quad T^*[\beta b, \varepsilon\zeta]W = (T[\beta b, \varepsilon\zeta]W)^*, \tag{2.25a,b}$$

and the 'shear' velocity V_{sh} by

$$V_{sh} = \int_z^{\varepsilon\zeta} \boldsymbol{\omega}_{\mu,h}^{\perp}, \tag{2.26}$$

we can rewrite the above identity under the form

$$(1-\mu T^*)V^* = \sqrt{\mu}T^*\overline{V} + V_{sh}^*. \tag{2.27}$$

It should be noted that this is an exact identity. Applying $(1+\mu T^*)$ on both sides, we obtain the following approximation of order $O(\mu^{3/2})$:

$$V^* = \sqrt{\mu}T^*\overline{V} + (1+\mu T^*)V_{sh}^* + O(\mu^{3/2}). \tag{2.28}$$

Together with (2.14), this yields the following $O(\mu)$ and $O(\mu^2)$ approximations for the horizontal velocity field $\boldsymbol{V}$:

$$\boldsymbol{V} = \overline{V} + \sqrt{\mu}V_{sh}^* + O(\mu), \tag{2.29}$$

$$\boldsymbol{V} = \overline{V} + \sqrt{\mu}V_{sh}^* + \mu T^*\overline{V} + \mu^{3/2}T^*V_{sh}^* + O(\mu^2). \tag{2.30}$$

Remark 4. Remark that even at order $O(\mu)$ the flow cannot be assumed to be columnar (i.e. that its horizontal velocity is independent of z), which is in sharp contrast to the rotational case.

Remark 5. Since $\overline{V}$ does not depend on z, one can compute explicitly

$$T^*\overline{V} = -\frac{1}{2}\left((z+1-\beta b)^2 - \frac{h^2}{3}\right)\nabla\nabla\cdot\overline{V} + \beta\left(z-\varepsilon\zeta+\frac{1}{2}h\right)[\nabla b\cdot\nabla\overline{V} + \nabla(\nabla b\cdot\overline{V})]. \tag{2.31}$$

Finally, for the vertical component $\boldsymbol{w}$ of the velocity, we use (2.28) together with (2.20) and (2.22) to obtain

$$\boldsymbol{w} = O(\mu), \tag{2.32}$$

$$\boldsymbol{w} = -\mu\nabla\cdot[(1+z-\beta b)\overline{V}] - \mu^{3/2}\nabla\cdot\int_{-1+\beta b}^{z} V_{sh}^* + O(\mu^2). \tag{2.33}$$

2.3. *An equation for the shear velocity*

The purpose of this section is to derive an approximate equation solved by the shear velocity V_{sh}^* defined in (2.26). We recall that the dimensionless vorticity equation is given by

$$\partial_t \boldsymbol{\omega}_\mu + \frac{\varepsilon}{\mu}\boldsymbol{U}^\mu \cdot \nabla^\mu \boldsymbol{\omega}_\mu = \frac{\varepsilon}{\mu}\boldsymbol{\omega}_\mu \cdot \nabla^\mu \boldsymbol{U}^\mu, \tag{2.34}$$

so that its horizontal component $\boldsymbol{\omega}_{\mu,h}$ solves

$$\partial_t \boldsymbol{\omega}_{\mu,h} + \varepsilon \boldsymbol{V} \cdot \nabla \boldsymbol{\omega}_{\mu,h} + \frac{\varepsilon}{\mu} \boldsymbol{w} \partial_z \boldsymbol{\omega}_{\mu,h} = \varepsilon \boldsymbol{\omega}_{\mu,h} \cdot \nabla \ \boldsymbol{V} + \frac{\varepsilon}{\sqrt{\mu}} \boldsymbol{\omega}_{\mu,v} \partial_z \boldsymbol{V}. \tag{2.35}$$

Using (2.29) and (2.33), this yields

$$\begin{aligned} &\partial_t \boldsymbol{\omega}_{\mu,h} + \varepsilon \overline{V} \cdot \nabla \boldsymbol{\omega}_{\mu,h} - \varepsilon \nabla \cdot [(1 + z - \beta b)\overline{V}]\partial_z \boldsymbol{\omega}_{\mu,h} \\ &\qquad = \varepsilon \boldsymbol{\omega}_{\mu,h} \cdot \nabla \overline{V} - \varepsilon (\nabla^\perp \cdot \overline{V}) \boldsymbol{\omega}_{\mu,h}^\perp + O(\varepsilon\sqrt{\mu}) \end{aligned} \tag{2.36}$$

(as explained in Remark 2, we keep track of the dependence on ε, but no smallness assumption is made on this parameter, and one can set $\varepsilon = 1$ everywhere). Recalling that $V_{sh} = \int_z^{\varepsilon\zeta} \boldsymbol{\omega}_{\mu,h}^\perp$, we can integrate this equation with respect to z to obtain

$$\begin{aligned} &\partial_t V_{sh} + \varepsilon \overline{V} \cdot \nabla V_{sh} + \varepsilon (\nabla \cdot \overline{V}) V_{sh} - (\partial_t \zeta + \varepsilon \overline{V} \cdot \nabla \zeta) \boldsymbol{\omega}^\perp_{\mu,h_{|surf}} \varepsilon [\nabla \cdot [(1 + z - \beta b)\overline{V}] \boldsymbol{\omega}_{\mu,h}^\perp]_z^{\varepsilon\zeta} \\ &= -\varepsilon V_{sh}^\perp \cdot \nabla \overline{V}^\perp - \varepsilon (\nabla^\perp \cdot \overline{V}) V_{sh}^\perp + O(\varepsilon\sqrt{\mu}). \end{aligned} \tag{2.37}$$

Using the fact that $\partial_t \zeta + \nabla \cdot (h\overline{V}) = 0$ we therefore obtain

$$\begin{aligned} &\partial_t V_{sh} + \varepsilon \overline{V} \cdot \nabla V_{sh} + \varepsilon (\nabla \cdot \overline{V}) V_{sh} + \varepsilon \nabla \cdot [(1 + z - \beta b)\overline{V}] \boldsymbol{\omega}_{\mu,h}^\perp \\ &\qquad = -\varepsilon V_{sh}^\perp \cdot \nabla \overline{V}^\perp - \varepsilon (\nabla^\perp \cdot \overline{V}) V_{sh}^\perp + O(\varepsilon\sqrt{\mu}). \end{aligned} \tag{2.38}$$

Recalling the vectorial identity

$$(\nabla \cdot A)B + B^\perp \cdot \nabla A^\perp + (\nabla^\perp \cdot A)B^\perp = B \cdot \nabla A, \tag{2.39}$$

this equation can be rewritten as

$$\partial_t V_{sh} + \varepsilon \overline{V} \cdot \nabla V_{sh} + \varepsilon V_{sh} \cdot \nabla \overline{V} - \varepsilon \nabla \cdot [(1 + z - \beta b)\overline{V}] \partial_z V_{sh} = O(\varepsilon\sqrt{\mu}). \tag{2.40}$$

Integrating this equation, one readily obtains the following equation for the average shear velocity $\overline{V}_{sh}$:

$$\partial_t \overline{V}_{sh} + \varepsilon \overline{V} \cdot \nabla \overline{V}_{sh} + \varepsilon \overline{V}_{sh} \cdot \nabla \overline{V} = O(\varepsilon\sqrt{\mu}). \tag{2.41}$$

Subtracting this equation from (2.40), we finally obtain the following evolution equation on V_{sh}^*:

$$\partial_t V_{sh}^* + \varepsilon \overline{V} \cdot \nabla V_{sh}^* + \varepsilon V_{sh}^* \cdot \nabla \overline{V} - \varepsilon \nabla \cdot [(1 + z - \beta b)\overline{V}] \partial_z V_{sh}^* = O(\varepsilon\sqrt{\mu}). \tag{2.42}$$

It should be noted that we will derive later a more precise evolution equation for (2.42) by making explicit the $O(\varepsilon\sqrt{\mu})$ terms (see § 4.5.1 below).

2.4. *Asymptotic expansion of the pressure field*

The vertical component of the Euler equation (2.7) is given by

$$\partial_t \boldsymbol{w} + \varepsilon \boldsymbol{V} \cdot \nabla \boldsymbol{w} + \frac{\varepsilon}{\mu} \boldsymbol{w} \partial_z \boldsymbol{w} = -\frac{1}{\varepsilon}(\partial_z P + 1). \tag{2.43}$$

Since, moreover, P vanishes at the surface, we obtain that

$$\begin{aligned} \frac{1}{\varepsilon} \nabla P &= \nabla \int_z^{\varepsilon\zeta} \left(-\frac{1}{\varepsilon} \partial_z P \right) \\ &= \nabla \zeta + \nabla \int_z^{\varepsilon\zeta} \left(\partial_t \boldsymbol{w} + \varepsilon \boldsymbol{V} \cdot \nabla \boldsymbol{w} + \frac{\varepsilon}{\mu} \boldsymbol{w} \partial_z \boldsymbol{w} \right) \end{aligned} \tag{2.44}$$

and therefore

$$\frac{1}{\varepsilon} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla P = h \nabla \zeta + \int_{-1+\beta b}^{\varepsilon\zeta} \nabla \int_z^{\varepsilon\zeta} \left(\partial_t \boldsymbol{w} + \varepsilon \boldsymbol{V} \cdot \nabla \boldsymbol{w} + \frac{\varepsilon}{\mu} \boldsymbol{w} \partial_z \boldsymbol{w} \right). \tag{2.45}$$

The first term on the right-hand side corresponds to the hydrostatic pressure. We still need an expansion of the non-hydrostatic terms with respect to μ; for the sake of clarity, this computation is performed in § 3.2 for the one-dimensional case with constant vorticity, in § 4.2 for the one-dimensional case with general vorticity and in § 5.2 for the two-dimensional case.

Remark 6. A direct consequence of (2.45) and (2.32) is that

$$\frac{1}{\varepsilon} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla P = h \nabla \zeta + O(\mu). \tag{2.46}$$

An order $O(\mu)$ approximation of (2.15) is therefore provided by the Saint-Venant (or nonlinear shallow water) equations

$$\left. \begin{aligned} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t (h\overline{V}) + h \nabla \zeta + \varepsilon \nabla \cdot (h\overline{V} \otimes \overline{V}) = 0; \end{aligned} \right\} \tag{2.47}$$

these equations are exactly the same as in the irrotational setting. As we shall see in the next sections, the rotational terms affect the $O(\mu)$ terms in (2.15) and, consequently, the Green–Naghdi equations differ from the standard irrotational version when vorticity is present.

3. The 1*d* Green–Naghdi equations with constant vorticity

We recall that throughout this paper, no smallness assumption is made on ε, so that it is possible to set $\varepsilon = 1$.

In dimension $d = 1$, one can consider flows with constant vorticity,

$$\boldsymbol{U}^\mu = \begin{pmatrix} \sqrt{\mu} v \\ 0 \\ \boldsymbol{w} \end{pmatrix}, \quad \boldsymbol{\omega}_\mu = \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} \tag{3.1a,b}$$

(so that $-\boldsymbol{\omega}_{\mu,h}^{\perp} = (\omega, 0)^{\mathrm{T}}$), and with $\omega = \partial_z v^* - \sqrt{\mu}\partial_x \tilde{\boldsymbol{w}} \equiv$ const. The fact that the vorticity remains constant implies that the wave does not affect the underlying current; however, as we show in this section, the current modifies the motion of the waves.

