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Asymptotic models for internal waves

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Abstract

Derived here in a systematic way, and for a large class of scaling regimes are asymptotic models for the propagation of internal waves at the interface between two layers of immiscible fluids of different densities, under the rigid lid assumption and with a flat bottom. The full (Euler) model for this situation is reduced to a system of evolution equations posed spatially on $\mathbb{R}^d$, $d = 1, 2$, which involve two nonlocal operators. The different asymptotic models are obtained by expanding the nonlocal operators with respect to suitable small parameters that depend variously on the amplitude, wave-lengths and depth ratio of the two layers. We rigorously derive classical models and also some model systems that appear to be new. Furthermore, the consistency of these asymptotic systems with the full Euler equations is established.

Résumé

Nous établissons ici de manière systématique, et pour une grande classe de régimes, des modèles asymptotiques pour la propagation d'ondes internes à l'interface de deux couches de fluides non miscibles de densités différentes, sous l'hypothèse de toit rigide et de fond plat. Les équations complètes pour cette situation (Euler) sont réduites à un système d'équations d'évolution posé dans le domaine spatial $\mathbb{R}^d$, $d = 1, 2$, et qui comprend deux opérateurs non locaux. Les divers modèles asymptotiques sont obtenus en développant les opérateurs non locaux par rapport à des petits paramètres convenables (dépendant de l'amplitude, de la longueur d'onde et du rapport de hauteur des deux couches). Nous établissons rigoureusement des modèles classiques ainsi que d'autres qui semblent nouveaux. De plus, on montre la consistance de ces systèmes asymptotiques avec les équations d'Euler.

Keywords: Water waves; Internal waves; Asymptotic models

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1. Introduction

1.1. General setting

The mathematical theory of waves on the interface between two layers of immiscible fluid of different densities has attracted interest because it is the simplest idealization for internal wave propagation and because of the challenging modeling, mathematical and numerical issues that arise in the analysis of this system. The recent survey article of Helfrich and Melville [20] provides a rather extensive bibliography and a good overview of the properties of steady internal solitary waves in such systems as well as for more general density stratifications. The compendium [22] of field observations comprised of synthetic aperture radar (SAR) images of large-amplitude internal waves in different oceans together with associated physical data shows just how varied can be the propagation of internal waves. This variety is reflected in the mathematical models for such phenomena. Because of the range of scaling regimes that come to the fore in real environments, the literature on internal wave models is markedly richer in terms of different types of model equations than is the case for surface wave propagation (see, e.g. [8,10] and the references therein).

The idealized system that will be the focus of the discussion here, when it is at rest, consists of a homogeneous fluid of depth d_1 and density ρ_1 lying over another homogeneous fluid of depth d_2 and density $\rho_2 > \rho_1$. The bottom on which both fluids rest is presumed to be horizontal and featureless while the top of fluid 1 is restricted by the rigid lid assumption, which is to say, the top is viewed as an impenetrable, bounding surface. This is a standard assumption, and is reckoned to be a good one when the pycnocline is far from the top, which is when d_1 is large relative to the wavelength of a disturbance. In the present work, two general classes of waves will be countenanced. Both of these require that the deviation of the interface be a graph over the flat bottom, so overturning waves are not within the purview of our theory (see Fig. 1 for a definition sketch). The first, which is referred to as the one-dimensional case, are long-crested waves that propagate principally along one axis, say along the x-direction in a standard x–y–z Cartesian frame in which z is directed opposite to the direction in which gravity acts. Such motions are taken to be sensibly independent of the y-coordinate and can be successfully modeled in the first instance by the two-dimensional Euler system involving only the independent variables x, z and of course time t. Because the interface is a graph over the bottom, these asymptotic models then depend only upon $x \in \mathbb{R}$ and t, and hence the appellation ‘one-dimensional’. Among one-dimensional models, the simplest are those in which one further assumes that the waves travel only in one direction, say in the direction of increasing values of x. Models which we will call ‘two-dimensional’ are not restricted by the long-crested presumption, and are consequently more general than the one-dimensional models. They are derived from the full three-dimensional Euler system and their dependent variables depend upon the spatial variable $X = (x, y) \in \mathbb{R}^2$ and time t.

One-dimensional, unidirectional, weakly nonlinear models such as the Korteweg–de Vries (KdV) equation, the Intermediate Long Wave (ILW) equation [23,25] or the Benjamin–Ono equation [5] have been extensively used and compared with laboratory experiments [24,31,35]. While much of our qualitative appreciation of the interaction between the competing effects of nonlinearity and dispersion in surface and internal wave propagation has been informed by these sorts of equations, they are of somewhat limited validity (cf. [1]). Weakly nonlinear models in two-dimensions have been derived by Camassa and Choi [14]. Nguyen and Dias [29] have derived and studied a Boussinesq-type system in a weakly nonlinear regime. Fully nonlinear models were obtained in the one-dimensional case by Matsuno [28], and in the two-dimensional case by Camassa and Choi [15]. We mention also the interesting paper by Camassa et al. [12] where the aforementioned models are compared, in the one-dimensional case, with experimental observations and numerical integrations of the full Euler system. In [14,15,28] the analysis commences with the full Euler system formulation and the asymptotic models are obtained by formally expanding the unknowns with respect to a small parameter. It is not easy using this approach to provide a rigorous justification of the asymptotic expansion, except perhaps within the setting of analytic functions. A different approach has been carried out by Craig, Guyenne and Kalisch [17] in the one-dimensional case. These authors use the Hamiltonian formulation of the Euler equations (due originally to Zakharov [36] for surface waves and to Benjamin and Bridges [6] for internal waves) and expand the Hamiltonian with respect to the relevant small parameters. This method provides a hierarchy of Hamiltonian systems which serve as approximations of the full Euler equations. Such systems are not always the best for modeling, analytical or numerical purposes, however. Indeed, they can even be linearly ill-posed in Hadamard’s classical sense. In such cases, it is necessary in the Hamiltonian framework to proceed one stage further in the expansion, leading to more complicated systems (which may still not be well posed).

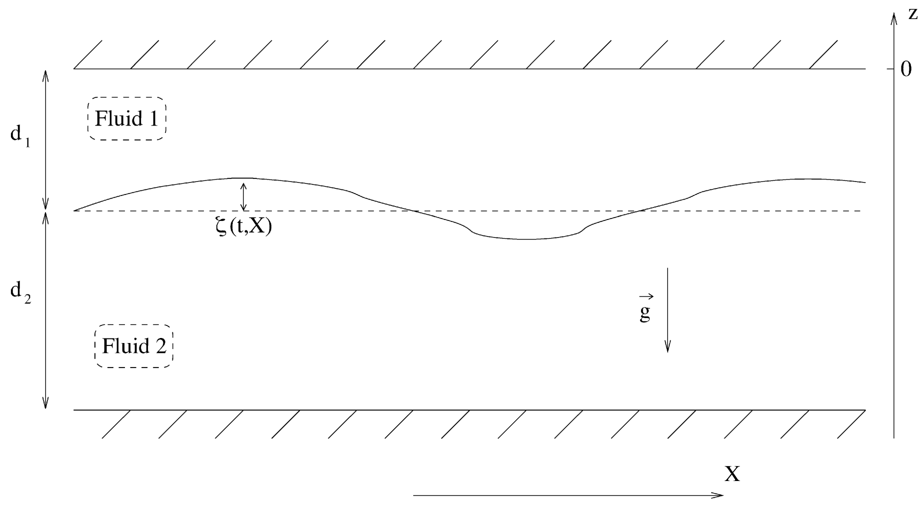

Fig. 1.

The strategy followed here is inspired by that initiated in [8–10]. Namely, following the procedure introduced in [17,19,36], we rewrite the full system as a system of two evolution equations posed on $\mathbb{R}^d$, where $d = 1$ or 2 depending upon whether a one- or two-dimensional model is being contemplated. The reformulated system, which depends only upon the spatial variable on the interface, involves two nonlocal operators, a Dirichlet-to-Neumann operator $G[\zeta]$, and what we term an "interface operator" $\mathbf{H}[\zeta]$, defined precisely below. Of course, the operator $\mathbf{H}[\zeta]$ does not appear in the theory of surface waves, and this is an interesting new aspect of the internal wave theory. A rigorously justified asymptotic expansion of the nonlocal operators with respect to dimensionless small parameters is then mounted. We consider both the "weakly nonlinear" case and the "fully nonlinear" situation and cover a variety of scaling regimes. For the considered scaling regimes, these expansions then lead to an asymptotic evolution system. As in [8–10], in each case a family of asymptotic models may then be inferred by using the "BBM trick" and suitable changes of the dependent variables. This analysis recovers most of the systems which have been introduced in the literature and also some interesting new ones. For instance, in certain of the two-dimensional regimes, a nonlocal operator appears whose analog is not present in any of the one-dimensional cases.

All the systems derived are proved to be consistent with the full Euler system. In rough terms, this means that any solution of the latter solves any of the asymptotic systems up to a small error. The systems are thus seen to be formally equivalent models in terms of the small parameters that arise in the expansions. The advantage of obtaining a family of equivalent asymptotic systems is clear from the modeling perspective. One can use the flexibility inherent in having a class of models to adjust the linearized dispersion relation to better fit the exact dispersion and can choose horizontal velocity variables that are well suited to the predictions in view. Mathematically, the choice will be among those that are well posed for the particular initial-value or initial-boundary-value problem under consideration. When it comes to computer simulation, some of the systems are far better suited to the construction of stable, accurate numerical schemes and these would naturally be favored.

The paper is organized as follows. In the next portion of the Introduction, the "Zakharov formulation" of the full system is written in dimensionless form and the different scaling regimes which will be studied enunciated. In Section 1.5, a compendium of the outcome of our analysis is offered to guide the reader through the rest of the paper. Section 2 is devoted to the rigorous asymptotic analysis of the nonlocal, Dirichlet-to-Neumann operator $G[\zeta]$ and the interface operator $\mathbf{H}[\zeta]$ mentioned earlier. The asymptotic models that result from the use of the expansions of these two operators are introduced (and proved to be consistent with the full Euler system) in Section 3. The somewhat technical proof of Proposition 3 is given in Appendix A.

In the present paper, we have refrained from pursuing the analysis to the point of obtaining convergence results for the asymptotic systems to the full internal waves system. Such a program has been fully achieved in the case of surface waves by combining the results of [2] and [10]. What is needed to complete the circle of ideas in the internal wave case is a stability analysis of the asymptotic models derived here (that is, an estimation of the remainders which comprise the difference between the Euler system and the models). Together with consistency, a straightforward analysis would then provide a convergence result to the full Euler system, assuming that the large time existence results obtained by Alvarez-Samaniego and Lannes in [2] for the surface wave system are valid for the internal waves system. The latter point is far from obvious; indeed, even the local well-posedness of the Euler equations in the two-fluid configuration seems to be an open problem in the absence of surface tension (cf. [30] for the rigorous derivation of the Benjamin–Ono equation for the two-fluid system in the presence of surface tension).