Since ω is constant, one deduces from (2.26) that $V_{sh}^* = (v_{sh}^*, 0)^{\mathrm{T}}$ with

$$v_{sh}^* = -\left((\varepsilon\zeta - z) - \tfrac{1}{2}h\right)\omega \tag{3.2}$$

and, writing $\overline{V} = (\overline{v}, 0)^{\mathrm{T}}$, (2.33) therefore takes the form

$$\boldsymbol{w} = -\mu\partial_x\big((1 + z - \beta b)\overline{v}\big) + \mu^{3/2}\tfrac{1}{2}\omega\partial_x\big((\varepsilon\zeta - z)(z + 1 - \beta b)\big). \tag{3.3}$$

3.1. *Computation of the 'rotational Reynolds tensor' contribution*

Using (3.2) and (3.3), the contribution of the 'rotational Reynolds tensor' to (2.15) can be written (dropping $O(\mu^2)$ terms) as

$$\begin{aligned}
\varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v^*|^2 &= \varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v_{sh}^*|^2 + 2\varepsilon\mu^{3/2}\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} v_{sh}^* T\overline{v} \\
&= \varepsilon\mu\omega^2\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} \left((\varepsilon\zeta - z) - \frac{1}{2}h\right)^2 \\
&\quad - 2\varepsilon\mu^{3/2}\omega\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} \left((\varepsilon\zeta - z) - \frac{1}{2}h\right) \int_z^{\varepsilon\zeta} \partial_x^2\big((1 + z' - \beta b)\overline{V}\big) \\
&= \frac{\varepsilon\mu}{12}\omega^2\partial_x(h^3) - \frac{\varepsilon\mu^{3/2}}{12}\omega\partial_x\left[h^3\left(h\partial_x^2\overline{v} - 2\beta\overline{v}\partial_x^2 b - 4\beta\partial_x\overline{v}\partial_x b\right)\right].
\end{aligned} \tag{3.4}$$

It should be noted that this expression only depends on ζ and $\overline{v}$ (and on the constant vorticity ω and the bottom parametrization b).

3.2. *Computation of the pressure contribution*

Similarly, we can use (3.2) and (3.3) to write the pressure contribution to (2.45) under the form (dropping again the $O(\mu^2)$ terms)

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x P &= h\partial_x\zeta + \int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} \left(\partial_t \boldsymbol{w} + \varepsilon v\partial_x \boldsymbol{w} + \frac{\varepsilon}{\mu}\,\boldsymbol{w}\partial_z \boldsymbol{w}\right) \\
&= h\partial_x\zeta + \mu h\mathscr{T}\left(\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\right) + \mu\varepsilon h\mathscr{Q}_1(\overline{v}) - \varepsilon\mu^{3/2}\frac{1}{2}\omega h\mathscr{T}(\beta\overline{v}\partial_x b - h\partial_x v) \\
&\quad - \varepsilon\mu^{3/2}\frac{1}{12}\omega\partial_x(h^4\partial_x^2\overline{v} + 6h^3\partial_x\zeta\,\partial_x\overline{v}) + \beta\mu^{3/2}\frac{1}{12} \\
&\quad \times [\omega h^2\partial_x\overline{v}\left(h\partial_x^2 b - 6\partial_x\zeta\,\partial_x b\right) + 3\beta h\overline{v}\partial_x b(h\partial_x^2 b + 2\partial_x b\partial_x\zeta)],
\end{aligned} \tag{3.5}$$

where $\mathscr{T}$ and $\mathscr{Q}_1$ are defined in (1.16) and (1.17), and where we used the fact that $\partial_t\zeta + \overline{v}\partial_x\zeta = \beta\overline{v}\partial_x b - h\partial_x v$. As for the Reynolds tensor contribution (3.4), this expression only depends on the variables ζ and $\overline{V}$.

Remark 7. We actually work here with the dimensionless form of the operators $\mathscr{T}$ and $\mathscr{Q}_1$, formally obtained by replacing ζ by $\varepsilon\zeta$, b by βb and with $h = 1 + \varepsilon\zeta - \beta b$. In dimension $d = 1$, this gives

$$\mathscr{T} v = -\frac{1}{3h}\partial_x(h^3\partial_x v) + \beta\frac{1}{2h}\left[\partial_x(h^2\partial_x b v) - h^2\partial_x b\partial_x v\right] + \beta^2(\partial_x b)^2 v, \tag{3.6}$$

$$\mathscr{Q}_1(v) = \frac{2}{3h}\partial_x\left(h^3(\partial_x v)^2\right) + \beta h(\partial_x v)^2\partial_x b + \frac{\beta}{2h}\partial_x\left(h^2 v^2\partial_x^2 b\right) + \beta^2 v^2\partial_x b\partial_x^2 b. \tag{3.7}$$

3.3. *The Green–Naghdi model*

Gathering (3.4) and (3.5), we obtain

$$\begin{aligned}
&\mu\varepsilon\partial_x\int_{-1}^{\varepsilon\zeta}|v^*|^2 + \frac{1}{\varepsilon}\int_{-1}^{\varepsilon\zeta}\partial_x P \\
&\quad = h\partial_x\zeta + \mu h\mathscr{T}(\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}) + \varepsilon\mu h\mathscr{Q}_1(\overline{v}) + \varepsilon\mu\frac{1}{12}\omega^2\partial_x(h^3) - \varepsilon\mu^{3/2}\frac{1}{6} \\
&\qquad \times\partial_x\left(2\omega h^4\partial_x^2\overline{v} + \partial_x(\omega h^4)\partial_x\overline{v}\right) + \varepsilon\beta\mu^{3/2}\frac{\omega}{3}\left[\partial_x\left(h^3\partial_x^2 b\overline{v}\right) + h^3\partial_x^2 b\partial_x\overline{v}\right]
\end{aligned} \tag{3.8}$$

(up to $O(\mu^2)$ terms); plugging this approximation into (2.15), we obtain the Green–Naghdi equations in dimension $d=1$ and with constant vorticity,

$$\left.\begin{aligned}
&\partial_t\zeta + \partial_x(h\overline{v}) = 0, \\
&(1+\mu\mathscr{T})\left(\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\right) + \partial_x\zeta + \varepsilon\mu\mathscr{Q}_1(\overline{v}) \\
&\quad + \varepsilon\mu\frac{1}{h}\partial_x\left(\frac{1}{12}h^3\omega^2\right) + \varepsilon\mu^{3/2}\mathscr{C}(\omega h, \overline{v}) + \varepsilon\beta\mu^{3/2}\mathscr{C}_b(\omega h, \overline{v}) = 0,
\end{aligned}\right\} \tag{3.9}$$

where $\mathscr{T}$ and $\mathscr{Q}_1$ are as in (1.16) and (1.17), and with

$$\forall v^\sharp, \forall\overline{v}, \quad \begin{aligned}
\mathscr{C}(v^\sharp, \overline{v}) &= -\frac{1}{6h}\partial_x\left(2h^3 v^\sharp\partial_x^2\overline{v} + \partial_x(h^3 v^\sharp)\partial_x\overline{v}\right), \\
\mathscr{C}_b(v^\sharp, \overline{v}) &= \frac{1}{3h}\left[\partial_x\left(h^2 v^\sharp\partial_x^2 b\overline{v}\right) + h^2 v^\sharp\partial_x^2 b\partial_x\overline{v}\right].
\end{aligned} \tag{3.10}$$

Remark 8. One can notice that

$$h\mathscr{C}(v^\sharp, \overline{v})\cdot\overline{v} = \partial_x\mathfrak{F}_{\mathscr{C}} \quad \text{and} \quad h\mathscr{C}_b(v^\sharp, \overline{v})\cdot\overline{v} = \partial_x\mathfrak{F}_{\mathscr{C}_b}, \tag{3.11a,b}$$

where the fluxes $\mathfrak{F}_{\mathscr{C}}$ and $\mathfrak{F}_{\mathscr{C}_b}$ are given by

$$\mathfrak{F}_{\mathscr{C}} = \frac{h^3}{6}\left((\partial_x\overline{v})^2 - \overline{v}\partial_x^2 v\right)v^\sharp - \frac{1}{6}\partial_x(h^3 v^\sharp\partial_x v)\overline{v} \quad \text{and} \quad \mathfrak{F}_{\mathscr{C}_b} = h^2 v^\sharp\partial_x^2 b\overline{v}^2. \tag{3.12a,b}$$

Remark 9. Remark 8 can be used to derive a local equation for the conservation of energy; one finds indeed that (1.19) can be generalized in the presence of a constant vorticity into

$$\partial_t\left(\mathfrak{e} + \mathfrak{e}_{rot}\right) + \partial_x\left(\mathfrak{F} + \mathfrak{F}_{rot}\right) = 0, \tag{3.13}$$

with (in dimensional form)

$$\mathfrak{e}_{rot} = \tfrac{1}{24}\omega^2 h^3 \quad \text{and} \quad \mathfrak{F}_{rot} = \tfrac{1}{8}\omega h^3 + \mathfrak{F}_{\mathscr{C}} + \mathfrak{F}_{\mathscr{C}_b}. \tag{3.14a,b}$$

4. The $1d$ Green–Naghdi equations with general vorticity

We recall that throughout this paper, no smallness assumption is made on ε, so that it is possible to set $\varepsilon = 1$.

In dimension $d = 1$ with a non-constant vorticity, we still use the notations (3.1), but $\omega = \omega(t, x, z)$ now depends on space and time. Contrary to what happens in the constant vorticity case studied in § 3, we show here that there is a non-trivial wave–current interaction in the sense that the underlying current is now affected by the motion of the waves.

Still denoting $V^*_{sh} = (v^*_{sh}, 0)^{\mathrm{T}}$, one deduces from (2.26) that

$$v^*_{sh} = -\left(\int_z^{\varepsilon\zeta} \omega\right)^*. \tag{4.1}$$

As in § 3 we compute the contribution to (2.15) of the 'rotational Reynolds tensor' and of the pressure; the difference from the case of constant vorticity addressed in § 3 is that the component $\boldsymbol{E}$ of the 'Reynolds' tensor cannot be computed explicitly; we show that it can be determined through the resolution of a finite cascade of equations.

In order to simplify the computations, we shall make from now on the following assumption on the amplitude of the bottom variations:

$$\text{Medium amplitude bottom variations: } \beta = O(\sqrt{\mu}). \tag{4.2}$$

This assumption leads to simpler models without neglecting any essential mechanisms of wave–current interaction; for instance, in the particular case of constant vorticity, this assumption allows one to neglect the term $\varepsilon\beta\mu^{3/2}\mathscr{C}_b(\omega h, \overline{v})$ in (3.9).

4.1. *Computation of the 'rotational Reynolds tensor' contribution*

Proceeding as for (3.4), we write (dropping $O(\mu^2)$ terms)

$$\begin{aligned}
\varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v^*|^2 &= \varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v^*_{sh}|^2 + 2\varepsilon\mu^{3/2}\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} v^*_{sh} T\overline{v} \\
&= \varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v^*_{sh}|^2 + 2\varepsilon\mu^{3/2}\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} v^*_{sh} \int_z^{\varepsilon\zeta} \partial_x^2((1+z'-\beta b)\overline{v}) \\
&= \varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v^*_{sh}|^2 - \varepsilon\mu^{3/2}\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} v^*_{sh}\partial_x^2((1+z'-\beta b)\overline{v}),
\end{aligned} \tag{4.3}$$

the last line stemming from the identity

$$2\int_{-1+\beta b}^{\zeta} W^* \int_z^{\varepsilon\zeta} \partial_x^2\big((1+z'-\beta b)\overline{W}\big) = -\int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} W^*\partial_x^2\big((1+z'-\beta b)\overline{W}\big). \tag{4.4}$$

Introducing

$$v^\sharp := -\frac{24}{h^3} \int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} \int_{-1+\beta b}^{z} v^*_{sh}, \tag{4.5}$$

one computes that

$$\int_{-1+\beta b}^{\varepsilon\zeta} \int_z^{\varepsilon\zeta} v^*_{sh}\partial_x^2\big((1+z'-\beta b)\overline{v}\big) = \frac{h^3}{12} v^\sharp \partial_x^2 \overline{v} + O(\beta), \tag{4.6}$$

so that under the assumption (4.2), one readily deduces that

$$\varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v^*|^2 = \varepsilon\mu\partial_x \int_{-1+\beta b}^{\varepsilon\zeta} |v_{sh}^*|^2 - \varepsilon\mu^{3/2}\frac{1}{12}\partial_x(h^3 v^\sharp \partial_x^2 \overline{v}) + O(\mu^2). \tag{4.7}$$

4.2. *Computation of the pressure contribution*

As for (3.5), we write

$$\begin{aligned}\frac{1}{\varepsilon}\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x P &= h\partial_x\zeta + \int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} \left(\partial_t \boldsymbol{w} + \varepsilon v\partial_x \boldsymbol{w} + \frac{\varepsilon}{\mu}\ \boldsymbol{w}\partial_z \boldsymbol{w}\right)\\ &= h\partial_x\zeta + \mu A_{(1)} + \mu^{3/2}A_{(3/2)} + O(\mu^2),\end{aligned} \tag{4.8}$$

with

$$A_{(1)} = h\mathscr{T}\left(\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\right) + \varepsilon h\mathscr{Q}_1(\overline{v}). \tag{4.9}$$

For $A_{(3/2)}$, it is convenient to introduce the operator $\widetilde{\mathscr{T}} = \widetilde{\mathscr{T}}[\varepsilon\zeta, \beta b]$ defined as

$$\widetilde{\mathscr{T}}[\varepsilon\zeta, \beta b]W = -\frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla \int_z^{\varepsilon\zeta} \nabla \cdot \int_{-1+\beta b}^{z} W; \tag{4.10}$$

in particular, if $W = \overline{W}$ does not depend on z, then $\widetilde{T}\overline{W} = \mathscr{T}\overline{W}$.

We then have, using the one-dimensional version of the above operator, namely, $\widetilde{\mathscr{T}}W = -1/h\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} \partial_x \int_{-1+\beta b}^{z} W$, that

$$\begin{aligned}A_{(3/2)} &= h\widetilde{\mathscr{T}}\partial_t v_{sh}^*\\ &\quad + \varepsilon\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} \left(\overline{v}\partial_x\left(-\partial_x\int_{-1+\beta b}^{z} v_{sh}^*\right) + v_{sh}^*\partial_x(-\partial_x((1+z'-\beta b)\overline{v}))\right)\\ &\quad + \varepsilon\int_{-1+\beta b}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} \left(-\partial_x((1+z'-\beta b)\overline{v})(-\partial_x v_{sh}^*) - \int_{-1+\beta b}^{z} \partial_x v_{sh}^*(-\partial_x\overline{v})\right),\end{aligned} \tag{4.11}$$

up to $O(\sqrt{\mu})$ terms. We can notice that the contribution of the bottom is of order $O(\beta)$ in $A_{(3/2)}$ and therefore of order $O(\mu^{3/2}\beta)$ in (4.8). For bottoms of medium amplitude as in (4.2), this contribution is of order $O(\mu^2)$ and can therefore be neglected at the precision of the model. We therefore take $b = 0$ in the following computations, which allows us to write (dropping $O(\sqrt{\mu}, \beta)$ terms)

$$\begin{aligned}A_{(3/2)} &= h\widetilde{\mathscr{T}}(\partial_t v_{sh}^* + \varepsilon\overline{v}\partial_x v_{sh}^*) + 2\varepsilon\int_{-1}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} (\partial_x\overline{v})\partial_x \int_{-1}^{z'} v_{sh}^*\\ &\quad - \varepsilon\int_{-1}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} v_{sh}^*\partial_x^2((1+z')\overline{v}) + \varepsilon\int_{-1}^{\varepsilon\zeta} \partial_x \int_z^{\varepsilon\zeta} (\partial_x v_{sh}^*)\partial_x((1+z')\overline{v}).\end{aligned} \tag{4.12}$$

In order to get a simpler expression for $B_{(3/2)}$, let us apply $\widetilde{\mathscr{T}}$ to (2.42) to obtain, up to $O(\sqrt{\mu})$ terms,

$$
\begin{aligned}
h\widetilde{\mathscr{T}}\big(\partial_t v_{sh}^* + \varepsilon\overline{v}\partial_x v_{sh}^* + \varepsilon v_{sh}^*\partial_x\overline{v}\big) &= -\varepsilon\int_{-1}^{\zeta}\partial_x\int_z^{\varepsilon\zeta}\partial_x\int_{-1}^{z'}\partial_x[(1+z)\overline{v}]\partial_z v_{sh}^* \\
&= \varepsilon\int_{-1}^{\zeta}\partial_x\int_z^{\varepsilon\zeta}\partial_x\int_{-1}^{z'}\partial_x\overline{v}v_{sh}^* \\
&\quad -\varepsilon\int_{-1}^{\zeta}\partial_x\int_z^{\varepsilon\zeta}\partial_x[\partial_x((1+z')\overline{v})v_{sh}^*],
\end{aligned} \tag{4.13}
$$

and therefore

$$
\begin{aligned}
h\widetilde{\mathscr{T}}\big(\partial_t v_{sh}^* + \varepsilon\overline{v}\partial_x v_{sh}^*\big) &= -2h\widetilde{\mathscr{T}}(v_{sh}^*\partial_x\overline{v}) - \varepsilon\int_{-1}^{\zeta}\partial_x\int_z^{\varepsilon\zeta}v_{sh}^*\partial_x^2((1+z')\overline{v}) \\
&\quad -\varepsilon\int_{-1}^{\zeta}\partial_x\int_z^{\varepsilon\zeta}\partial_x((1+z')\overline{v})\partial_x v_{sh}^*.
\end{aligned} \tag{4.14}
$$

Plugging this identity into the above expression for $A_{(3/2)}$, we obtain

$$
\begin{aligned}
A_{(3/2)} &= -2h\widetilde{\mathscr{T}}(v_{sh}^*\partial_x\overline{v}) + 2\varepsilon\int_{-1}^{\varepsilon\zeta}\partial_x\int_z^{\varepsilon\zeta}(\partial_x\overline{v})\partial_x\int_{-1}^{z'}v_{sh}^* \\
&\quad -2\varepsilon\int_{-1}^{\varepsilon\zeta}\partial_x\int_z^{\varepsilon\zeta}v_{sh}^*\partial_x^2((1+z')\overline{v}).
\end{aligned} \tag{4.15}
$$

Using (4.6) together with the identities

$$
-8h\widetilde{\mathscr{T}}(v_{sh}^*\partial_x\overline{v}) = -\tfrac{1}{3}\partial_x(h^3\partial_x(v^\sharp\partial_x\overline{v})) - \partial_x(h^2\partial_x h v^\sharp\partial_x\overline{v}) + O(\beta), \tag{4.16}
$$

$$
-4\partial_x\int_{-1}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\int_{-1}^{z'}\partial_x\overline{v}\partial_x v_{sh}^* = \partial_x\left(\left(\frac{h^2\partial_x h}{2}v^\sharp + \frac{h^3}{6}\partial_x v^\sharp\right)\partial_x\overline{v}\right) + O(\beta), \tag{4.17}
$$

we finally obtain

$$
A_{(3/2)} = -\tfrac{1}{4}\partial_x\big(h^3 v^\sharp\partial_x^2\overline{v}\big) - \tfrac{1}{6}\partial_x\big(\partial_x(h^3 v^\sharp)\partial_x\overline{v}\big). \tag{4.18}
$$

4.3. *Wave–current interaction in the velocity equation*

Due to (4.7) and (4.8)–(4.12), we deduce from the momentum equation in (2.15) that

$$
(1+\mu\mathscr{T})\big(\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\big) + \partial_x\zeta + \varepsilon\mu\mathscr{Q}_1(\overline{v}) + \varepsilon\mu\frac{1}{h}\partial_x\int_{-1+\beta b}^{\varepsilon\zeta}|v_{sh}^*|^2 + \mu^{3/2}\frac{1}{h}B_{(3/2)} = O(\mu^2), \tag{4.19}
$$

with

$$
B_{(3/2)} = h\widetilde{\mathscr{T}}\big(\partial_t v_{sh}^* + \varepsilon\overline{v}\partial_x v_{sh}^*\big) - 2\varepsilon\int_{-1}^{\varepsilon\zeta}\partial_x\int_z^{\varepsilon\zeta}v_{sh}^*\partial_x^2((1+z')\overline{v}) - \partial_x\big(h^2\partial_x\overline{v}\partial_x h v_{sh|_{z=\varepsilon\zeta}}^*\big). \tag{4.20}
$$

To sum the above computations up, we have obtained the following evolution equations for $\overline{v}$ (dropping $O(\mu^2)$ terms):

$$
(1+\mu\mathscr{T})\big(\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\big) + \partial_x\zeta + \varepsilon\mu\mathscr{Q}_1(\overline{v}) + \varepsilon\mu\frac{1}{h}\partial_x\boldsymbol{E} + \varepsilon\mu^{3/2}\mathscr{C}(v^\sharp,\overline{v}) = 0, \tag{4.21}
$$

with

$$\boldsymbol{E}=\int_{-1+\beta b}^{\varepsilon\zeta}|v_{sh}^{*}|^{2} \quad \text{and} \quad v^{\sharp}=-\frac{24}{h^{3}}\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta}\int_{-1+\beta b}^{z}v_{sh}^{*}, \tag{4.22a,b}$$

and $\mathscr{C}(v^{\sharp},\overline{v})$ as defined in (3.10).

When the vorticity is constant, one readily computes from (3.2) that $\boldsymbol{E}=(h^{3}/12)\omega^{2}$ and $v^{\sharp}=\omega h$, so that (4.21) coincides, as expected, with (3.9) (provided that we drop in (3.9) the term $\varepsilon\beta\mu^{3/2}\mathscr{C}_{b}(\omega h,\overline{v})$, which is of size $O(\mu^{2})$ under the medium-amplitude bottom assumption (4.2) we assumed to derive (4.21)). In particular, (3.9) forms a closed system of equations in ζ and $\overline{V}$. This is no longer the case in the general (variable vorticity) case since (4.21) involves the quantities $\boldsymbol{E}$ and $v^{\sharp}$ which need to be determined using the vorticity equation. The presence of these terms renders stronger wave–current interactions than in the case of constant vorticity.

The necessary closure equations on $v^{\sharp}$ and $\boldsymbol{E}$ are derived in the following two subsections.

4.4. *Closure equation for* $v^{\sharp}$

For later use, we perform directly the computations in the two-dimensional case $d=2$ here, as the adaptations to the case $d=1$ are straightforward.

Since $v^{\sharp}$ appears only in the $O(\mu^{3/2})$ terms in (4.21), it is sufficient to derive an equation for $v^{\sharp}$ at precision $O(\sqrt{\mu})$ so that the overall $O(\mu^{2})$ precision of (4.21) is respected.