Note finally that the analysis of the present paper could be extended to the case of a seabed with structure (a non-flat bottom, see [13] for the case of surface waves) and to the case of a two-layer system where the upper surface is free rather than restricted by the rigid lid hypothesis (see [3,19,28] for a derivation of asymptotic models in this situation). These issues are under study and an analysis will be reported separately. Of special interest is a comparison of the problem considered with the rigid-lid condition at the top and the problem wherein the upper surface is left free in the case where d_1 is relatively large.

Notation. Denote by X the d-dimensional horizontal variable as described earlier, where $d = 1, 2$. Thus, $X = x$ when $d = 1$ and $X = (x, y)$ when $d = 2$. We continue to use z for the vertical variable.

The usual symbols ∇ and Δ connote the gradient and Laplace operator in the horizontal variables, whereas $\nabla_{X,z}$ and $\Delta_{X,z}$ are their $(d+1)$-variable version (the gradient in both or all three variables, depending on whether $d = 1$ or 2 and similarly for the Laplacian). For $\mu > 0$, it is very convenient to also introduce scaled versions of the gradient and Laplace operators, namely $\nabla^{\mu}_{X,z} = (\sqrt{\mu}\nabla^T, \partial_z)^T$ and $\Delta^{\mu}_{X,z} = \nabla^{\mu}_{X,z} \cdot \nabla^{\mu}_{X,z} = \mu\Delta + \partial_z^2$.

For any tempered distribution u, denote by $\widehat{u}$ or $\mathcal{F}u$ its Fourier transform. If f and u are two functions defined on $\mathbb{R}^d$, we use the Fourier multiplier notation $f(D)u$ which is defined in terms of Fourier transforms, viz.

$$\widehat{f(D)u} = f\widehat{u}.$$

The projection onto gradient fields in $L^2(\mathbb{R}^d)^d$ is written Π and is defined by the formula:

$$\Pi = -\frac{\nabla\nabla^T}{|D|^2}.$$

(Note that $\Pi = Id$ when $d = 1$.) The operator $\Lambda = (1 - \Delta)^{1/2}$ is equivalently defined using the Fourier multiplier notation to be $\Lambda = (1 + |D|^2)^{1/2}$. Appearing frequently are the Fourier multipliers $\mathbf{T}_\mu$ and $\mathbf{T}_{\mu_2}$, given by:

$$\mathbf{T}_\mu = \tanh\big(\sqrt{\mu}|D|\big) \quad \text{and} \quad \mathbf{T}_{\mu_2} = \tanh\big(\sqrt{\mu}_2|D|\big),$$

where $\mu, \mu_2 > 0$.

The standard notation $H^s(\mathbb{R}^d)$, or simply H^s if the underlying domain is clear from the context, is used for the L^2-based Sobolev spaces; their norm is written $|\cdot|_{H^s}$.

The planar strip $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$ appears often. The unadorned norm $\|\cdot\|$ will always be the usual norm of $L^2(\mathcal{S})$.

1.2. The equations

The Euler system of equations for our system is reviewed here. As in Fig. 1, the origin of the vertical coordinate z is taken at the rigid top of the two-fluid system. Assuming each fluid is incompressible and each flow irrotational, there exists velocity potentials Φ_i $(i = 1, 2)$ associated to both the upper and lower fluid layers which satisfy:

$$\Delta_{X,z}\Phi_i = 0 \quad \text{in } \Omega^i_t, \tag{1}$$

for all time t, where Ω^i_t denotes the region occupied by fluid i at time t, $i = 1, 2$. As above, fluid 1 refers to the upper fluid layer whilst fluid 2 is the lower layer (see again Fig. 1). Assuming that the densities ρ_i, $i = 1, 2$, of both fluids are constant, we also have two Bernouilli equations, namely,

$$\partial_t\Phi_i + \frac{1}{2}|\nabla_{X,z}\Phi_i|^2 = -\frac{P}{\rho_i} - gz \quad \text{in } \Omega^i_t, \tag{2}$$

where g denotes the acceleration of gravity and P the pressure inside the fluid. These equations are complemented by two boundary conditions stating that the velocity must be horizontal at the two rigid surfaces $\Gamma_1 := \{z = 0\}$ and $\Gamma_2 := \{z = -(d_1 + d_2)\}$, which is to say,

$$\partial_z\Phi_i = 0 \quad \text{on } \Gamma_i \ (i = 1, 2). \tag{3}$$

Finally, as mentioned earlier, it is presumed that the interface is given as the graph of a function $\zeta(t,X)$ which expresses the deviation of the interface from its rest position $(X,-d_1)$ at the spatial coordinate X at time t. The interface $\Gamma_t := \{z = -d_1 + \zeta(t,X)\}$ between the fluids is taken to be a bounding surface, or equivalently it is assumed that no fluid particle crosses the interface. This condition, written for fluid i, is classically expressed by the relation $\partial_t \zeta = \sqrt{1+|\nabla\zeta|^2} v_n^i$, where v_n^i denotes the upwards normal derivative of the velocity of fluid i at the surface. Since this equation must of course be independent of which fluid is being contemplated, it follows that the normal component of the velocity is continuous at the interface. The two equations,

$$\partial_t \zeta = \sqrt{1+|\nabla\zeta|^2}\partial_n \Phi_1 \quad \text{on } \Gamma_t, \tag{4}$$

and

$$\partial_n \Phi_1 = \partial_n \Phi_2 \quad \text{on } \Gamma_t, \tag{5}$$

with

$$\partial_n := \mathbf{n}\cdot\nabla_{X,z} \quad \text{and} \quad \mathbf{n} := \frac{1}{\sqrt{1+|\nabla\zeta|^2}}(-\nabla\zeta, 1)^T,$$

follow as a consequence. A final condition is needed on the pressure to close this set of equations, namely,

$$P \text{ is continuous at the interface.} \tag{6}$$

1.3. Transformation of the equations

In this subsection, a new set of equations is deduced from the internal-wave equations (1)–(6). Introduce the trace of the potentials Φ_1 and Φ_2 at the interface,

$$\psi_i(t,X) := \Phi_i\big(t, X, -d_1 + \zeta(t,X)\big) \quad (i = 1,2).$$

One can evaluate Eq. (2) at the interface and use (4) and (5) to obtain a set of equations coupling ζ to ψ_i $(i = 1,2)$, namely

$$\partial_t \zeta - \sqrt{1+|\nabla\zeta|^2}\partial_n \Phi_i = 0, \tag{7}$$

$$\rho_i\left(\partial_t \psi_i + g\zeta + \frac{1}{2}|\nabla\psi_i|^2 - \frac{(\sqrt{1+|\nabla\zeta|^2}(\partial_n\Phi_i) + \nabla\zeta\cdot\nabla\psi_i)^2}{2(1+|\nabla\zeta|^2)}\right) = -P, \tag{8}$$

where in (7) and (8), $(\partial_n \Phi_i)$ and P are both evaluated at the interface $z = -d_1 + \zeta(t,X)$. Notice that $\partial_n \Phi_1$ is fully determined by ψ_1 since Φ_1 is uniquely given as the solution of Laplace's equation (1) in the upper fluid domain, the Neumann condition (3) on Γ_1 and the Dirichlet condition $\Phi_1 = \psi_1$ at the interface. Following the formalism introduced for the study of surface water waves in [18,19,36], we can therefore define the Dirichlet–Neumann operator $G[\zeta]\cdot$ by

$$G[\zeta]\psi_1 = \sqrt{1+|\nabla\zeta|^2}(\partial_n \Phi_1)\big|_{z=-d_1+\zeta}.$$

Similarly, one remarks that ψ_2 is determined up to a constant by ψ_1 since Φ_2 is given (up to a constant) by the resolution of the Laplace equation (1) in the lower fluid domain, with Neumann boundary conditions (3) on Γ_2 and $\partial_n\Phi_2 = \partial_n\Phi_1$ at the interface (this latter being provided by (5)). It follows that ψ_1 fully determines $\nabla\psi_2$ and we may thus define the operator $\mathbf{H}[\zeta]\cdot$ by

$$\mathbf{H}[\zeta]\psi_1 = \nabla\psi_2.$$

Using the continuity of the pressure at the interface expressed in (6), the left-hand sides of $(8)_1$ and $(8)_2$ may be equated using the operators $G[\zeta]$ and $\mathbf{H}[\zeta]$ just defined. This yields the equation

$$\partial_t(\psi_2 - \gamma\psi_1) + g(1-\gamma)\zeta + \frac{1}{2}\big(|\mathbf{H}[\zeta]\psi_1|^2 - \gamma|\nabla\psi_1|^2\big) + \mathcal{N}(\zeta,\psi_1) = 0,$$

where $\gamma = \rho_1/\rho_2$ and

$$\mathcal{N}(\zeta,\psi_1) := \frac{\gamma(G[\zeta]\psi_1 + \nabla\zeta\cdot\nabla\psi_1)^2 - (G[\zeta]\psi_1 + \nabla\zeta\cdot\mathbf{H}[\zeta]\psi_1)^2}{2(1+|\nabla\zeta|^2)}.$$

Taking the gradient of this equation and using (7) then gives the system of equations

$$\begin{cases} \partial_t\zeta - G[\zeta]\psi_1 = 0, \\ \partial_t\big(\mathbf{H}[\zeta]\psi_1 - \gamma\nabla\psi_1\big) + g(1-\gamma)\nabla\zeta + \frac{1}{2}\nabla\big(\big|\mathbf{H}[\zeta]\psi_1\big|^2 - \gamma|\nabla\psi_1|^2\big) + \nabla\mathcal{N}(\zeta,\psi_1) = 0, \end{cases} \tag{9}$$

for ζ and ψ_1. This is the system of equations that will be used in the next sections to derive asymptotic models.

Remark 1. More precise definitions of the operators $G[\zeta]$ and $\mathbf{H}[\zeta]$ will be presented in Sections 1.4 and 2.

Remark 2. Setting $\rho_1 = 0$, and thus $\gamma = 0$, in the above equations, one recovers the usual surface water-wave equations written in terms of ζ and ψ as in [18,19,36].

1.4. Non-dimensionalization of the equations

The asymptotic behavior of (9) is more transparent when these equations are written in dimensionless variables. Denoting by a a typical amplitude of the deformation of the interface in question, and by λ a typical wavelength, the dimensionless independent variables

$$\widetilde{X} := \frac{X}{\lambda}, \qquad \widetilde{z} := \frac{z}{d_1}, \qquad \widetilde{t} := \frac{t}{\lambda/\sqrt{gd_1}},$$

are introduced. Likewise, we define the dimensionless unknowns

$$\widetilde{\zeta} := \frac{\zeta}{a}, \qquad \widetilde{\psi}_1 := \frac{\psi_1}{a\lambda\sqrt{g/d_1}},$$

as well as the dimensionless parameters

$$\gamma := \frac{\rho_1}{\rho_2}, \qquad \delta := \frac{d_1}{d_2}, \qquad \varepsilon := \frac{a}{d_1}, \qquad \mu := \frac{d_1^2}{\lambda^2}.$$

Though they are redundant, it is also notationally convenient to introduce two other parameters ε_2 and μ_2 defined as

$$\varepsilon_2 = \frac{a}{d_2} = \varepsilon\delta, \qquad \mu_2 = \frac{d_2^2}{\lambda^2} = \frac{\mu}{\delta^2}.$$

Remark 3. The parameters ε_2 and μ_2 correspond to ε and μ with d_2 rather than d_1 taken as the unit of length in the vertical direction.

Before writing (9) in dimensionless variables, a dimensionless Dirichlet–Neumann operator $G^\mu[\varepsilon\zeta]\cdot$ is needed, associated to the non-dimensionalized upper fluid domain,

$$\Omega_1 = \big\{(X,z)\in\mathbb{R}^{d+1}\colon -1+\varepsilon\zeta(X) < z < 0\big\}.$$

Throughout the discussion, it will be presumed that this domain remains connected, so there is a positive value H_1 such that

$$1 - \varepsilon\zeta \geqslant H_1 \quad \text{on } \mathbb{R}^d. \tag{10}$$

Definition 1. Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that (10) is satisfied and let $\psi_1 \in H^{3/2}(\mathbb{R}^d)$. If Φ_1 is the unique solution in $H^2(\Omega_1)$ of the boundary value problem,

$$\begin{cases} \mu\Delta\Phi_1 + \partial_z^2\Phi_1 = 0 \quad \text{in } \Omega_1, \\ \partial_z\Phi_1|_{z=0} = 0, \qquad \Phi_1|_{z=-1+\varepsilon\zeta(X)} = \psi_1, \end{cases} \tag{11}$$

then $G^\mu[\varepsilon\zeta]\psi_1 \in H^{1/2}(\mathbb{R}^d)$ is defined by

$$G^\mu[\varepsilon\zeta]\psi_1 = -\mu\varepsilon\nabla\zeta\cdot\nabla\Phi_1|_{z=-1+\varepsilon\zeta} + \partial_z\Phi_1|_{z=-1+\varepsilon\zeta}.$$

Remark 4. Another way to approach G^μ is to define

$$G^\mu[\varepsilon\zeta]\psi_1 = \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_1|_{z=-1+\varepsilon\zeta},$$

where $\partial_n\Phi_1|_{z=-1+\varepsilon\zeta}$ stands for the upper conormal derivative associated to the elliptic operator $\mu\Delta\Phi_1+\partial_z^2\Phi_1$.

In the same vein, one may define a dimensionless operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ associated to the non-dimensionalized lower fluid domain

$$\Omega_2 = \{(X,z)\in\mathbb{R}^{d+1}, -1-1/\delta < z < -1+\varepsilon\zeta(X)\},$$

where it is assumed as in (10) that there is an $H_2>0$ such that

$$1+\varepsilon\delta\zeta \geqslant H_2 \quad \text{on } \mathbb{R}^d. \tag{12}$$

Definition 2. Let $\zeta\in W^{2,\infty}(\mathbb{R}^d)$ be such that (10) and (12) are satisfied, and suppose that $\psi_1\in H^{3/2}(\mathbb{R}^d)$ is given. If the function Φ_2 is the unique solution (up to a constant) of the boundary value problem

$$\begin{cases} \mu\Delta\Phi_2+\partial_z^2\Phi_2 = 0 \quad \text{in } \Omega_2, \\ \partial_z\Phi_2|_{z=-1-1/\delta} = 0, \qquad \partial_n\Phi_2|_{z=-1+\varepsilon\zeta(X)} = \dfrac{1}{(1+\varepsilon^2|\nabla\zeta|^2)^{1/2}}G^\mu[\varepsilon\zeta]\psi_1, \end{cases} \tag{13}$$

then the operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ is defined on ψ_1 by,

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \nabla(\Phi_2|_{z=-1+\varepsilon\zeta}) \in H^{1/2}(\mathbb{R}^d).$$

Remark 5. In the statement above, $\partial_n\Phi_2|_{z=-1+\varepsilon\zeta}$ stands here for the upwards conormal derivative associated to the elliptic operator $\mu\Delta\Phi_2+\partial_z^2\Phi_2$, so that

$$\sqrt{1+\varepsilon^2|\nabla\zeta|^2}\partial_n\Phi_2|_{z=-1+\varepsilon\zeta} = -\mu\varepsilon\nabla\zeta\cdot\nabla\Phi_2|_{z=-1+\varepsilon\zeta} + \partial_z\Phi_2|_{z=-1+\varepsilon\zeta}.$$

The Neumann boundary condition of (13) at the interface can also be stated as $\partial_n\Phi_2|_{z=-1+\varepsilon\zeta} = \partial_n\Phi_1|_{z=-1+\varepsilon\zeta}$.

Remark 6. Of course, the solvability of (13) requires the condition $\int_\Gamma \partial_n\Phi_2\,d\Gamma = 0$ (where $d\Gamma = \sqrt{1+\varepsilon^2|\nabla\zeta|^2}\,dX$ is the Lebesgue measure on the surface $\Gamma = \{z=-1+\varepsilon\zeta\}$). This is automatically satisfied thanks to the definition of $G^\mu[\varepsilon\zeta]\psi_1$. Indeed, applying Green's identity to (11), one obtains

$$\int_\Gamma \partial_n\Phi_2\,d\Gamma = \int_\Gamma \partial_n\Phi_1\,d\Gamma = -\int_{\Omega_1}(\mu\Delta\Phi_1+\partial_z^2\Phi_1) = 0.$$

Example 1. The operators $G^\mu[\varepsilon\zeta]\cdot$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$ have explicit expressions when the interface is flat (i.e. when $\zeta=0$). In that case, taking the horizontal Fourier transform of the Laplace equations (11) and (13) transforms them into ordinary differential equations with respect to z which can easily be solved to obtain

$$G^\mu[0]\psi = -\sqrt{\mu}|D|\tanh(\sqrt{\mu}|D|)\psi \quad \text{and} \quad \mathbf{H}^{\mu,\delta}[0]\psi = -\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\frac{\sqrt{\mu}}{\delta}|D|)}\nabla\psi.$$

Eqs. (9) can therefore be written in dimensionless variables as

$$\begin{cases} \partial_{\tilde t}\tilde\zeta - \dfrac{1}{\mu}G^\mu[\varepsilon\tilde\zeta]\tilde\psi_1 = 0, \\ \partial_{\tilde t}(\mathbf{H}^{\mu,\delta}[\varepsilon\tilde\zeta]\tilde\psi_1 - \gamma\nabla\tilde\psi_1) + (1-\gamma)\nabla\tilde\zeta + \dfrac{\varepsilon}{2}\nabla(|\mathbf{H}^{\mu,\delta}[\varepsilon\tilde\zeta]\tilde\psi_1|^2 - \gamma|\nabla\tilde\psi_1|^2) + \varepsilon\nabla\mathcal{N}^{\mu,\delta}(\varepsilon\tilde\zeta,\tilde\psi_1) = 0, \end{cases} \tag{14}$$

where $\mathcal{N}^{\mu,\delta}$ is defined for all pairs (ζ,ψ) smooth enough by the formula

$$\mathcal{N}^{\mu,\delta}(\zeta,\psi) := \mu\frac{\gamma(\frac{1}{\mu}G^\mu[\zeta]\psi+\nabla\zeta\cdot\nabla\psi)^2 - (\frac{1}{\mu}G^\mu[\zeta]\psi+\nabla\zeta\cdot\mathbf{H}^{\mu,\delta}[\zeta]\psi)^2}{2(1+\mu|\nabla\zeta|^2)}.$$

Our work centers around the study of the asymptotics of the non-dimensionalized equations (14) in various physical regimes corresponding to different relationships among the dimensionless parameter‘s ε, μ and δ.

Notation 1. The tildes which indicate the non-dimensional quantities will be systematically dropped henceforth.

Remark 7. Linearizing Eqs. (14) around the rest state, one finds the equations:

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}G^{\mu}[0]\psi_1 = 0, \\ \partial_t\big(\mathbf{H}^{\mu,\delta}[0]\psi_1 - \gamma\nabla\psi_1\big) + (1-\gamma)\nabla\zeta = 0. \end{cases}$$

The explicit formulas in Example 1 thus allow one to calculate the linearized dispersion relation,

$$\omega^2 = (1-\gamma)\frac{|\mathbf{k}|}{\sqrt{\mu}}\frac{\tanh(\sqrt{\mu}|\mathbf{k}|)\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}{\tanh(\sqrt{\mu}|\mathbf{k}|) + \gamma\tanh(\frac{\sqrt{\mu}}{\delta}|\mathbf{k}|)}, \tag{15}$$

corresponding to plane-wave solutions $e^{i\mathbf{k}\cdot X - i\omega t}$. In particular, the expected instability is found when $\gamma > 1$, corresponding to the case wherein the heavier fluid lies over the lighter one. One also checks that the classical dispersion relation,

$$\omega^2 = \frac{1}{\sqrt{\mu}}|\mathbf{k}|\tanh\big(\sqrt{\mu}|\mathbf{k}|\big),$$

for surface water waves is recovered when $\gamma = 0$ and $\delta = 1$.

1.5. Principal results

The overall goal here is to propose model systems of equations for describing the motion of internal waves by obtaining the asymptotic form of Eqs. (14) in various regimes corresponding to different values of the parameters ε, δ and μ. All these asymptotic models are $(1+d)$-dimensional systems coupling the surface elevation ζ to the variable $\mathbf{v}$ defined to be,

$$\mathbf{v} := \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1. \tag{16}$$

(For the surface water-wave problem formally recovered by taking $\gamma = 0$ and $\delta = 1$, $\mathbf{v}$ is the horizontal velocity evaluated at the free surface.) We will often refer to $\mathbf{v}$ as the velocity variable, though its precise interpretation will vary. Note that $\mathbf{v}$ is essentially the gradient of the second canonical variable in the Hamiltonian formulation of (14), (see for instance [6]).

It will be rigorously established that the internal-wave equations (14) are *consistent* with the asymptotic models for $(\zeta, \mathbf{v})$ derived in this paper in the following precise sense.

Definition 3. The internal-wave equations (14) are *consistent* with a system S of $d+1$ equations for ζ and $\mathbf{v}$ if for all sufficiently smooth solutions (ζ, ψ_1) of (14) such that (10) and (12) are satisfied, the pair $(\zeta, \mathbf{v} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \gamma\nabla\psi_1)$ solves S up to a small residual called the *precision* of the asymptotic model.

Remark 8. It is worth emphasis that the above definition does not require the well-posedness of the internal wave equations (14). Indeed, these can be subject to Kelvin–Helmholtz type instabilities (see for instance [4] and [21]), although one might expect a “stability of the instability” result even in the face of such instabilities (see [16]). *Consistency* is only concerned with the properties of smooth solutions to the system (which do exist in the classical configuration of the Kelvin–Helmholtz problem, even when instabilities manifest themselves; see e.g. [33,32]). In fact, the two-layer water-wave system is known to be well-posed in Sobolev spaces in the presence of surface tension [21]. In consequence, one could simply add a small amount of surface tension at the interface between the two homogeneous layers to put oneself in a well-posed situation. The resulting analysis would be exactly the same and would, in fact, lead to the same asymptotic models. (Such an approach is used in [30] for the Benjamin–Ono equation.) As the resulting model systems do not change, such a regularization has been eschewed here.