Such an equation is obtained by applying the triple integration operator $\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta}\int_{-1+\beta b}^{z}$ to (2.42). The resulting equation is

$$\partial_{t}(h^{3}V^{\sharp})+\varepsilon\overline{V}\cdot\nabla(h^{3}V^{\sharp})+\varepsilon h^{3}V^{\sharp}\cdot\nabla\overline{V}=-3\varepsilon h^{3}V^{\sharp}\nabla\cdot\overline{V}+O(\varepsilon\sqrt{\mu}). \tag{4.23}$$

Making use of the identity $\partial_{t}h+\varepsilon\nabla\cdot(h\overline{V})=0$, this yields

$$\partial_{t}V^{\sharp}+\varepsilon\overline{V}\cdot\nabla V^{\sharp}+\varepsilon V^{\sharp}\cdot\nabla\overline{V}=O(\varepsilon\sqrt{\mu}), \tag{4.24}$$

which is the desired closure equation.

4.5. *Closure equations for* $\boldsymbol{E}$

Since $\boldsymbol{E}$ appears in an order $O(\mu)$ term in (4.21), we need to derive an equation for $\boldsymbol{E}$ at precision $O(\mu)$ to preserve the overall $O(\mu^{2})$ precision of (4.21). The first step is to derive a more precise equation for the shear velocity V_{sh}^{*} that takes into account the $O(\sqrt{\mu})$ neglected in (2.42). An equation for $\boldsymbol{E}$ is then deduced from this equation.

4.5.1. *An equation for* V_{sh}^{*} *at order* $O(\mu)$

For later use, we perform directly the computations in the two-dimensional case $d=2$ here, as the adaptations to the case $d=1$ are straightforward.

A more precise version of (2.36) is obtained by making more explicit the $O(\varepsilon\sqrt{\mu})$ term on the right-hand side; more precisely, we substitute $O(\varepsilon\sqrt{\mu})$ in (2.36) by

$$\varepsilon\sqrt{\mu}\Big\{-V_{sh}^{*}\cdot\nabla\boldsymbol{\omega}_{\mu,h}+\Big(\nabla\cdot\int_{-1+\beta b}^{z}V_{sh}^{*}\Big)\partial_z\boldsymbol{\omega}_{\mu,h}+\boldsymbol{\omega}_{\mu,h}\cdot\nabla V_{sh}^{*}$$
$$+\nabla^{\perp}\cdot\overline{V}\partial_z T^{*}\overline{V}+\nabla^{\perp}\cdot V_{sh}^{*}\partial_z V_{sh}^{*}\Big\}+O(\varepsilon\mu). \tag{4.25}$$

Consequently, we include the $O(\sqrt{\mu})$ terms in (2.40) to obtain

$$\partial_t V_{sh}+\varepsilon\overline{V}\cdot\nabla V_{sh}+\varepsilon V_{sh}\cdot\nabla\overline{V}-\varepsilon\nabla\cdot[(1+z-\beta b)\overline{V}]\partial_z V_{sh}=\varepsilon\sqrt{\mu}C+O(\varepsilon\mu), \tag{4.26}$$

with

$$C\;=\;-\int_z^{\varepsilon\zeta}V_{sh}^{*}\cdot\nabla\boldsymbol{\omega}_{\mu,h}^{\perp}-\int_z^{\varepsilon\zeta}(\nabla\cdot V_{sh}^{*})\boldsymbol{\omega}_{\mu,h}^{\perp}+\Big[\Big(\nabla\cdot\int_{-1+\beta b}^{z}V_{sh}^{*}\Big)\boldsymbol{\omega}_{\mu,h}^{\perp}\Big]_z^{\varepsilon\zeta}$$
$$+\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_{\mu,h}\cdot\nabla(V_{sh}^{*})^{\perp}+\int_z^{\varepsilon\zeta}\nabla^{\perp}\cdot\overline{V}(\partial_z T^{*}\overline{V})^{\perp}+\int_z^{\varepsilon\zeta}\nabla^{\perp}\cdot V_{sh}^{*}\partial_z V_{sh}^{*\perp}. \tag{4.27}$$

Using (2.39) and since $\boldsymbol{\omega}_{\mu,h}^{\perp}=-\partial_z V_{sh}^{*}$, this yields

$$C\;=\;-\int_z^{\varepsilon\zeta}V_{sh}^{*}\cdot\nabla\boldsymbol{\omega}_{\mu,h}^{\perp}-\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_{\mu,h}^{\perp}\cdot\nabla V_{sh}^{*}$$
$$+\Big[\Big(\nabla\cdot\int_{-1+\beta b}^{z}V_{sh}^{*}\Big)\boldsymbol{\omega}_{\mu,h}^{\perp}\Big]_z^{\varepsilon\zeta}+\int_z^{\varepsilon\zeta}\nabla^{\perp}\cdot\overline{V}(\partial_z T^{*}\overline{V})^{\perp}. \tag{4.28}$$

Using again that $\boldsymbol{\omega}_{\mu,h}^{\perp}=-\partial_z V_{sh}^{*}$, we deduce that

$$C\;=\;\int_z^{\varepsilon\zeta}\partial_z\big(V_{sh}^{*}\cdot\nabla V_{sh}^{*}\big)+\Big[\Big(\nabla\cdot\int_{-1+\beta b}^{z}V_{sh}^{*}\Big)\boldsymbol{\omega}_{\mu,h}^{\perp}\Big]_z^{\varepsilon\zeta}+\int_z^{\varepsilon\zeta}\nabla^{\perp}\cdot\overline{V}(\partial_z T^{*}\overline{V})^{\perp}$$
$$=\;-V_{sh}^{*}\cdot\nabla V_{sh}^{*}+\overline{V}_{sh}\cdot\nabla\overline{V}_{sh}+\Big(\nabla\cdot\int_{-1+\beta b}^{z}V_{sh}^{*}\Big)\partial_z V_{sh}^{*}-\nabla^{\perp}\cdot\overline{V}(T\overline{V})^{\perp}. \tag{4.29}$$

Finally, we have, therefore, the following higher-order version of (2.40):

$$\partial_t V_{sh}+\varepsilon\overline{V}\cdot\nabla V_{sh}+\varepsilon V_{sh}\cdot\nabla\overline{V}+\varepsilon\sqrt{\mu}\Big(V_{sh}^{*}\cdot\nabla V_{sh}-V_{sh}\cdot\nabla\overline{V}_{sh}\Big)$$
$$=\varepsilon\nabla\cdot\Big[\int_{-1+\beta b}^{z}(\overline{V}+\sqrt{\mu}V_{sh}^{*})\Big]\partial_z V_{sh}^{*}-\varepsilon\sqrt{\mu}\nabla^{\perp}\cdot\overline{V}(T\overline{V})^{\perp}+O(\varepsilon\mu). \tag{4.30}$$

Integrating this yields the following equation for the shear velocity $\overline{V}_{sh}$:

$$\partial_t\overline{V}_{sh}+\varepsilon\overline{V}\cdot\nabla\overline{V}_{sh}+\varepsilon\overline{V}_{sh}\cdot\nabla\overline{V}+\varepsilon\sqrt{\mu}\frac{1}{h}\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta}V_{sh}^{*}\otimes V_{sh}^{*}-\varepsilon\sqrt{\mu}\overline{V}_{sh}\cdot\nabla\overline{V}_{sh}$$
$$=-\varepsilon\sqrt{\mu}\nabla^{\perp}\cdot\overline{V}(\overline{TV})^{\perp}+O(\varepsilon\mu). \tag{4.31}$$

Taking the difference of these two equations, and dropping the $O(\mu)$ terms, we obtain the following higher-order generalization of (2.42):

$$\partial_t V_{sh}^{*}+\varepsilon\overline{V}\cdot\nabla V_{sh}^{*}+\varepsilon V_{sh}^{*}\cdot\nabla\overline{V}+\varepsilon\sqrt{\mu}\left(V_{sh}^{*}\cdot\nabla V_{sh}^{*}-\frac{1}{h}\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta}V_{sh}^{*}\otimes V_{sh}^{*}\right)$$
$$=\varepsilon\nabla\cdot\Big[\int_{-1+\beta b}^{z}(\overline{V}+\sqrt{\mu}V_{sh}^{*})\Big]\partial_z V_{sh}^{*}-\varepsilon\sqrt{\mu}\nabla^{\perp}\cdot\overline{V}(T^{*}\overline{V})^{\perp}. \tag{4.32}$$

4.5.2. *An equation for* $\boldsymbol{E}$

In dimension $d=1$, (4.32) takes the form

$$\begin{aligned}\partial_t v_{sh}^* + \varepsilon\overline{v}\partial_x v_{sh}^* + \varepsilon v_{sh}^*\partial_x\overline{v} + \varepsilon\sqrt{\mu}\left(v_{sh}^*\partial_x v_{sh}^* - \frac{1}{h}\partial_x\int_{-1+\beta b}^{\varepsilon\zeta}|v_{sh}^*|^2\right)\\ = \varepsilon\partial_x\left[\int_{-1+\beta b}^{z}(\overline{v}+\sqrt{\mu}v_{sh}^*)\right]\partial_z v_{sh}^*.\end{aligned} \tag{4.33}$$

An equation for $\boldsymbol{E}$ is simply obtained by multiplying this equation by v_{sh}^* and integrating in z,

$$\partial_t\boldsymbol{E} + \varepsilon\overline{v}\partial_x\boldsymbol{E} + 3\varepsilon\boldsymbol{E}\partial_x\overline{v} + \varepsilon\sqrt{\mu}\partial_x\boldsymbol{F} = 0 \tag{4.34}$$

(up to $O(\varepsilon\mu)$ terms), with

$$\boldsymbol{F} = \int_{-1+\beta b}^{\varepsilon\zeta}(v_{sh}^*)^3. \tag{4.35}$$

Clearly, $\boldsymbol{F}$ cannot be determined in terms of ζ, $\overline{v}$, $v^\sharp$ and $\boldsymbol{E}$, and a last equation is therefore needed.

4.5.3. *An equation for* $\boldsymbol{F}$

Since $\boldsymbol{F}$ appears only in the $O(\sqrt{\mu})$ term in (4.34), we just need to determine an evolution equation for $\boldsymbol{F}$ up to $O(\sqrt{\mu})$ terms. This equation is easily obtained by multiplying (2.42) by $|v_{sh}^*|^2$ and integrating vertically,

$$\partial_t\boldsymbol{F} + \varepsilon\overline{v}\partial_x\boldsymbol{F} + 4\varepsilon\boldsymbol{F}\partial_x\overline{v} = 0 \tag{4.36}$$

(up to $O(\varepsilon\sqrt{\mu})$ terms).