Here is a summary of the different asymptotic regimes investigated in this paper. It is convenient to organize the discussion around the parameters ε and $\varepsilon_2 = \varepsilon\delta$ (the nonlinearity, or amplitude, parameters for the upper and lower fluids, respectively), and in terms of μ and $\mu_2 = \frac{\mu}{\delta^2}$ (the long-wavelength parameters for the upper and lower fluids). Notice that the assumptions made about δ are therefore implicit.

The interfacial wave is said to be of *small amplitude* for the upper fluid layer (resp., the lower layer) if $\varepsilon \ll 1$ (resp., $\varepsilon_2 \ll 1$) and the upper (resp., lower) layer is said to be *shallow* if $\mu \ll 1$ (resp., $\mu_2 \ll 1$). This terminology is consistent with the usual one for surface water waves (recovered by taking $\rho_1 = 0$ and $\delta = 1$). In the discussion below, the notation *regime 1/regime 2* means that the wave motion is such that the upper layer is in regime 1 (small amplitude or shallow water) and the lower one is in regime 2.

1. The small-amplitude/small-amplitude regime: $\varepsilon \ll 1$, $\varepsilon_2 \ll 1$. This regime corresponds to interfacial deformations which are small for both the upper and lower fluid domains. Various sub-regimes are defined by making further assumptions about the size of μ and μ_2.
   (a) The Full Dispersion/Full Dispersion (FD/FD) regime: $\varepsilon \sim \varepsilon_2 \ll 1$ and $\mu \sim \mu_2 = O(1)$ (and thus $\delta \sim 1$). In this regime, investigated in Section 3.1.1, the shallowness parameters are not small for either of the fluid domains, and the full dispersive effects must therefore be kept for both regions; the asymptotic model corresponding to this situation is given in (26).
   (b) The Boussinesq/Full Dispersion (B/FD) regime: $\mu \sim \varepsilon \ll 1$, $\mu_2 \sim 1$. This regime is studied in Section 3.1.2 and corresponds to the case where the flow has a Boussinesq structure in the upper part (and thus dispersive effects of the same order as nonlinear effects), but with a shallowness parameter not small in the lower fluid domain. This configuration occurs when $\delta^2 \sim \varepsilon$, that is, when the lower region is much larger than the upper one. A further analysis of the asymptotic model yields a three-parameter family of equivalent systems (see (27) below).
   (c) The Boussinesq/Boussinesq (B/B) regime: $\mu \sim \mu_2 \sim \varepsilon \sim \varepsilon_2 \ll 1$. In this regime, investigated in Section 3.1.3, one has $\delta \sim 1$ and the flow has a Boussinesq structure in both the upper and lower fluid domains. Here again, a three-parameter family of asymptotic systems is obtained (see (28) below).
2. The Shallow Water/Shallow Water (SW/SW) regime: $\mu \sim \mu_2 \ll 1$. This regime, which allows relatively large interfacial amplitudes ($\varepsilon \sim \varepsilon_2 = O(1)$), does not belong to the regimes singled out above. The structure of the flow is then of shallow water type in both regions; in particular, the asymptotic model (see Section 3.2) is a nonlinear, but non-dispersive system, given in (29), which degenerates into the usual shallow water equations when $\gamma = 0$ and $\delta = 1$. It is very interesting in this case that a nonlocal term arises when $d = 2$. Such a nonlocal term does not appear in the one-dimensional case, nor in the two-dimensional shallow water equations for surface waves.
3. The Shallow Water/Small Amplitude (SW/SA) regime: $\mu \ll 1$ and $\varepsilon_2 \ll 1$. In this regime, the upper layer is shallow (but with possibly large surface deformations), and the surface deformations are small for the lower layer (but it can be deep). Various sub-regimes arise in this case also.
   (a) The Shallow Water/Full Dispersion (SW/FD) regime: $\mu \sim \varepsilon_2^2 \ll 1$, $\varepsilon \sim \mu_2 \sim 1$. This regime is investigated in Section 3.3.1. The dispersive effects are negligible in the upper fluid, but the full dispersive effects must be kept in the lower one (see system (31) below).
   (b) The Intermediate Long Waves (ILW) regime: $\mu \sim \varepsilon^2 \sim \varepsilon_2 \ll 1$, $\mu_2 \sim 1$. In this regime, the interfacial deformations are also small for the upper fluid (which is not the case in the SW/FD regime). This allows some simplifications, as shown in Section 3.3.2. It is also possible (see (32)) to derive a one-parameter family of equivalent systems.
   (c) The Benjamin–Ono (BO) regime: $\mu \sim \varepsilon^2 \ll 1$, $\mu_2 = \infty$. A formal study of this regime is performed in Section 3.3.3. It is shown in particular how to recover the Benjamin–Ono equation as the unidirectional limit in the one-dimensional case $d = 1$. The Benjamin–Ono equation is also shown to be a particular case of a one-parameter family of *regularized Benjamin–Ono equations*, given in (34).

The range of validity of these regimes is summarized in Table 1.

Remark 9. The small amplitude/shallow water regime is not investigated here. It corresponds to the situation where the upper fluid domain is much larger than the lower one, which is more of an atmospheric configuration than an oceanographic case.

Table 1

	$\varepsilon = O(1)$	$\varepsilon \ll 1$
$\mu = O(1)$	Full equations	$\delta \sim 1$: FD/FD eqs.
$\mu \ll 1$	$\delta \sim 1$: SW/SW eqs. $\delta^2 \sim \mu \sim \varepsilon_2^2$: SW/FD eqs.	$\mu \sim \varepsilon$ and $\delta^2 \sim \varepsilon$: B/FD eqs. $\mu \sim \varepsilon$ and $\delta \sim 1$: B/B eqs. $\delta^2 \sim \mu \sim \varepsilon^2$: ILW eqs. $\delta = 0$ and $\mu \sim \varepsilon^2$: BO eqs.

2. Asymptotic expansions of the operators

In this section, asymptotic expansions are given of the central operators defined in the Introduction. The discussion begins with the Dirichlet–Neumann operator.

2.1. Asymptotic expansion of the Dirichlet–Neumann operator $G^\mu[\varepsilon\zeta]\cdot$

The following lemma connects ζ with the vertically integrated horizontal velocity via the Dirichlet–Neumann operator $G^\mu[\varepsilon\zeta]\cdot$.

Lemma 1. *Let $\zeta \in W^{2,\infty}(\mathbb{R}^d)$ be such that* (10) *is satisfied and let $\psi \in H^{3/2}(\mathbb{R}^d)$ and Φ_1 be the solution of* (11) *with $\psi_1 = \psi$. If V^μ is defined by*

$$V^\mu[\varepsilon\zeta]\psi := \int_{-1+\varepsilon\zeta}^{0} \left(\sqrt{\mu}\nabla\Phi_1\right) dz,$$

then it follows that

$$G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla \cdot \left(V^\mu[\varepsilon\zeta]\psi\right).$$

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a test function. Using Green's identity, and with the notation of Remark 6, one obtains

$$\int_{\mathbb{R}^d} G^\mu[\varepsilon\zeta]\psi\varphi = \int_\Gamma \partial_n\Phi_1\varphi\, d\Gamma = -\int_{\Omega_1} \left(\sqrt{\mu}\nabla\right)\Phi_1 \cdot \left(\sqrt{\mu}\nabla\right)\varphi$$
$$= -\int_{\mathbb{R}^d} \int_{-1+\varepsilon\zeta}^{0} \left(\sqrt{\mu}\nabla\Phi_1\right) dz \cdot \sqrt{\mu}\nabla\varphi.$$

Defining $V^\mu[\varepsilon\zeta]\psi$ as in the statement of the lemma, one finds that

$$\int_{\mathbb{R}^d} G^\mu[\varepsilon\zeta]\psi\varphi = -\sqrt{\mu}\int_{\mathbb{R}^d} V^\mu[\varepsilon\zeta]\psi \cdot \nabla\varphi = \sqrt{\mu}\int_{\mathbb{R}^d} \nabla \cdot \left(V^\mu[\varepsilon\zeta]\psi\right)\varphi.$$

Since the above identity is true for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, the result follows. □

Remark 10. In Sections 2.1.1 and 2.1.2 below, asymptotic expansions are obtained of $V^\mu[\varepsilon\zeta]\psi$ in terms of ε and μ, respectively. Because of Lemma 1, asymptotic expansions of $G^\mu[\varepsilon\zeta]\psi$ then follow immediately.

2.1.1. Asymptotic expansion of $V^\mu[\varepsilon\zeta]\cdot$ when $\boldsymbol{\varepsilon} \ll 1$

When $\varepsilon \ll 1$, the approach to obtaining an asymptotic expansion of $V^\mu[\varepsilon\zeta]\psi$ is to make a Taylor expansion in terms of the interface deformation around the rest state, viz.

$$V^\mu[\varepsilon\zeta]\psi = V^\mu[0]\psi + \varepsilon\left(d_0\left(V^\mu[\cdot]\right)\zeta\right)\psi + \cdots.$$

(Note, however, that the expansion of $V^\mu[\varepsilon\zeta]\psi$ itself, and not only the consequent expansion of $G^\mu[\varepsilon\zeta]\psi$, is needed so that the elliptic estimate of Proposition 3 will be used in the proof of Corollary 1.)

Proposition 1. *Let $s > d/2$ and $\zeta \in H^{s+3/2}(\mathbb{R}^d)$ be such that* (10) *is satisfied. Then for ψ such that $\nabla\psi \in H^{s+1/2}(\mathbb{R}^d)$, the inequality,*

$$\big|V^\mu[\varepsilon\zeta]\psi - \big[\mathcal{T}_{0,\mu}\nabla\psi + \varepsilon\sqrt{\mu}\big(-\zeta + \mathcal{T}_{1,\mu}[\zeta]\big)\nabla\psi\big]\big|_{H^s} \leqslant \varepsilon^2 C\left(\frac{1}{H_1}, \varepsilon\sqrt{\mu}, |\zeta|_{H^{s+3/2}}, |\nabla\psi|_{H^{s+1/2}}\right),$$

holds for all $\varepsilon \in [0,1]$ and $\mu > 0$, where $\mathcal{T}_{0,\mu} = \frac{\tanh(\sqrt{\mu}|D|)}{|D|}$, $\mathcal{T}_{1,\mu}[\zeta] = -\nabla\mathcal{T}_{0,\mu}(\zeta\mathcal{T}_{0,\mu}\nabla^T)$, and $V^\mu[\varepsilon\zeta]\psi$ is as defined in Lemma 1 *(so that $G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla\cdot V^\mu[\varepsilon\zeta]\psi$).*

The key ingredient in the proof is an *explicit* formula of the derivative of the mapping $\zeta \mapsto V^\mu[\varepsilon\zeta]\psi$, which generalizes the formula obtained in [26] for the shape derivative of Dirichlet–Neumann operators. This interesting technical point is the subject of the next lemma.

Lemma 2. *Let $s > d/2$ and suppose that ψ is such that $\nabla\psi \in H^{s+1/2}(\mathbb{R}^d)$. The mapping $H^{s+3/2}(\mathbb{R}^d) \ni \zeta \mapsto V^\mu[\varepsilon\zeta]\psi \in H^{s+1/2}(\mathbb{R}^d)^d$ is differentiable. Moreover, for all $\zeta, \zeta' \in H^s(\mathbb{R}^d)$, the derivative of $V^\mu[\varepsilon\cdot]\psi$ at ζ in the direction ζ' is given by the formula*

$$d_\zeta\big(V^\mu[\varepsilon\cdot]\psi\big)\zeta' = -\varepsilon V^\mu[\varepsilon\zeta]\big(\zeta' Z^\mu[\varepsilon\zeta]\psi\big) - \varepsilon\zeta'\big(\sqrt{\mu}\nabla\psi - \varepsilon\sqrt{\mu}\nabla\zeta Z^\mu[\varepsilon\zeta]\psi\big),$$

where $Z^\mu[\varepsilon\zeta]\psi := \frac{1}{1+\varepsilon^2\mu|\nabla\zeta|^2}(G^\mu[\varepsilon\zeta]\psi + \varepsilon\mu\nabla\zeta\cdot\nabla\psi)$.

Proof of the lemma. First, define another Dirichlet–Neumann operator $\mathcal{G}^\mu[\varepsilon\zeta]\cdot$ by

$$\mathcal{G}^\mu[\varepsilon\zeta]\psi = \mathbf{e}_z\cdot P^\mu[\varepsilon\zeta]\nabla^{\mu_2}_{X,z}\Phi|_{z=0}, \tag{17}$$

where Φ solves

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot P^\mu[\varepsilon\zeta]\nabla^{\mu_2}_{X,z}\Phi = 0 & \text{in } -1 < z < 0, \\ \Phi|_{z=0} = \psi, \qquad \partial_z\Phi|_{z=-1} = 0, \end{cases} \tag{18}$$

and where

$$P^\mu[\varepsilon\zeta] = \begin{pmatrix} (1+\varepsilon\zeta)I_{d\times d} & -\varepsilon\sqrt{\mu}(z+1)\nabla\zeta \\ -\varepsilon\sqrt{\mu}(z+1)\nabla\zeta^T & \frac{1+\varepsilon^2\mu(1+z^2)|\nabla\zeta|^2}{1+\varepsilon\zeta} \end{pmatrix}.$$

This operator is the classical Dirichlet–Neumann operator often used for the study of the surface water-wave equations and for which an explicit expression exists for the derivative of the mapping $\zeta \mapsto \mathcal{G}^\mu[\varepsilon\zeta]\psi$ (see, e.g., Theorem 3.20 of [26] and Theorem 3.1 of [2]). Studying the transformation of the fluid domain into the flat strip $-1 < z < 0$ (flattening of the domain) reveals that $G^\mu[\varepsilon\zeta]\psi = -\mathcal{G}^\mu[-\varepsilon\zeta]\psi$ (see Proposition 2.7 of [26] and Section 2.2 below where the same kind of transformation is performed). It will be convenient to consider the operator $\mathcal{G}^\mu[-\varepsilon\zeta]\cdot$ rather than $G^\mu[\varepsilon\zeta]\cdot$, because this allow us to take over intact some elements of the proof of Theorem 3.20 in [26]. Moreover, for the sake of clarity, we take $\varepsilon = \mu = 1$ in this proof, setting $P^1 = P$, and leave to the reader the straightforward modifications for the general case. The proof is divided into 5 steps.

Step 1. One has that $\mathcal{G}[\zeta]\psi = -\nabla\cdot(\mathcal{V}[\zeta])$, with $\mathcal{V}[\zeta] = \int_{-1}^0 P_I[\zeta]\nabla_{X,z}\Phi\,dz$, and where $P_I[\zeta]$ is the $d\times(d+1)$ matrix obtained by taking the last row off $P[\zeta]$. The proof of this result is more or less identical to the proof of Lemma 1.

Step 2. Denoting by $\mathcal{V}'$ the derivative of $\mathcal{V}[\cdot]\psi$ at ζ and in the direction ζ', one computes that

$$\mathcal{V}' = \int_{-1}^{0} P_I'\nabla_{X,z}\Phi\,dz + \int_{-1}^{0} P_I[\zeta]\nabla_{X,z}\Phi'\,dz,$$

where P_I' and Φ' stand, respectively, for the derivative at ζ and in the direction ζ' of the mappings $\zeta \mapsto P_I[\zeta]$ and $\zeta \mapsto \Phi$.

Step 3. Defining $\chi = (z+1)\frac{\zeta'}{1+\zeta}\partial_z \Phi$, one has

$$\int_{-1}^{0} P_I[\zeta]\nabla_{X,z}(\Phi' - \chi) = -\mathcal{V}[\zeta]\big(\zeta'\mathcal{Z}[\zeta]\psi\big),$$

with $\mathcal{Z}[\zeta]\psi = \frac{\mathcal{G}[\zeta]\psi + \nabla\zeta\cdot\nabla\psi}{1+|\nabla\zeta|^2}$. To prove this result, first remark that $w := \Phi' - \chi$ solves the boundary-value problem

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot P[\zeta]\nabla^{\mu_2}_{X,z} w = 0 & \text{in } -1 < z < 0, \\ w|_{z=0} = -\zeta'\mathcal{Z}[\zeta]\psi, & \partial_z w|_{z=-1} = 0, \end{cases}$$

as a consequence of Lemma 3.22 of [26]. The result then follows directly from the definition of $\mathcal{V}[\zeta]\cdot$.

Step 4. The identity,

$$\int_{-1}^{0} \big(P_I'\nabla_{X,z}\Phi + P_I[\zeta]\nabla_{X,z}\chi\big)\,dz = \zeta'\big(\nabla\psi - \mathcal{Z}[\zeta]\psi\nabla\zeta\big),$$

also holds. To establish this, first compute that

$$P_I'\nabla_{X,z}\Phi + P_I[\zeta]\nabla_{X,z}\chi = \zeta'\partial_z\big((z+1)\nabla\Phi\big) - \nabla\zeta\,\partial_z\bigg(\frac{(z+1)^2}{1+\zeta}\zeta'\partial_z\Phi\bigg).$$

The result then follows upon integrating with respect to z.

Step 5. It now remains simply to put together the pieces. It is deduced from Steps 2–4 that

$$\mathcal{V}' = \zeta'\big(\nabla\psi - \mathcal{Z}[\zeta]\psi\nabla\zeta\big) - \mathcal{V}[\zeta]\big(\zeta'\mathcal{Z}[\zeta]\psi\big).$$

The result then follows from the observation that if $V[\zeta]\psi$ is as defined in Lemma 1, then $V[\zeta]\psi = \mathcal{V}[-\zeta]\psi$. $\square$

Proof of Proposition 1. A second order Taylor expansion reveals that

$$V^\mu[\varepsilon\zeta]\psi = V^\mu[0]\psi + d_0\big(V^\mu[\varepsilon\cdot]\psi\big)\zeta + \int_0^1 (1-z)d^2_{z\zeta}\big(V^\mu[\varepsilon\cdot]\psi\big)(\zeta,\zeta)\,dz.$$

Lemma 2 therefore implies that

$$V^\mu[\varepsilon\zeta]\psi = V^\mu[0]\psi - \varepsilon V^\mu[0]\big(\zeta G^\mu[0]\psi\big) - \varepsilon\sqrt{\mu}\zeta\nabla\psi + \int_0^1 (1-z)d^2_{z\zeta}\big(V^\mu[\varepsilon\cdot]\psi\big)(\zeta,\zeta)\,dz.$$

We saw in Example 1 that $G^\mu[0]\psi = -\sqrt{\mu}|D|\tanh(\sqrt{\mu}|D|)\psi$. Similarly, one can check that $V^\mu[0]\psi = \frac{\tanh(\sqrt{\mu}|D|)}{|D|}\nabla\psi$. The proof of the proposition is now clear after appreciating that

$$\bigg|\int_0^1 (1-z)d^2_{z\zeta}\big(V^\mu[\varepsilon\cdot]\psi\big)(\zeta,\zeta)\,dz\bigg|_{H^s} \leqslant \varepsilon^2 C\bigg(\frac{1}{H_1}, \varepsilon\sqrt{\mu}, |\zeta|_{H^{s+3/2}}, |\nabla\psi|_{H^{s+1/2}}\bigg),$$

a fact which is obtained exactly as in Proposition 3.3 of [2]. $\square$

2.1.2. Asymptotic expansion of $V^\mu[\varepsilon\zeta]\cdot$ for large-amplitude waves and shallow depth ($\varepsilon = O(1)$ and $\mu \ll 1$)

For larger amplitude waves, the expansion of the Dirichlet–Neuman operator $G^\mu[\varepsilon\zeta]\psi$ (and also of $V^\mu[\varepsilon\zeta]\psi$) around the rest state no longer provides an accurate approximation. However, if $\mu \ll 1$, which is what we have earlier called the shallow water regime for the upper fluid, it is possible to obtain an expansion of $V^\mu[\varepsilon\zeta]\psi$ (and thus of $G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla\cdot V^\mu[\varepsilon\zeta]\psi$) with respect to μ which is uniform with respect to $\varepsilon \in [0,1]$.

Proposition 2. *Let $s > d/2$ and $\zeta \in H^{s+3/2}(\mathbb{R}^d)$. Then for all $\mu \in (0,1)$ and ψ such that $\nabla\psi \in H^{s+5/2}(\mathbb{R}^d)$, one has*

$$\big|\sqrt{\mu}V^\mu[\varepsilon\zeta]\psi - \mu(1-\varepsilon\zeta)\nabla\psi\big|_{H^s} \leqslant \mu^2 C\big(|\zeta|_{H^{s+3/2}}, |\nabla\psi|_{H^{s+5/2}}\big),$$

uniformly with respect to $\varepsilon \in [0,1]$, where $V^\mu[\varepsilon\zeta]\psi$ is as defined in Lemma 1 (so that $G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla\cdot V^\mu[\varepsilon\zeta]\psi$).

Remark 11. As in Proposition 3.8 of [2], one can carry out the expansion explicitly to second order in μ, thereby obtaining

$$\sqrt{\mu}V^\mu[\varepsilon\zeta]\psi = \mu(1-\varepsilon\zeta)\nabla\psi + \frac{\mu^2}{3}\Delta\nabla\psi + O\big(\mu^3, \varepsilon\mu^2\big).$$

Proof. Recall that $G^\mu[\varepsilon\zeta]\psi = -\mathcal{G}^\mu[-\varepsilon\zeta]\psi$, where $\mathcal{G}^\mu[\varepsilon\zeta]\cdot$ is defined in (17), and that $\mathcal{G}^\mu[\varepsilon\zeta]\psi = -\sqrt{\mu}\nabla\cdot V^\mu[-\varepsilon\zeta]\psi$ (see the proof of Lemma 2). Proposition 3.8 of [2] shows that

$$\big|\mathcal{G}^\mu[\varepsilon\zeta]\psi - \nabla\cdot\big(-\mu(1+\varepsilon\zeta)\nabla\psi\big)\big|_{H^s} \leqslant \mu^2 C\big(|\zeta|_{H^{s+3/2}}, |\nabla\psi|_{H^{s+5/2}}\big).$$

An obvious adaptation of the proof shows that the estimate given in the statement of the proposition can be obtained in the same way. □

2.2. *Asymptotic expansions of* $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$

Attention is now turned to the interface operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$.