4.6. *The Green–Naghdi model*

We are now able to give the Green–Naghdi equations in dimension $d=1$, with general vorticity, and for non-flat bottoms of medium amplitude (i.e. $\beta = O(\sqrt{\mu})$). These equations are an order $O(\mu^2)$ approximation of (2.15), where the momentum equation is approximated by (4.21) which involves a rotational energy $\boldsymbol{E}$ determined through the finite cascade (4.34), (4.36). More precisely, we have, dropping $O(\mu^2)$ terms,

$$\left.\begin{aligned}\partial_t\zeta + \partial_x(h\overline{v}) = 0,\\ (1+\mu\mathscr{T})\big(\partial_t\overline{v}+\varepsilon\overline{v}\partial_x\overline{v}\big) + \partial_x\zeta + \varepsilon\mu\mathscr{Q}_1(\overline{v}) + \varepsilon\mu\frac{1}{h}\partial_x\boldsymbol{E} + \varepsilon\mu^{3/2}\mathscr{C}(v^\sharp,\overline{v}) = 0,\\ \partial_t v^\sharp + \varepsilon\overline{v}\partial_x v^\sharp + \varepsilon v^\sharp\partial_x\overline{v} = 0,\\ \partial_t\boldsymbol{E} + \varepsilon\overline{v}\partial_x\boldsymbol{E} + 3\varepsilon\boldsymbol{E}\partial_x\overline{v} + \varepsilon\sqrt{\mu}\partial_x\boldsymbol{F} = 0,\\ \partial_t\boldsymbol{F} + \varepsilon\overline{v}\partial_x\boldsymbol{F} + 4\varepsilon\boldsymbol{F}\partial_x\overline{v} = 0,\end{aligned}\right\} \tag{4.37}$$

where $\mathscr{T}$, $\mathscr{Q}_1$ and $\mathscr{C}$ are defined in (1.16), (1.17) and (3.10) respectively, while we recall that $\boldsymbol{E}$, $\boldsymbol{F}$ and $v^\sharp$ stand for

$$\boldsymbol{E} = \int_{-1+\beta b}^{\varepsilon\zeta}(v_{sh}^*)^2, \quad \boldsymbol{F} = \int_{-1+\beta b}^{\varepsilon\zeta}(v_{sh}^*)^3 \quad \text{and} \quad v^\sharp = -\frac{24}{h^3}\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\int_{-1+\beta b}^{z} v_{sh}^*. \tag{4.38a–c}$$

Remark 10. By simple integrations by parts, it is possible to rewrite $v^\sharp$ as a second-order momentum,

$$v^\sharp = \frac{12}{h^3}\int_{-1+\beta b}^{\varepsilon\zeta}(z+1-\beta b)^2 v_{sh}^*. \tag{4.39}$$

Remark 11. As in Remark 9 for the case of constant vorticity, a local equation for the conservation of energy can be derived, which generalizes (1.19) and (3.13), namely,

$$\partial_t\big(\mathfrak{e}+\mathfrak{e}_{rot}\big)+\partial_x\big(\mathfrak{F}+\mathfrak{F}_{rot}\big)=0, \tag{4.40}$$

with (in dimensional form)

$$\mathfrak{e}_{rot}=\tfrac{1}{2}\boldsymbol{E} \quad \text{and} \quad \mathfrak{F}_{rot}=\tfrac{3}{2}\boldsymbol{E}\,\overline{v}+\tfrac{1}{2}\boldsymbol{F}+\mathfrak{F}_{\mathscr{C}}, \tag{4.41a,b}$$

the flux $\mathfrak{F}_{\mathscr{C}}$ being as in Remark 8.

Remark 12. No smallness assumption on ε has been made to derive (4.37). Assuming that $\varepsilon=\sqrt{\mu}$ (medium-amplitude waves with the terminology of Lannes (2013)), one can simplify (4.37) by dropping $O(\mu^2)$ terms. This yields

$$\left.\begin{aligned}&\partial_t\zeta+\partial_x(h\overline{v})=0,\\ &(1+\mu\mathscr{T})\big(\partial_t\overline{v}+\varepsilon\overline{v}\partial_x\overline{v}\big)+\partial_x\zeta+\varepsilon\mu\mathscr{Q}_1(\overline{v})+\varepsilon\mu\partial_x\boldsymbol{E}=0,\\ &\partial_t\boldsymbol{E}+\varepsilon\overline{v}\partial_x\boldsymbol{E}+3\varepsilon\boldsymbol{E}\partial_x\overline{v}=0.\end{aligned}\right\} \tag{4.42}$$

5. The $2d$ Green–Naghdi equations with general vorticity

We recall that throughout this paper, no smallness assumption is made on ε, so that it is possible to set $\varepsilon=1$.

We deal here with the derivation of Green–Naghdi type equations in the general two-dimensional case ($d=2$). One of the main new phenomena compared with the one-dimensional case is the interaction between the horizontal and vertical components of the vorticity.

As in § 4, we assume throughout this section that (4.2) holds, i.e. that the bottom variations are of medium amplitude, and we also use the following notations for tensors.

Notation 2. Given $K\in\mathbb{N}$ vectors, $a^k\in\mathbb{R}^2$, with $k=1,2,\ldots,K$ and with coordinates a_i^k for $i=1,2$, we define the $\underbrace{2\times 2\times\cdots\times 2}_{P}$ tensor, for $P\in\mathbb{N}$, by the expression

$$\big(a^{i_1}\otimes a^{i_2}\otimes\cdots\otimes a^{i_P}\big)_{j_1,j_2,\ldots,j_P}=a_{j_1}^{i_1}a_{j_2}^{i_2}\cdots a_{j_P}^{i_P}, \tag{5.1}$$

where $i_l\in\{1,2,\ldots,K\}$ for $l=1,2,\ldots,P$ and $j_n=1,2$ for $n=1,2,\ldots,P$.

The gradient of a vector $a\in\mathbb{R}^2$ is given by

$$(\nabla a)_{ij}=\partial_i a_j \tag{5.2}$$

for $i,j=1,2$.

The divergence of a matrix $\boldsymbol{A}\in\mathbb{R}^2\times\mathbb{R}^2$ is the vector $\nabla\cdot\boldsymbol{A}$ with coordinates

$$(\nabla\cdot\boldsymbol{A})_i=\partial_j A^{ji}\equiv\sum_{j=1}^{2}\partial_j A^{ji}\quad(i=1,2) \tag{5.3}$$

(where we also use Einstein's summation convention on repeated indices).

5.1. *Computation of the 'rotational Reynolds tensor' contribution*

Proceeding as for (3.4) and (4.7), we write, up to $O(\mu^2)$ terms,

$$
\begin{aligned}
\varepsilon\mu\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta} V^*\otimes V^* = {}& \varepsilon\mu\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta} V^*_{sh}\otimes V^*_{sh} \\
& - \varepsilon\mu^{3/2}\frac{1}{2}\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\nabla\nabla\cdot((1+z-\beta b)\overline{V})\otimes V^*_{sh} \\
& - \varepsilon\mu^{3/2}\frac{1}{2}\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta} V^*_{sh}\otimes\nabla\nabla\cdot((1+z-\beta b)\overline{V}).
\end{aligned}
\tag{5.4}
$$

Introducing

$$
V^\sharp = \frac{24}{h^3}\int_{-1+\beta b}^{\varepsilon\zeta}\int_z^{\varepsilon\zeta}\int_{-1+\beta b}^{z} V^*_{sh} \tag{5.5}
$$

and proceeding as in § 4.1, we obtain

$$
\begin{aligned}
\varepsilon\mu\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta} V^*\otimes V^* = \varepsilon\mu\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta} V^*_{sh}\otimes V^*_{sh} \\
- \varepsilon\mu^{3/2}\frac{1}{24}\nabla\cdot h^3 V^\sharp\otimes\nabla\nabla\cdot\overline{V} + \nabla\nabla\cdot\overline{V}\otimes V^\sharp + O(\mu^2).
\end{aligned}
\tag{5.6}
$$

5.2. *Computation of the pressure contribution*

As for (3.5) and (4.8), we have

$$
\begin{aligned}
\frac{1}{\varepsilon}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla p &= h\nabla\zeta + \int_{-1+\beta b}^{\varepsilon\zeta}\nabla\int_z^{\varepsilon\zeta}\left(\partial_t w + \varepsilon V\cdot\nabla w + \frac{\varepsilon}{\mu} w\partial_z w\right) \\
&= h\nabla\zeta + \mu A_{(1)} + \mu^{3/2}A_{(3/2)} + O(\mu^2),
\end{aligned}
\tag{5.7}
$$

where, as in the irrotational case,

$$
A_{(1)} = h\mathscr{T}\left(\partial_t\overline{V} + \varepsilon\overline{V}\cdot\nabla\overline{V}\right) + \varepsilon h\mathscr{Q}_1(\overline{V}), \tag{5.8}
$$

and $\mathscr{Q}_1(\overline{V})$ as in (1.17). For the $O(\mu^{3/2})$ component, we have

$$
\begin{aligned}
A_{(3/2)} = {}& h\widetilde{\mathscr{T}}(\partial_t V^*_{sh} + \varepsilon\overline{V}\cdot\nabla V^*_{sh}) + 2\varepsilon\int_{-1}^{\varepsilon\zeta}\nabla\int_z^{\varepsilon\zeta}\left(\int_{-1}^{z}\nabla\cdot V^*_{sh}\right)\nabla\cdot\overline{V} \\
& - \varepsilon\int_{-1}^{\varepsilon\zeta}\nabla\int_z^{\varepsilon\zeta} V^*_{sh}\cdot\nabla\left(\nabla\cdot((1+z')\overline{V})\right) + \varepsilon\int_{-1}^{\varepsilon\zeta}\nabla\int_z^{\varepsilon\zeta}\nabla\cdot\left((1+z)\overline{V}\right)\nabla\cdot V^*_{sh} \\
& + \varepsilon\int_{-1}^{\varepsilon\zeta}\nabla\int_z^{\varepsilon\zeta}\left([\mathrm{div}, \overline{V}\cdot\nabla] - (\nabla\cdot\overline{V})\nabla^{\mathrm{T}}\right)\int_{-1}^{z} V^*_{sh}.
\end{aligned}
\tag{5.9}
$$

Applying $\widetilde{\mathscr{T}}$ to (2.42) to handle the first term on the right-hand side, we obtain, proceeding as in § 4.2, that

$$\begin{aligned}
A_{(3/2)} &= -\varepsilon h\widetilde{\mathscr{T}}(V_{sh}^{*}\cdot\nabla\overline{V}+V_{sh}^{*}\nabla\cdot\overline{V})+2\varepsilon\int_{-1}^{\varepsilon\zeta}\nabla\int_{z}^{\varepsilon\zeta}\left(\int_{-1}^{z}\nabla\cdot V_{sh}^{*}\right)\nabla\cdot\overline{V}\\
&\quad -2\varepsilon\int_{-1}^{\varepsilon\zeta}\nabla\int_{z}^{\varepsilon\zeta}V_{sh}^{*}\cdot\nabla\big(\nabla\cdot((1+z')\overline{V})\big)\\
&\quad +\varepsilon\int_{-1}^{\varepsilon\zeta}\nabla\int_{z}^{\varepsilon\zeta}\big([\mathrm{div},\overline{V}\cdot\nabla]-(\nabla\cdot\overline{V})\nabla^{\mathrm{T}}\big)\int_{-1}^{z}V_{sh}^{*}\\
&= -\varepsilon\nabla\left[\frac{h^{3}}{4}V_{sh}^{*}\cdot\nabla\nabla\cdot\overline{V}+\frac{1}{12}\nabla\cdot\overline{V}\nabla\cdot(h^{3}V_{sh}^{*})+\frac{1}{12}\mathrm{Tr}\big(\nabla\overline{V}\nabla(h^{3}V^{\sharp})\big)\right].
\end{aligned} \tag{5.10}$$