The boundary-value problem (13) plays a key role in the analysis of the operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\cdot$. The analysis of this problem is easier if we first transform it into a variable-coefficient, boundary-value problem on the flat strip $\mathcal{S} := \mathbb{R}^d \times (-1,0)$ using the diffeomorphism

$$\sigma: \begin{array}{l} \mathcal{S} \to \Omega_2, \\ (X,z) \mapsto \sigma(X,z) := \Big(X, (1+\varepsilon\delta\zeta)\dfrac{z}{\delta} + (-1+\varepsilon\zeta)\Big). \end{array}$$

As shown in Proposition 2.7 of [26] (see also Section 2.2 of [2]), Φ_2 solves (13) if and only if $\underline{\Phi}_2 := \Phi_2 \circ \sigma$ solves

$$\begin{cases} \nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_2 = 0 & \text{in } \mathcal{S}, \\ \partial_n\underline{\Phi}_2|_{z=0} = \dfrac{1}{\delta}G^\mu[\varepsilon\zeta]\psi_1, & \partial_n\underline{\Phi}_2|_{z=-1} = 0, \end{cases} \tag{19}$$

with

$$Q^{\mu_2}[\varepsilon_2\zeta] = \begin{pmatrix} (1+\varepsilon_2\zeta)I_{d\times d} & -\sqrt{\mu_2}\varepsilon_2(z+1)\nabla\zeta \\ -\sqrt{\mu_2}\varepsilon_2(z+1)\nabla\zeta^T & \frac{1+\mu_2\varepsilon_2^2(z+1)^2|\nabla\zeta|^2}{1+\varepsilon_2\zeta} \end{pmatrix},$$

and where, as before, $\varepsilon_2 = \varepsilon\delta$, $\mu_2 = \frac{\mu}{\delta^2}$, and $\nabla^{\mu_2}_{X,z} = (\sqrt{\mu_2}\nabla, \partial_z)^T$.

Remark 12. As always in the present exposition, $\partial_n\underline{\Phi}_2$ stands for the *upward* conormal derivative associated to the elliptic operator involved in the boundary-value problem,

$$\partial_n\underline{\Phi}_2|_{z=0 \text{ or } z=-1} = \mathbf{e}_z\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_2|_{z=0 \text{ or } z=-1},$$

where $\mathbf{e}_z$ is the upward-pointing unit vector along the vertical axis.

An asymptotic expansion of

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \nabla(\underline{\Phi}_2|_{z=0}) \tag{20}$$

is obtained by finding an approximation $\underline{\Phi}_{app}$ to the solution of (19) and then using the formal relationship $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 \sim \nabla(\underline{\Phi}_{app}|_{z=0})$. This procedure is justified in the following proposition, whose proof is postponed to Appendix A so as not to interrupt the flow of the development. The proposition is used in both Sections 2.2.1 and 2.2.2 to give explicit asymptotic expansions of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$. To state the result, it is useful to have in place the spaces

$$H^{s,k}(\mathcal{S}) = \big\{f \in \mathcal{D}'(\overline{\mathcal{S}}):\ \|f\|_{H^{s,k}} < \infty\big\},$$

for $s \in \mathbb{R}$ and $k \in \mathbb{N}$, where $\|f\|_{H^{s,k}} = \sum_{j=0}^k \|\Lambda^{s-j}\partial_z^j f\|$.

Proposition 3. *Let $s_0 > d/2$, $s \geqslant s_0 + 1/2$, and $\zeta \in H^{s+3/2}(\mathbb{R}^d)$ be such that* (10) *and* (12) *are satisfied* (*the interface does not touch the horizontal boundaries*). *If $\mathbf{h} \in H^{s+1/2,1}(\mathcal{S})^{d+1}$ and $V \in H^{s+1}(\mathbb{R}^d)^d$ are given, then the boundary value problem*

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = \nabla^{\mu_2}_{X,z} \cdot \mathbf{h} & \text{in } \mathcal{S}, \\ \partial_n u|_{z=0} = \sqrt{\mu_2}\,\nabla \cdot V + \mathbf{e}_z \cdot \mathbf{h}|_{z=0}, & \partial_n u|_{z=-1} = \mathbf{e}_z \cdot \mathbf{h}|_{z=-1}, \end{cases} \tag{21}$$

admits a unique solution u. Moreover, the solution u obeys the inequality

$$|\nabla u|_{z=0}|_{H^s} \leqslant \frac{1}{\sqrt{\mu_2}} C\left(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{s+3/2}}\right)\left(\|\mathbf{h}\|_{H^{s+1/2,1}} + |V|_{H^{s+1}}\right),$$

uniformly with respect to $\varepsilon_2 \in [0, \varepsilon_2^{\max}]$ and $\mu_2 \in (0, \mu_2^{\max})$.

Remark 13. In the case of a flat interface ($\zeta = 0$), Example 1 shows that $\frac{1}{\delta}G^\mu[0]\psi_1 = \sqrt{\mu_2}\,\nabla \cdot V$ with $V = \frac{\nabla}{|D|}\tanh(\sqrt{\mu}|D|)\psi_1$. Consequently, (19), (20) and Proposition 3 (with $\mathbf{h} = 0$) show that

$$\left|\mathbf{H}^{\mu,\delta}[0]\psi_1\right|_{H^s} \lesssim \left|\frac{\tanh(\sqrt{\mu}|D|)}{\sqrt{\mu_2}|D|}\nabla\psi_1\right|_{H^{s+1}} \lesssim \delta|\nabla\psi_1|_{H^{s+1}},$$

which is exactly the estimate one could have deduced from the explicit expression for $\mathbf{H}^{\mu,\delta}[0]\cdot$ given in Example 1 (except that using the latter approach gives an estimate in H^s rather than in H^{s+1}. The H^s-type result does not in fact carry over to the general case of non-flat interfaces).

Remark 14. Suppose we take $\mathbf{h} = 0$ and $V = V^\mu[\varepsilon\zeta]\psi$ in Proposition 3. By Lemma 1, one has $G^\mu[\varepsilon\zeta]\psi = \sqrt{\mu}\nabla \cdot V^\mu[\varepsilon\zeta]\psi$, and so it follows that $\nabla u|_{z=0} = \mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi$. The proposition thus provides an estimate of the operator norm of $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$.

2.2.1. The small-amplitude/small-amplitude regime: $\varepsilon \ll 1$, $\varepsilon_2 \ll 1$

In this regime, it is assumed that the interface deformations are of small amplitude for both the upper and lower fluids. The asymptotic expansion of the operator $H^{\mu,\delta}[\varepsilon\zeta]$ is thus made in terms of ε and $\varepsilon_2 = \varepsilon\delta$. We proceed by first constructing formally an approximate solution $\underline{\Phi}_{app}$ to (19) in the form:

$$\underline{\Phi}_{app} = \Phi^{(0)} + \varepsilon_2\Phi^{(1)}.$$

This formal approximation is then justified rigorously in Corollary 1 below.

Using the expression for $Q^{\mu_2}[\varepsilon_2\zeta]$, write

$$\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} = \Delta^{\mu_2}_{X,z} + \varepsilon_2\nabla^{\mu_2}_{X,z} \cdot Q_1\nabla^{\mu_2}_{X,z} + \varepsilon_2^2\nabla^{\mu_2}_{X,z} \cdot Q_2\nabla^{\mu_2}_{X,z},$$

with

$$Q_1 = \begin{pmatrix} \zeta I_{d\times d} & -\sqrt{\mu_2}(z+1)\nabla\zeta \\ -\sqrt{\mu_2}(z+1)\nabla\zeta^T & -\zeta \end{pmatrix},$$

and

$$Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\zeta^2 + \mu_2(z+1)^2|\nabla\zeta|^2}{1+\varepsilon_2\zeta} \end{pmatrix}.$$

It follows that

$$\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_{app} = \Delta^{\mu_2}_{X,z}\Phi^{(0)} + \varepsilon_2\left(\Delta^{\mu_2}_{X,z} \cdot \Phi^{(1)} + \nabla^{\mu_2}_{X,z} \cdot Q_1\nabla^{\mu_2}_{X,z}\Phi^{(0)}\right) + O\left(\varepsilon_2^2\right).$$

Similarly, we obtain

$$\partial_n\underline{\Phi}_{app}|_{z=0/-1} = \partial_z\Phi^{(0)}|_{z=0/-1} + \varepsilon_2\left(\mathbf{e}_z \cdot Q_1\nabla^{\mu_2}_{X,z}\Phi^{(0)} + \partial_z\Phi^{(1)}\right)|_{z=0/-1} + O\left(\varepsilon_2^2\right).$$

Since it is known from Proposition 1 that

$$\frac{1}{\delta}G^{\mu}[\varepsilon\zeta]\psi_1 = \sqrt{\mu_2}\,\nabla\cdot(\mathcal{T}_{0,\mu}\nabla\psi_1) + \varepsilon_2\mu_2\nabla\cdot\big(-\zeta+\mathcal{T}_{1,\mu}[\zeta]\big)\nabla\psi_1 + O\bigg(\frac{1}{\delta^2}\varepsilon_2^2\mu_2\bigg),$$

one therefore deduces that $\underline{\Phi}_{app}$ solves (19) up to order $O(\varepsilon_2^2+\frac{1}{\delta^2}\varepsilon_2^2\mu_2)$ provided that $\Phi^{(0)}$ and $\Phi^{(1)}$ solve

$$\begin{cases}\Delta^{\mu_2}_{X,z}\Phi^{(0)} = 0,\\ \partial_z\Phi^{(0)}|_{z=0} = \sqrt{\mu_2}\,\nabla\cdot(\mathcal{T}_{0,\mu}\nabla\psi_1), \qquad \partial_z\Phi^{(0)}|_{z=-1}=0,\end{cases}$$

which is obviously solved by $\Phi^{(0)}(X,z) = -\frac{\cosh(\sqrt{\mu_2}(z+1)|D|)}{\cosh(\sqrt{\mu_2}|D|)}\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)}\psi_1$, and

$$\begin{cases}\Delta^{\mu_2}_{X,z}\Phi^{(1)} = -\nabla^{\mu_2}_{X,z}\cdot Q_1\nabla^{\mu_2}_{X,z}\Phi^{(0)},\\ \partial_z\Phi^{(1)}|_{z=0} = A, \qquad \partial_z\Phi^{(1)}|_{z=-1}=0,\end{cases}$$

with $A=\mu_2\nabla\cdot(-\zeta+\mathcal{T}_{1,\mu}[\zeta])\nabla\psi_1 - \mathbf{e}_z\cdot Q_1\nabla^{\mu_2}_{X,z}\Phi^{(0)}|_{z=0}$. Because $-\nabla^{\mu_2}_{X,z}\cdot Q_1\nabla^{\mu_2}_{X,z}\Phi^{(0)} = \Delta^{\mu_2}_{X,z}[(z+1)\zeta\,\partial_z\Phi^{(0)}]$ and

$$A = \mu_2\nabla\cdot\big[-\zeta+\mathcal{T}_{1,\mu}[\zeta]\nabla\psi_1\big] + \mu_2\nabla\cdot\big(\zeta\nabla\Phi^{(0)}\big) + \partial_z\big((z+1)\zeta\,\partial_z\Phi^{(0)}\big)|_{z=0},$$

it results that $\Phi^{(1)} = (z+1)\zeta\,\partial_z\Phi^{(0)} + u$, where u solves the boundary value problem

$$\begin{cases}\Delta^{\mu_2}_{X,z}u = 0,\\ \partial_z u|_{z=0} = \mu_2\nabla\cdot\big[-\zeta+\mathcal{T}_{1,\mu}[\zeta]\nabla\psi_1\big] + \mu_2\nabla\cdot\big(\zeta\nabla\Phi^{(0)}\big), \qquad \partial_z u|_{z=-1}=0.\end{cases}$$

This latter boundary-value problem can be explicitly solved by taking the Fourier transform in the horizontal variables and solving the resulting ordinary differential equation in the variable z. One obtains from this calculation that

$$\begin{aligned}\nabla u|_{z=0} = {}& \sqrt{\mu_2}\frac{|D|}{\tanh(\sqrt{\mu_2}|D|)}\Pi\bigg[\zeta\bigg(1+\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)}\bigg)\nabla\psi_1\bigg]\\ &+\sqrt{\mu_2}\,\nabla\bigg[\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)}\bigg(\zeta\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\Delta\psi_1\bigg)\bigg].\end{aligned}$$

Since $\Phi^{(1)} = u + (z+1)\zeta\,\partial_z\Phi^{(0)}$ and

$$\nabla\big[(z+1)\zeta\,\partial_z\Phi^{(0)}\big]\big|_{z=0} = \sqrt{\mu_2}\,\nabla\bigg(\zeta\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\Delta\psi_1\bigg),$$

it is deduced immediately that $\nabla\Phi^{(1)}|_{z=0} = B(\zeta,\nabla\psi_1)$, where

$$\begin{aligned}B(\zeta,\nabla\psi_1) = {}& \sqrt{\mu_2}\frac{|D|}{\tanh(\sqrt{\mu_2}|D|)}\Pi\bigg[\zeta\bigg(1+\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)}\bigg)\nabla\psi_1\bigg]\\ &+\sqrt{\mu_2}\,\nabla\bigg[\bigg(1+\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)}\bigg)\bigg(\zeta\frac{\tanh(\sqrt{\mu}|D|)}{|D|}\Delta\psi_1\bigg)\bigg].\end{aligned}\tag{22}$$

The rigorous result concerning the asymptotic expansion of the operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the present regime, which is a corollary of Proposition 3, may now be stated and proved.

Corollary 1 *(Full dispersion/Full dispersion regime). Let $t_0 > d/2$, $s \geqslant t_0+1/2$, and $\zeta\in H^{s+3/2}(\mathbb{R}^d)$ be such that* (10) *and* (12) *are satisfied. Then, for all ψ_1 such that $\nabla\psi_1\in H^{s+5/2}(\mathbb{R}^d)$,*

$$\begin{aligned}&\bigg|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 - \bigg(-\frac{\tanh(\sqrt{\mu}|D|)}{\tanh(\sqrt{\mu_2}|D|)}\nabla\psi_1 + \varepsilon_2 B(\zeta,\nabla\psi_1)\bigg)\bigg|_{H^s}\\ &\quad\leqslant \frac{\varepsilon_2^2+\varepsilon^2}{\sqrt{\mu_2}}C\bigg(\frac{1}{H_1},\frac{1}{H_2},\delta^{\max},\mu^{\max},\mu_2^{\max},|\zeta|_{H^{s+3/2}}\bigg)|\nabla\psi_1|_{H^{s+5/2}},\end{aligned}$$

where the bilinear mapping $B(\cdot,\cdot)$ is defined in (22)*. This estimate is uniform with respect to $\varepsilon\in[0,1]$, $\mu\in(0,\mu^{\max})$ and $\delta\in(0,\delta^{\max})$ such that $\mu_2=\frac{\mu}{\delta^2}\in(0,\mu_2^{\max})$.*

Proof. The computations above show that

$$\nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_{app}=\varepsilon_2^2\nabla^{\mu_2}_{X,z}\cdot\mathbf{h},$$

with $\mathbf{h}=Q_1\nabla^{\mu_2}_{X,z}\Phi^{(1)}+Q_2\nabla^{\mu_2}_{X,z}(\Phi^{(0)}+\varepsilon_2\Phi^{(1)})$. It is also easy to check that

$$\begin{aligned}\partial_n\underline{\Phi}_{app}|_{z=0}&=\sqrt{\mu_2}\,\nabla\cdot(\mathcal{T}_{0,\mu}\nabla\psi_1)+\varepsilon_2\mu_2\nabla\cdot\big(-\zeta+\mathcal{T}_{1,\mu}[\zeta]\big)\nabla\psi_1+\varepsilon_2^2\mathbf{e}_z\cdot\mathbf{h}|_{z=0},\\ \partial_n\underline{\Phi}_{app}|_{z=-1}&=\varepsilon_2^2\mathbf{e}_z\cdot\mathbf{h}|_{z=-1}.\end{aligned}$$

Therefore, the difference $v=\underline{\Phi}_{app}-\underline{\Phi}_2$ satisfies the boundary-value problem

$$\begin{cases}\nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}v=\varepsilon_2^2\nabla^{\mu_2}_{X,z}\cdot\mathbf{h},\\ \partial_n v|_{z=0}=\sqrt{\mu_2}\,\nabla\cdot V+\varepsilon_2^2\mathbf{e}_z\cdot\mathbf{h}|_{z=0},\qquad \partial_n v|_{z=-1}=\varepsilon_2^2\mathbf{e}_z\cdot\mathbf{h}|_{z=-1},\end{cases}$$

with $V=(\mathcal{T}_{0,\mu}\nabla\psi_1)+\varepsilon_2\sqrt{\mu_2}(-\zeta+\mathcal{T}_{1,\mu}[\zeta])\nabla\psi_1-V^\mu[\varepsilon\zeta]\psi_1$, and where $V^\mu[\varepsilon\zeta]\psi_1$ is given by Lemma 1. Applying Proposition 3 in this situation, it is immediately deduced that $|\nabla v|_{z=0}|_{H^s}$ is bounded from above by the quantity

$$C\bigg(\frac{1}{H_2},\delta^{\max},\mu_2^{\max},|\zeta|_{H^{s+3/2}}\bigg)\bigg(\frac{\varepsilon_2^2}{\sqrt{\mu_2}}\|\mathbf{h}\|_{H^{s+1/2,1}}+\frac{1}{\sqrt{\mu_2}}|V|_{H^{s+1}}\bigg).$$

The stated result is thus a direct consequence of Proposition 1 and the observation that $\|\mathbf{h}\|_{H^{s+1/2,1}}\leqslant C(\frac{1}{H_2},\delta^{\max},\mu_2^{\max},|\zeta|_{H^{s+3/2}})|\nabla\psi_1|_{H^{s+3/2}}$. □

This section concludes with two specializations of Corollary 1 that obtain when additional smallness assumptions are made on the parameters μ, μ_2 or on δ. These simple consequences of Corollary 1 will be useful presently. The two additional regimes we have in mind are the following.

1. The Boussinesq/Full dispersion regime: This regime is obtained by assuming that $\mu\sim\varepsilon$ and $\mu_2\sim 1$ (and thus $\delta\sim\varepsilon^{1/2}$) in addition to the assumptions $\varepsilon\ll 1$ and $\varepsilon_2\ll 1$ which are required if one wants Corollary 1 to provide a good approximation.
2. The Boussinesq/Boussinesq regime: Here, it is assumed in addition to $\varepsilon\ll 1$ and $\varepsilon_2\ll 1$ that $\mu\sim\varepsilon$ and $\mu_2\sim\varepsilon_2$ (and thus $\delta\sim 1$).

Corollary 2 *(Boussinesq/Full dispersion regime). Let $t_0>d/2$, $s\geqslant t_0+1/2$, and $\zeta\in H^{s+3/2}(\mathbb{R}^d)$ be such that* (10) *and* (12) *are satisfied. Then, for all ψ_1 such that $\nabla\psi_1\in H^{s+5/2}(\mathbb{R}^d)$, the inequality*

$$\begin{aligned}&\bigg|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1-\sqrt{\mu}|D|\coth\big(\sqrt{\mu_2}|D|\big)\bigg[-\nabla\psi_1-\frac{\mu}{3}\Delta\nabla\psi_1+\varepsilon\Pi(\zeta\nabla\psi_1)\bigg]\bigg|_{H^s}\\ &\quad\leqslant\bigg(\frac{\varepsilon_2^2+\varepsilon^2}{\sqrt{\mu_2}}+\varepsilon\mu+\varepsilon\mu^{1/2}\delta\bigg)C\bigg(\frac{1}{H_1},\frac{1}{H_2},\delta^{\max},\mu^{\max},\mu_2^{\max},|\zeta|_{H^{s+3/2}}\bigg)|\nabla\psi_1|_{H^{s+5/2}},\end{aligned}$$

where $\Pi=-\frac{\nabla\nabla^T}{|D|^2}$, holds uniformly with respect to $\varepsilon\in[0,1]$, $\mu\in(0,\mu^{\max})$ and $\delta\in(0,\delta^{\max})$ such that $\mu_2=\frac{\mu}{\delta^2}\in(0,\mu_2^{\max})$.

Remark 15. When $\varepsilon\ll 1$, $\mu\sim\varepsilon$, $\mu_2\sim 1$ (and thus $\delta\sim\varepsilon^{1/2}$), the three components of the error estimate are all of the same size $O(\varepsilon^2)$.

Proof. The result is obtained by using $\tanh(\sqrt{\mu}|D|)\sim\sqrt{\mu}|D|-\mu\sqrt{\mu}\frac{1}{3}|D|^3$ when μ is small in Corollary 1. □

Similarly, one may also deduce from Corollary 1 the following result in the Boussinesq/Boussinesq regime.

Corollary 3 *(Boussinesq/Boussinesq regime). Let $t_0>d/2$, $s\geqslant t_0+1/2$, and $\zeta\in H^{s+3/2}(\mathbb{R}^d)$ be such that* (10) *and* (12) *are satisfied. Then, for all ψ_1 such that $\nabla\psi_1\in H^{s+5/2}(\mathbb{R}^d)$, we have*

$$\left|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1-\left(-\delta\nabla\psi_1-\frac{\delta}{3}\mu\left(1-\frac{1}{\delta^2}\right)\Delta\nabla\psi_1+\varepsilon_2(1+\delta)\Pi(\zeta\nabla\psi_1)\right)\right|_{H^s}$$
$$\leqslant\left(\frac{\varepsilon_2^2+\varepsilon^2}{\sqrt{\mu_2}}+\mu^2+\varepsilon^2\right)C\left(\frac{1}{H_1},\frac{1}{H_2},\frac{1}{\delta^{\min}},\delta^{\max},\mu^{\max},|\zeta|_{H^{s+3/2}}\right)|\nabla\psi_1|_{H^{s+5/2}},$$

where $\Pi=-\frac{\nabla\nabla^T}{|D|^2}$. *Moreover, this estimate is uniform with respect to* $\varepsilon\in[0,1]$, $\mu\in(0,\mu^{\max})$ *and* $\delta\in(\delta^{\min},\delta^{\max})$.