5.3. *Wave–current interaction in the velocity equation*

From the momentum equation in (2.15) and (5.6)–(5.8) and (5.10), we obtain

$$(1+\mu\mathscr{T})\left(\partial_t\overline{V}+\varepsilon\overline{V}\cdot\nabla\overline{V}\right)+\mu\varepsilon\mathscr{Q}_1(\overline{V})+\varepsilon\mu\frac{1}{h}\nabla\cdot\boldsymbol{E}+\varepsilon\mu^{3/2}\mathscr{C}(V^{\sharp},\overline{V})=0, \tag{5.11}$$

where

$$\boldsymbol{E}=\int_{-1+\beta b}^{\varepsilon\zeta}V_{sh}^{*}\otimes V_{sh}^{*}\quad\text{and}\quad V^{\sharp}=-\frac{24}{h^{3}}\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta}\int_{-1+\beta b}^{z}V_{sh}^{*}, \tag{5.12a,b}$$

and where $\mathscr{C}(V^{\sharp},\overline{V})$ is the two-dimensional generalization of (3.10),

$$\begin{aligned}
\mathscr{C}(V^{\sharp},\overline{V}) &= -\frac{1}{24h}\nabla\cdot\big(h^{3}(V^{\sharp}\otimes\nabla\nabla\cdot\overline{V}+\nabla\nabla\cdot\overline{V}\otimes V^{\sharp})\big)\\
&\quad -\frac{1}{4h}\nabla\left[h^{3}V^{\sharp}\cdot\nabla\nabla\cdot\overline{V}+\frac{1}{3}\nabla\cdot\overline{V}\nabla\cdot(h^{3}V^{\sharp})+\frac{1}{3}\mathrm{Tr}\big(\nabla\overline{V}\nabla(h^{3}V^{\sharp})\big)\right].
\end{aligned} \tag{5.13}$$

As in the one-dimensional case (see § 4.3), closure equations are needed for $V^{\sharp}$ and $\boldsymbol{E}$. The closure equation has already been derived in the two-dimensional case in § 4.4; it is given by

$$\partial_t V^{\sharp}+\varepsilon(\overline{V}\cdot\nabla)V^{\sharp}+\varepsilon(V^{\sharp}\cdot\nabla)\overline{V}=0. \tag{5.14}$$

The derivation of the closure equation for $\boldsymbol{E}$ is addressed in the following section.

5.4. *Closure equation for $\boldsymbol{E}$*

We first derive an equation for $\boldsymbol{E}$, which involves a third-order tensor $\boldsymbol{F}$, for which we also derive an equation that closes the system.

5.4.1. *An equation for $\boldsymbol{E}$*

Recall first that we obtained in (4.32) an evolution equation for V_{sh}^{*}, namely

$$\partial_t V_{sh}^{*}+\varepsilon\overline{V}\cdot\nabla V_{sh}^{*}+\varepsilon V_{sh}^{*}\cdot\nabla\overline{V}=\varepsilon\nabla\cdot\big((1+z-\beta b)\overline{V}\big)\partial_z V_{sh}^{*}+\varepsilon\sqrt{\mu}C^{*}, \tag{5.15}$$

with

$$C^{*}=-V_{sh}^{*}\cdot\nabla V_{sh}^{*}+\frac{1}{h}\nabla\cdot\int_{-1+\beta b}^{\varepsilon\zeta}V_{sh}^{*}\otimes V_{sh}^{*}+\nabla\cdot\left(\int_{-1+\beta b}^{z}V_{sh}^{*}\right)\partial_z V_{sh}^{*}-\nabla^{\perp}\cdot\overline{V}(T^{*}\overline{V})^{\perp}. \tag{5.16}$$

Time differentiating the tensor $\boldsymbol{E}$, we obtain therefore

$$\begin{aligned}\partial_t \boldsymbol{E} &= \partial_t(\varepsilon\zeta)(V_{sh}^*\otimes V_{sh}^*)_{|surf} + \int_{-1+\beta b}^{\varepsilon\zeta}\left(\partial_t V_{sh}^*\otimes V_{sh}^* + V_{sh}^*\otimes \partial_t V_{sh}^*\right)\\ &= \partial_t(\varepsilon\zeta)(V_{sh}^*\otimes V_{sh}^*)_{|surf} + I_1 + I_2 + I_3 + \varepsilon\sqrt{\mu}\boldsymbol{G},\end{aligned}\tag{5.17}$$

with

$$\begin{aligned}I_1 &= -\varepsilon\int_{-1+\beta b}^{\varepsilon\zeta}\left((\overline{V}\cdot\nabla)V_{sh}^*\otimes V_{sh}^* + V_{sh}^*\otimes(\overline{V}\cdot\nabla)V_{sh}^*\right)\\ &= -\varepsilon\overline{V}\cdot\nabla\boldsymbol{E} + \varepsilon\big(\overline{V}\cdot\nabla(\varepsilon\zeta)(V_{sh}^*\otimes V_{sh}^*)_{|surf} - \overline{V}\cdot\nabla(\beta b)(V_{sh}^*\otimes V_{sh}^*)_{|bott}\big),\end{aligned}\tag{5.18}$$

while

$$\begin{aligned}I_2 &= \int_{-1+\beta b}^{\varepsilon\zeta}\nabla\cdot\big((1+z-\beta b)\overline{V}\big)\left(\partial_z V_{sh}^*\otimes V_{sh}^* + V_{sh}^*\otimes\partial_z V_{sh}^*\right)\\ &= -\boldsymbol{E} + h(V_{sh}^*\otimes V_{sh}^*)_{|surf} - \nabla(\beta b)\cdot\overline{V}\big((V_{sh}^*\otimes V_{sh}^*)_{|surf} - (V_{sh}^*\otimes V_{sh}^*)_{|bott}\big)\end{aligned}\tag{5.19}$$

and

$$\begin{aligned}I_3 &= -\varepsilon\int_{-1+\beta b}^{\varepsilon\zeta}\big((V_{sh}^*\cdot\nabla\overline{V})\otimes V_{sh}^* + V_{sh}^*\otimes(V_{sh}^*\cdot\nabla)\overline{V}\big)\\ &= -(\nabla\overline{V})^{\perp}\boldsymbol{E} - \ \boldsymbol{E}\nabla\overline{V}.\end{aligned}\tag{5.20}$$

Finally, the matrix $\boldsymbol{G}$ is given by

$$\begin{aligned}\boldsymbol{G} &= \int_{-1+\beta b}^{\varepsilon\zeta} C^*\otimes V_{sh}^* + V_{sh}^*\otimes C^*\\ &= -\int_{-1+\beta b}^{\varepsilon\zeta}(V_{sh}^*\cdot\nabla V_{sh}^*)\otimes V_{sh}^* + V_{sh}^*\otimes(V_{sh}^*\cdot\nabla V_{sh}^*)\\ &\quad - \int_{-1+\beta b}^{\varepsilon\zeta}(\nabla\cdot V_{sh}^*)V_{sh}^*\otimes V_{sh}^* - \varepsilon\nabla\zeta\cdot V_{sh}^*(V_{sh}^*\otimes V_{sh}^*)_{|surf} + \beta\nabla b\cdot V_{sh}^*(V_{sh}^*\otimes V_{sh}^*)_{|bott}\\ &\quad + \frac{h^4}{24}\big(\nabla^{\perp}\cdot\overline{V}\nabla^{\perp}\nabla\cdot\overline{V}\otimes V^{\sharp} + V^{\sharp}\otimes\nabla^{\perp}\cdot\overline{V}\nabla^{\perp}\nabla\cdot\overline{V}\big).\end{aligned}\tag{5.21}$$

Gathering all these computations, we finally obtain that

$$\partial_t\boldsymbol{E} + \varepsilon\overline{V}\cdot\nabla\boldsymbol{E} + \varepsilon\nabla\cdot\overline{V}\boldsymbol{E} + \varepsilon\nabla\overline{V}^{\mathrm{T}}\boldsymbol{E} + \varepsilon\boldsymbol{E}\nabla\overline{V} + \varepsilon\sqrt{\mu}\nabla\cdot\boldsymbol{F} = \varepsilon\sqrt{\mu}\mathscr{D}(V^{\sharp},\overline{V}),\tag{5.22}$$

where

$$\boldsymbol{F} = \int_{-1+\beta b}^{\varepsilon\zeta} V_{sh}^*\otimes V_{sh}^*\otimes V_{sh}^*\tag{5.23}$$

and

$$\mathscr{D}(V^{\sharp},\overline{V}) = \frac{h^3}{24}\nabla^{\perp}\cdot\overline{V}\big(\nabla^{\perp}\nabla\cdot\overline{V}\otimes V^{\sharp} + V^{\sharp}\otimes\nabla^{\perp}\nabla\cdot\overline{V}\big).\tag{5.24}$$

Remark 13. The tensor $\boldsymbol{E}$ is symmetric, and therefore $\boldsymbol{E}=\boldsymbol{E}^{\mathrm{T}}$. Equation (5.22) is therefore the same as the equation governing the evolution of the Reynolds tensor in barotropic turbulent compressible fluids (Mohammadi & Pironneau 1994; Pope 2005; Gavrilyuk & Gouin 2012), with a source term $\boldsymbol{S}=-\varepsilon\sqrt{\mu}\nabla\cdot\boldsymbol{F}+\varepsilon\sqrt{\mu}\mathscr{D}(V^\sharp,\overline{V})$. The structure of the source term $\boldsymbol{S}$ is central in turbulence theory and is still under intense investigation; in the present case, it has a well-defined structure and the system of equations can be closed by deriving evolution equations on $V^\sharp$ and $\boldsymbol{F}$. We refer to Gavrilyuk & Gouin (2012) for an instructive geometric study of the equation in the case $\boldsymbol{S}=0$.

5.4.2. *An equation for* $\boldsymbol{F}$

To close the system we need an equation for the third-order tensor $\boldsymbol{F}$ up to order $O(\sqrt{\mu})$. We just compute, for $i,j,k=1,2$,

$$\partial_t F_{ijk}=\varepsilon\partial_t\zeta\,(V^*_{sh})_i(V^*_{sh})_j(V^*_{sh})_{k_{|surf}}+\int_{-1+\beta b}^{\varepsilon\zeta}\partial_t\big((V^*_{sh})_i(V^*_{sh})_j(V^*_{sh})_k\big) \tag{5.25}$$

and use (2.42) to obtain, up to $O(\sqrt{\mu})$ terms and with Einstein's summation convention on repeated indices,

$$\begin{aligned}\int_{-1+\beta b}^{\varepsilon\zeta}\partial_t(V^*_{sh})_i(V^*_{sh})_j(V^*_{sh})_k&=-\varepsilon\int_{-1+\beta b}^{\varepsilon\zeta}\overline{V}\cdot\nabla((V^*_{sh})_i(V^*_{sh})_j(V^*_{sh})_k)\\ -F_{lkj}\partial_l\overline{V}_i-F_{ilk}\partial_l\overline{V}_j-F_{ijl}\partial_l\overline{V}_k&+\varepsilon\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\cdot\big((1+z-\beta b)\overline{V}\big)\partial_z((V^*_{sh})_i(V^*_{sh})_j(V^*_{sh})_k).\end{aligned} \tag{5.26}$$

One then readily deduces the following equation:

$$\partial_t F_{ijk}+\varepsilon\overline{V}\cdot\nabla F_{ijk}+\varepsilon F_{lkj}\partial_l\overline{V}_i+\varepsilon F_{ilk}\partial_l\overline{V}_j+\varepsilon F_{ijl}\partial_l\overline{V}_k+\varepsilon\nabla\cdot\overline{V}F_{ijk}=0 \tag{5.27}$$

$(i,j,k=1,2)$, which closes the system of equations on ζ, $\overline{V}$, $V^\sharp$, $\boldsymbol{E}$ and $\boldsymbol{F}$.

5.5. *The Green–Naghdi model*

We can now give the two-dimensional generalization of the Green–Naghdi equations (4.37) with general vorticity. More precisely, we have, dropping $O(\mu^2)$ terms,

$$\left.\begin{aligned}&\partial_t\zeta+\nabla\cdot(h\overline{V})=0,\\ &(1+\mu\mathscr{T})\left(\partial_t\overline{V}+\varepsilon\overline{V}\cdot\nabla\overline{V}\right)+\nabla\zeta+\mu\varepsilon\mathscr{Q}_1(\overline{V})+\varepsilon\mu\frac{1}{h}\nabla\cdot\boldsymbol{E}+\varepsilon\mu^{3/2}\mathscr{C}(V^\sharp,\overline{V})=0,\\ &\partial_t V^\sharp+\varepsilon(\overline{V}\cdot\nabla)V^\sharp+\varepsilon(V^\sharp\cdot\nabla)\overline{V}=0,\\ &\partial_t\boldsymbol{E}+\varepsilon\overline{V}\cdot\nabla\boldsymbol{E}+\varepsilon\nabla\cdot\overline{V}\boldsymbol{E}+\varepsilon\nabla\overline{V}^{\mathrm{T}}\boldsymbol{E}+\varepsilon\boldsymbol{E}\nabla\overline{V}+\varepsilon\sqrt{\mu}\nabla\cdot\boldsymbol{F}=\varepsilon\sqrt{\mu}\mathscr{D}(V^\sharp,\overline{V}),\\ &\partial_t F_{ijk}+\varepsilon\overline{V}\cdot\nabla F_{ijk}+\varepsilon F_{lkj}\partial_l\overline{V}_i+\varepsilon F_{ilk}\partial_l\overline{V}_j+\varepsilon F_{ijl}\partial_l\overline{V}_k+\varepsilon\nabla\cdot\overline{V}F_{ijk}=0\end{aligned}\right\} \tag{5.28}$$

$(i,j,k=1,2)$, where $\mathscr{T}$, $\mathscr{Q}_1$ and $\mathscr{C}$ are defined in (1.16), (1.17) and (5.13) respectively, while we recall that $\boldsymbol{E}$, $\boldsymbol{F}$ and $V^\sharp$ stand for

$$\boldsymbol{E}=\int_{-1+\beta b}^{\varepsilon\zeta}V^*_{sh}\otimes V^*_{sh},\quad \boldsymbol{F}=\int_{-1+\beta b}^{\varepsilon\zeta}V^*_{sh}\otimes V^*_{sh}\otimes V^*_{sh} \tag{5.29a,b}$$

and

$$V^\sharp = -\frac{24}{h^3}\int_{-1+\beta b}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta}\int_{-1+\beta b}^{z} V^*_{sh} \tag{5.30}$$

(as in Remark 10, $V^\sharp$ can equivalently be written as a second-order momentum of the shear velocity fluctuation), and where

$$\mathscr{D}(V^\sharp,\overline{V}) = \frac{h^3}{24}\nabla^\perp\cdot\overline{V}\big(\nabla^\perp\nabla\cdot\overline{V}\otimes V^\sharp + V^\sharp\otimes\nabla^\perp\nabla\cdot\overline{V}\big). \tag{5.31}$$

Remark 14. In dimension $d=2$, the structural property given in Remark 8 can be generalized; it holds that

$$h\mathscr{C}(V^\sharp,\overline{V})\cdot\overline{V} = \nabla\cdot\mathfrak{F}_{\mathscr{C}} + \tfrac{1}{2}\mathrm{Tr}(\mathscr{D}(V^\sharp,\overline{V})), \tag{5.32}$$

where the flux $\mathfrak{F}_{\mathscr{C}}$ is given by the vector

$$\begin{aligned}\mathfrak{F}_{\mathscr{C}} \;=\; & -\frac{h^3}{12}\left(\frac{1}{2}V^\sharp\cdot\overline{V}(\nabla\nabla\cdot V) + \frac{1}{2}\overline{V}\cdot\nabla\nabla\cdot VV^\sharp + 3V^\sharp\cdot\nabla\nabla\cdot\overline{VV} - (\nabla\cdot\overline{V})^2V^\sharp\right)\\ & + \frac{1}{12}\Big(\nabla\cdot\overline{V}\nabla\cdot(h^3V^\sharp)\overline{V} + \mathrm{Tr}\big(\nabla\overline{V}\nabla(h^3V^\sharp)\big)\overline{V}\Big)\\ & + \frac{h^3}{12}\Big((\nabla\cdot\overline{V})(V_1^\sharp\nabla^\perp\overline{V}_2 - V_2^\sharp\nabla^\perp\overline{V}_1)\Big).\end{aligned} \tag{5.33}$$

Contrary to the one-dimensional case, the quantity $h\mathscr{C}(V^\sharp,\overline{V})\cdot\overline{V}$ is not the divergence of a flux. The presence of the term $\mathrm{Tr}(\mathscr{D}(V^\sharp,\overline{V}))/2$ renders a new mechanism of energy flux due to the interaction of the vertical and horizontal components of the vorticity.

Remark 15. Using Remark 14, the local equation for the conservation of energy derived in Remark 11 in one dimension can be generalized in two dimensions; one has

$$\partial_t\big(\mathfrak{e}+\mathfrak{e}_{rot}\big) + \nabla\cdot\big(\mathfrak{F}+\mathfrak{F}_{rot}\big) = 0, \tag{5.34}$$

with (in dimensional form)

$$\mathfrak{e}_{rot} = \frac{1}{2}\mathrm{Tr}\,\boldsymbol{E} \quad\text{and}\quad \mathfrak{F}_{rot} = \frac{1}{2}(\mathrm{Tr}\,\boldsymbol{E})\overline{V} + \boldsymbol{E}\overline{V} + \frac{1}{2}\begin{pmatrix}F^{111}+F^{122}\\ F^{211}+F^{222}\end{pmatrix} + \mathfrak{F}_{\mathscr{C}}, \tag{5.35a,b}$$

the flux $\mathfrak{F}_{\mathscr{C}}$ being as in Remark 14.

6. Reconstruction of the velocity profile and vorticity dynamics

6.1. *First-order (Saint-Venant) reconstruction of the velocity*

As noted in Remark 6, a first-order approximation of the averaged Euler equations (2.15) is provided by the Saint-Venant (or nonlinear shallow water) equations

$$\left.\begin{aligned}\partial_t\zeta + \nabla\cdot(h\overline{V}) = 0,\\ \partial_t(h\overline{V}) + h\nabla\zeta + \varepsilon\nabla\cdot(h\overline{V}\otimes\overline{V}) = 0;\end{aligned}\right\} \tag{6.1}$$

these equations are the same as in the irrotational case. However, when reconstructing the velocity field $\boldsymbol{V}(t, X, z)$ inside the fluid domain, the effects of the vorticity cannot be neglected. According to (2.29) and (2.32), the horizontal and vertical velocities are given at first order by

$$\boldsymbol{V} = \overline{V} + \sqrt{\mu} V_{sh}^*, \quad \text{and} \quad \frac{1}{\mu}\boldsymbol{w} = -\nabla \cdot \left[(1 + z - \beta b)\overline{V}\right] - \mu^{1/2} \nabla \cdot \int_{-1+\beta b}^{z} V_{sh}^*. \quad (6.2a,b)$$

We also know that V_{sh}^* is found through the resolution of (2.42). This equation is a variable coefficient linear equation cast on the fluid domain Ω_t. Although Ω_t is known at this step (through the resolution of the Saint-Venant equation), it is still a moving $(d+1)$-dimensional domain and the numerical computation of the solutions to (2.42) is time consuming. We propose here a simpler approach consisting in deriving a simple scalar d-dimensional equation determining the horizontal velocity on each level line Γ_θ, with

$$\Gamma_\theta = \{(X, z), z = -1 + \beta b(X) + \theta h(t, X)\} \quad (\theta \in [0, 1]), \tag{6.3}$$

so that Γ_0 corresponds to the bottom and Γ_1 to the surface. With the notation

$$V_\theta^*(X) = V_{sh|_{\Gamma_\theta}}^*(X) = V_{sh}^*(X, -1 + \beta b(X) + \theta h(t, X)), \tag{6.4}$$

one has

$$\left.\begin{aligned} \boldsymbol{V}_{|_{\Gamma_\theta}} &= \overline{V} + \sqrt{\mu} V_\theta^*, \\ \frac{1}{\mu}\boldsymbol{w}_{|_{\Gamma_\theta}} &= -\nabla \cdot \left(h(\theta\overline{V} + \sqrt{\mu} Q_\theta)\right) + \nabla(-1 + \beta b + \theta h) \cdot (\overline{V} + \sqrt{\mu} V_\theta^*), \end{aligned}\right\} \tag{6.5}$$

with

$$Q_\theta = \frac{1}{h}\int_{-1+\beta b}^{-1+\beta b+\theta h} V_{sh}^* \mathrm{d}z = \int_0^\theta V_{\theta'}^* \mathrm{d}\theta', \tag{6.6}$$

and one easily obtains from (2.42) that

$$\partial_t V_\theta^* + \varepsilon \overline{V} \cdot \nabla V_\theta^* + \varepsilon V_\theta^* \cdot \nabla \overline{V} = 0 \tag{6.7}$$

and that

$$\partial_t Q_\theta + \varepsilon \overline{V} \cdot \nabla Q_\theta + \varepsilon Q_\theta \cdot \nabla \overline{V} = 0. \tag{6.8}$$

The velocity field in the fluid domain can therefore be fully determined at the precision of the model by the following decoupled procedure.

(*a*) Solve the Saint-Venant equations (2.47) to obtain ζ and $\overline{V}$ on the desired time interval.

(*b*) Obtain the quantities V_θ^* and Q_θ from their initial values by solving (6.7) and (6.8).

(*c*) Reconstruct the velocity field on each level line Γ_θ $(0 \leqslant \theta \leqslant 1)$ by using the formulae (6.5).

Although the first-order (Saint-Venant) approximation is the same as in the irrotational case, the velocity in the fluid domain therefore differs from the irrotational theory by the shear component $\sqrt{\mu} V_\theta^*$, which is essentially advected at the mean velocity $\overline{V}$.

6.2. *Second-order (Green–Naghdi) reconstruction of the velocity*

As seen in § 5.5, a second-order approximation of the averaged Euler equations (2.15) is provided by the Green–Naghdi equations with vorticity (5.28). According to (2.30) and (2.33), the velocity in the fluid domain is given by

$$\boldsymbol{V} = \overline{V} + \sqrt{\mu} V^*_{sh} + \mu T^* \overline{V} + O(\mu^{3/2}) \tag{6.9}$$

and

$$\frac{1}{\sqrt{\mu}} \boldsymbol{w} = -\nabla \cdot \left[(1 + z - \beta b) \overline{V} \right] - \mu^{1/2} \nabla \cdot \int_{-1+\beta b}^{z} V^*_{sh} - \mu \nabla \cdot \int_{-1+\beta b}^{z} T^* \overline{V} + O(\mu^{3/2}). \tag{6.10}$$

Using the same notations as in the previous section, and defining T^*_θ as

$$T^*_\theta \overline{V} = -\tfrac{1}{2} \left(\theta^2 - \tfrac{1}{3} \right) h^2 \nabla \nabla \cdot \overline{V} + \beta \left(\theta - \tfrac{1}{2} \right) h \big(\nabla b \cdot \nabla \overline{V} + \nabla (\nabla b \cdot \overline{V}) \big), \tag{6.11}$$

we therefore obtain, up to $O(\mu^{3/2})$ terms,

$$\left. \begin{aligned} \boldsymbol{V}_{|\Gamma_\theta} &= \overline{V} + \sqrt{\mu} V^*_\theta + \mu T^*_\theta \overline{V}, \\ \frac{1}{\mu} \boldsymbol{w}_{|\Gamma_\theta} &= -\nabla \cdot \big(h(\theta \overline{V} + \sqrt{\mu} Q_\theta) \big) + \nabla (-1 + \beta b + \theta h) \cdot (\overline{V} + \sqrt{\mu} V^*_\theta) \\ &\quad - \mu \nabla \left(h \cdot \int_0^\theta T^*_{\theta'} \overline{V} \mathrm{d}\theta' \right) + \mu \nabla (-1 + \beta b + \theta h) T^*_\theta \overline{V}. \end{aligned} \right\} \tag{6.12}$$

Remark 16. It is actually possible to reconstruct the velocity at order $O(\mu^2)$, but this requires us to include the $O(\mu)$ terms in the equations for V^*_θ; for the sake of simplicity, we therefore stick to order $O(\mu^{3/2})$.

We also deduce from (4.32) that

$$\partial_t V^*_\theta + \overline{V} \cdot \nabla V^*_\theta + V^*_\theta \cdot \nabla \overline{V} + \varepsilon \sqrt{\mu} V^*_\theta \cdot \nabla V^*_\theta = \varepsilon \sqrt{\mu} (\mathscr{S}_1 + \mathscr{S}_2), \tag{6.13}$$

where the source terms $\mathscr{S}_1$ and $\mathscr{S}_2$ are given by

$$\mathscr{S}_1 = \frac{1}{h} \nabla \cdot \boldsymbol{E} + \frac{q_\theta}{h} \nabla \cdot (h Q_\theta), \tag{6.14}$$

$$\mathscr{S}_2 = -\frac{h^2}{6} (1 - 3\theta^2) \nabla^\perp \cdot \overline{V} \nabla^\perp \nabla \cdot \overline{V}, \tag{6.15}$$

the quantities q_θ and Q_θ being defined as

$$q_\theta = h \partial_z V^*_{sh|\Gamma_\theta} = \partial_\theta V^*_\theta \quad \text{and} \quad Q_\theta = \frac{1}{h} \int_{-1+\beta b}^{-1+\beta b + \theta h} V^*_{sh} \mathrm{d}z = \int_0^\theta V^*_{\theta'} \mathrm{d}\theta'. \tag{6.16a,b}$$

These two quantities can be straightforwardly computed from their initial values by solving the equations

$$\left. \begin{aligned} \partial_t q_\theta + \varepsilon q_\theta \cdot \nabla \overline{V} + \varepsilon \overline{V} \cdot \nabla q_\theta = 0, \\ \partial_t Q_\theta + \varepsilon Q_\theta \cdot \nabla \overline{V} + \varepsilon \overline{V} \cdot \nabla Q_\theta = 0. \end{aligned} \right\} \tag{6.17}$$

The velocity field in the fluid domain can therefore be fully determined at the precision of the model by the following decoupled procedure.

(*a*) Solve the Green–Naghdi equations (5.28) to obtain ζ and $\overline{V}$ on the desired time interval.

(*b*) Obtain the quantities V_θ^*, q_θ and Q_θ from their initial values by solving (6.13) and (6.17).

(*c*) Reconstruct the velocity field on each level line Γ_θ ($0 \leqslant \theta \leqslant 1$) by using the formulae (6.12).

There are several important differences to be underlined if one compares this second-order approximation with the first-order (Saint-Venant) approximation considered in the previous section.

(i) The equations for ζ and $\overline{V}$ are not the same as in the irrotational theory.

(ii) The quadratic term on the left-hand side and the source term $\mathscr{S}_1$ in (6.13) render a more complex behaviour of the shear velocity due to the horizontal vorticity.

(iii) The above reconstruction procedure exhibits a mechanism of creation of shear velocity from vertical vorticity. Even if we start from an initial zero horizontal vorticity (and therefore $V^*_{sh|_{t=0}} = 0$), the shear velocity does not remain equal to zero during the evolution of the flow. Indeed, due to the source term $\mathscr{S}_2$ in (6.13) (itself proportional to the vertical vorticity $\nabla^\perp \cdot \overline{V}$), the quantity V_θ^* departs from its zero initial value.

6.3. *The dynamics of the vertical vorticity*

The dynamics of the vertical component $\boldsymbol{\omega}_{\mu,v} = \nabla^\perp \cdot \boldsymbol{V}$ of the vorticity is very important for the study of rip currents (see, for instance, Hammack *et al.* (1991) and Chen *et al.* (1999)). One directly obtains from (6.9) that, up to $O(\mu^{3/2})$ terms,

$$\boldsymbol{\omega}_{\mu,v} = \nabla^\perp \cdot \overline{V} + \sqrt{\mu}\nabla^\perp \cdot V^*_{sh} + \mu\frac{h}{3}\nabla^\perp h \cdot \nabla(\nabla \cdot \overline{V}). \tag{6.18}$$

In order to make more explicit the possible creation of vertical vorticity from horizontal vorticity, we consider here the time evolution of the vertically averaged vertical velocity $\overline{\boldsymbol{\omega}}_{\mu,v}$, defined as

$$\overline{\boldsymbol{\omega}}_{\mu,v}(t, X) = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} \boldsymbol{\omega}_{\mu,v}(t, X, z)\mathrm{d}z, \tag{6.19}$$

so that

$$\overline{\boldsymbol{\omega}}_{\mu,v}(t, X) = \overline{\omega}_0 + \sqrt{\mu}\overline{\omega}_1, \tag{6.20}$$

with

$$\left.\begin{aligned} \overline{\omega}_0 &= \nabla^\perp \cdot \overline{V} + \mu\frac{h}{3}\nabla^\perp h \cdot \nabla(\nabla \cdot \overline{V}), \\ \overline{\omega}_1 &= -\frac{1}{h}\left(\varepsilon\nabla^\perp\zeta \cdot V_1^* - \beta\nabla^\perp b V_0^*\right). \end{aligned}\right\} \tag{6.21}$$

(According to the notation (6.4), V_0^* and V_1^* correspond to the evaluation of V^*_{sh} at the bottom and at the surface respectively.) We have already seen that all the V_θ^* can be recovered from their initial values through (6.13); considering the particular cases $\theta = 0, 1$ we therefore obtain $\overline{\omega}_1$. For $\overline{\omega}_0$, we apply $\nabla^\perp$ to the second equation of (5.28), so that

$$\partial_t\overline{\omega}_0 + \nabla \cdot (\overline{\omega}_0\overline{V}) + \varepsilon\mu\nabla^\perp \cdot \left(\frac{1}{h}\nabla \cdot \boldsymbol{E}\right) + \varepsilon\mu^{3/2}\nabla^\perp \cdot \mathscr{C}(V^\sharp, \overline{V}) = 0. \tag{6.22}$$

The procedure to recover the vertical vorticity is therefore the following.

(*a*) Solve the Green–Naghdi equations (5.28) to obtain ζ, $\overline{V}$, $\boldsymbol{E}$ and $V^\sharp$ on the desired time interval.
(*b*) Obtain $\overline{\omega}_0$ from its initial value by solving (6.22).
(*c*) Obtain the quantities V_0^* and V_1^* from their initial values by solving (6.13) and deduce $\underline{\omega}_1$ from (6.21).
(*d*) Recover the averaged vorticity $\overline{\boldsymbol{\omega}}_{\mu,v}$ through (6.20).

One important aspect to underline here is that it is possible to start with a zero averaged vertical vorticity but that this quantity becomes non-zero with the evolution of the flow provided that some (horizontal) vorticity is present. This mechanism of transfer from the horizontal to the vertical vorticity is likely to play an important role in the study of rip currents; its detailed study is left for future works.

7. Conclusion

We have derived here several fully nonlinear models generalizing the Green–Naghdi equations in the presence of vorticity. We have based our formal computations on the rigorous estimates derived in Castro & Lannes (2014), thereby ensuring the validity of the approximations made throughout this article. For the sake of clarity, we have presented these models by increasing complexity (constant vorticity in dimension $d = 1$, general vorticity in dimension $d = 1$, general vorticity in dimension $d = 2$). The most remarkable feature of these models is that they do not require the coupling with a $(d + 1)$-dimensional equation for the vorticity; indeed, they differ from the irrotational theory by the coupling, reminiscent of turbulence theory, with a finite cascade of equations. This cascade gives the evolution of the components of the 'Reynolds' tensor describing the self-interaction of the shear velocity induced by the vorticity on the one hand, and its interaction with the non-hydrostatic vertical variations typical of Boussinesq type models on the other hand. The reconstruction of the velocity profile in the $(d+1)$-dimensional fluid domain can then be performed by a completely decoupled and simple procedure. The most striking phenomena here are a mechanism of creation of horizontal shear from vertical vorticity and, conversely, a mechanism of transfer of horizontal to vertical vorticity, which is likely to play a key role in rip currents, for instance.

A natural perspective opened by this work is therefore to allow for the possibility of shear flows in the numerous codes developed recently for the numerical simulation of (irrotational) Green–Naghdi systems (see, for instance, Chen *et al.* (2000), Cienfuegos *et al.* (2006), Le Métayer *et al.* (2010), Bonneton *et al.* (2011*b*), Kazolea *et al.* (2012), Dutykh *et al.* (2014), Lannes & Marche (2014), Ricchiuto & Filippini (2014) and the review Bonneton *et al.* (2011*a*)) and the modelling of rip currents using the models derived in this article.

In view of describing rip currents, another natural perspective is to take into account the creation of vorticity by mechanisms such as wave breaking, surface and bottom boundary layers, etc. The present work deals, indeed, with a conservative framework (as shown, for instance, by the (5.34) for the local conservation of energy). Our goal here was to understand the coupling between surface waves and underlying vortical flows. It complements in this respect the recent works by Richard & Gavrilyuk (2012) and Richard & Gavrilyuk (2013). These authors work, indeed, at the level of the Saint-Venant equations (i.e. they neglect the non-hydrostatic terms of the Green–Naghdi equations), but provide a thorough description of vorticity generation in roll waves and hydraulic jumps, for instance. In their approach, the classical Saint-Venant equations are coupled with a third equation describing the creation of enstrophy by wave breaking and by the bottom boundary layer. This enstrophy is closely related to the tensor $\boldsymbol{E}$ in (5.28); it seems therefore possible to combine our approach and that of Richard and Gavrilyuk, leading to a complete Green–Naghdi model describing both the coupling between surface waves and underlying flows and the creation of vorticity. This is left for future work.

Another direction of research would be to investigate the behaviour of some of the specific structures observed for steady waves with constant vorticity (the 'cat's eye' patterns studied in Wahlen (2009) and Constantin & Varvaruca (2011), for instance). Our new models can, for instance, be used to describe the dynamical behaviour of small perturbations (with possibly non-constant vorticity).

Acknowledgements

A.C. is supported by the grant MTM2011-266696 (Spain), ICMAT Severo Ochoa project SEV-2011-0087 and ERC grant 307179-GFTIPFD. D.L. acknowledges support from the ANR-13-BS01-0003-01 DYFICOLTI, the ANR BOND and the INSU-CNRS LEFE-MANU project Soli.

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