Remark 16. When $\varepsilon\sim\varepsilon_2\sim\mu\sim\mu_2\ll 1$ (and thus $\delta\sim 1$), the last two components of the error estimate are of size $O(\varepsilon^2)$, but the first is of size $O(\varepsilon^{3/2})$. This loss of precision is not seen at the formal level. It comes from the $1/\sqrt{\mu_2}$ term in the elliptic estimate provided by Proposition 3.

2.2.2. The Shallow-water/Shallow-water regime: $\mu\ll 1$, $\mu_2\ll 1$

In this regime, large amplitude waves are allowed for the upper fluid ($\varepsilon=O(1)$) and for the lower fluid ($\varepsilon_2=O(1)$). Assuming that $\mu\ll 1$ and $\mu_2\ll 1$ raises the prospect of making asymptotic expansions of shallow-water type, in terms of μ and μ_2. As before, the plan is to formally construct an approximate solution $\underline{\Phi}_{app}$ to (19) having the form

$$\underline{\Phi}_{app}=\Phi^{(0)}+\mu_2\Phi^{(1)}.$$

The formal approximation is then rigorously justified (Corollary 4 below) and the desired expansion results. From the expression for $Q^{\mu_2}[\varepsilon_2\zeta]$, we may write

$$\nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}=\frac{1}{h_2}\partial_z^2+\mu_2\nabla_{X,z}\cdot Q_1\nabla_{X,z},$$

with $h_2=1+\varepsilon_2\zeta$, and

$$Q_1=\begin{pmatrix} h_2 I_{d\times d} & -\varepsilon_2(z+1)\nabla\zeta\\ -\varepsilon_2(z+1)\nabla\zeta^T & \frac{\varepsilon_2^2(z+1)^2|\nabla\zeta|^2}{h_2}\end{pmatrix}.$$

It follows readily that

$$\nabla^{\mu_2}_{X,z}\cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_{app}=\frac{1}{h_2}\partial_z^2\Phi^{(0)}+\mu_2\left(\nabla_{X,z}\cdot Q_1\nabla_{X,z}\Phi^{(0)}+\frac{1}{h_2}\partial_z^2\Phi^{(1)}\right)+O\big(\mu_2^2\big).$$

Similarly, one infers that at $z=0$ and $z=-1$,

$$\partial_n\underline{\Phi}_{app}=\frac{1}{h_2}\partial_z\Phi^{(0)}+\mu_2\left(\mathbf{e}_z\cdot Q_1\nabla_{X,z}\Phi^{(0)}+\frac{1}{h_2}\partial_z\Phi^{(1)}\right)+O\big(\mu_2^2\big).$$

Since it is known from Proposition 2 that

$$\frac{1}{\delta}G^{\mu}[\varepsilon\zeta]\psi_1=\delta\mu_2\nabla\cdot(h_1\nabla\psi_1)+O\left(\frac{\mu^2}{\delta}\right)$$

(with $h_1=1-\varepsilon\zeta$), it is clearly the case that $\underline{\Phi}_{app}$ solves (19) up to order $O(\mu_2^2+\frac{\mu^2}{\delta})$ provided that $\Phi^{(0)}$ and $\Phi^{(1)}$ solve

$$\begin{cases}\partial_z^2\Phi^{(0)}=0,\\ \partial_z\Phi^{(0)}|_{z=0}=0,\qquad \partial_z\Phi^{(0)}|_{z=-1}=0,\end{cases}$$

(which is obviously solved by any $\Phi^{(0)}(X,z)=\Phi^{(0)}(X)$ independent of z) which also satisfies

$$\begin{cases}\partial_z^2\Phi^{(1)}=-h_2^2\Delta\Phi^{(0)},\\ \partial_z\Phi^{(1)}|_{z=0}=h_2\big(\varepsilon_2\nabla\zeta\cdot\nabla\Phi^{(0)}+\delta\nabla\cdot(h_1\nabla\psi_1)\big),\qquad \partial_z\Phi^{(1)}|_{z=-1}=0,\end{cases}$$

where we have used the fact that $\Phi^{(0)}$ does not depend on z. Solving this second order ordinary differential equation in the variable z with the boundary condition at $z=0$ yields (up to a function independent of z which we take equal to 0 for the sake of simplicity),

$$\Phi^{(1)}=-\frac{z^2}{2}h_2^2\Delta\Phi^{(0)}+z(\partial_z\Phi_1|_{z=0}).$$

Matching the boundary condition at $z=-1$ leads to the restriction

$$\nabla\cdot\big(h_2\nabla\Phi^{(0)}\big)=-\delta\nabla\cdot(h_1\nabla\psi_1),$$

which implies that $\Pi(h_2\nabla\Phi^{(0)})=\Pi(-\delta h_1\nabla\psi_1)$, where $\Pi=-\frac{\nabla\nabla^T}{|D|^2}$ is the orthogonal projector onto the gradient vector fields of $L^2(\mathbb{R}^d)^d$ defined earlier. We will solve this equation thanks to the following lemma.

Lemma 3. *Assume that $\zeta\in L^\infty(\mathbb{R}^d)$ is such that $|\varepsilon_2\zeta|_\infty<1$. Let also $W\in L^2(\mathbb{R}^d)^d$. Then*

(i) *one can define the mapping $\mathfrak{Q}[\varepsilon_2\zeta]$ as*

$$\mathfrak{Q}[\varepsilon_2\zeta]:\begin{array}{rl} L^2\big(\mathbb{R}^d\big)^d & \to L^2\big(\mathbb{R}^d\big)^d,\\ U & \mapsto \sum_{n=0}^\infty(-1)^n\big(\Pi(\varepsilon_2\zeta\Pi\cdot)\big)^n(\Pi U);\end{array}$$

(ii) *there exists a unique solution $V\in L^2(\mathbb{R}^d)^d$ to the equation,*

$$\nabla\cdot(h_2V)=\nabla\cdot W\quad(h_2=1+\varepsilon_2\zeta),$$

such that $\Pi V=V$ and one has $V=\mathfrak{Q}[\varepsilon_2\zeta]W$;

(iii) *if moreover $\zeta\in H^s(\mathbb{R}^d)$ and $W\in H^s(\mathbb{R}^d)^d$ $(s>d/2+1)$ then $\mathfrak{Q}[\varepsilon_2\zeta]W\in H^s(\mathbb{R}^d)^d$ and*

$$\big|\mathfrak{Q}[\varepsilon_2\zeta]W\big|_{H^s}\leqslant C\bigg(|\varepsilon_2\zeta|_{H^s},\frac{1}{1-|\varepsilon_2\zeta|_\infty}\bigg)|W|_{H^s}.$$

Remark 17. In dimension $d=1$, one has Π is the identity map and the first point of the lemma simplifies into $V=\frac{1}{h_2}W$ so that the proof is trivial.

Proof. (i) The result follows from the observation that under the assumptions of the lemma, one has

$$\big\|\Pi(\varepsilon_2\zeta\Pi\cdot)\big\|_{L^2\to L^2}\leqslant|\varepsilon_2\zeta|_\infty<1, \tag{23}$$

so that the series used to define $\mathfrak{Q}[\varepsilon_2\zeta]U$ converges in $L^2(\mathbb{R}^d)^d$.

(ii) Let us first check that $V=\mathfrak{Q}[\varepsilon_2\zeta]W$ is indeed a solution of the equation stated in the lemma. Since $V=\Pi V$, it transpires that

$$\begin{aligned}\nabla\cdot(\varepsilon_2\zeta V)=\nabla\cdot\big(\Pi(\varepsilon_2\zeta\Pi V)\big)&=-\nabla\cdot\sum_{n=1}^\infty(-1)^n\big(\Pi(\varepsilon_2\zeta\Pi\cdot)\big)^n(\Pi W)\\&=-\nabla\cdot(V-\Pi W),\end{aligned}$$

from which one deduces easily that $\nabla\cdot(h_2V)=\nabla\cdot W$.

Let us now turn to proving uniqueness of the solution by showing that one has necessarily $V=0$ if $W=0$. To check that this is the case, just remark that from the equation $\nabla\cdot(h_2V)=0$ and the requirement that $\Pi V=V$, one has

$$V=-\Pi(\varepsilon_2\zeta\Pi V).$$

Since $\|\Pi(\varepsilon_2\zeta\Pi V)\|_{L^2\to L^2}\leqslant|\varepsilon_2\zeta|_\infty<1$, it follows that $V=0$.

(iii) It is clear from (23) that $|\mathfrak{Q}[\varepsilon_2\zeta]W|_2\leqslant\frac{1}{1-|\varepsilon_2\zeta|_\infty}$. Now, applying Λ^s to the equations, one gets

$$\nabla\cdot\big(h_2\Lambda^sV\big)=\nabla\cdot\widetilde{W},$$

with $\widetilde{W}=\Lambda^sW+[\Lambda^s,\varepsilon_2\zeta]V$. The result follows therefore from the L^2-estimate, a standard commutator estimate and a simple induction. □

If Lemma 3 is applied with $V=\nabla\Phi^{(0)}$, $W=-\delta h_1\nabla\psi_1$, there results the equation

$$\nabla\Phi^{(0)}=-\delta\mathfrak{Q}[\varepsilon_2\zeta](h_1\nabla\psi_1).$$

Note that when $d = 1$, this reduces to

$$\partial_x \Phi^{(0)} = -\delta \frac{h_1}{h_2} \partial_x \psi_1.$$

The following corollary of Proposition 3, which gives the needed asymptotic expansion of the operator $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ in the present regime, now comes into view.

Corollary 4 *(Shallow water/Shallow water regime). Let $t_0 > d/2$, $s \geqslant t_0 + 1/2$, and $\zeta \in H^{s+3/2}(\mathbb{R}^d)$ be such that* (10) *and* (12) *are satisfied. Let $h_1 = 1 - \varepsilon\zeta$ and $h_2 = 1 + \varepsilon_2\zeta$ and let ψ_1 be such that $\nabla\psi_1 \in H^{s+5/2}(\mathbb{R}^d)$. Then it follows that*

$$\begin{aligned} &\big|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 + \delta\mathfrak{Q}[\varepsilon_2\zeta](h_1\nabla\psi_1)\big|_{H^s} \\ &\quad \leqslant \delta(\mu + \mu_2) C\bigg(\big(1 - \delta(1 - H_1)\big)^{-1}, \frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{s+3/2}}\bigg)|\nabla\psi_1|_{H^{s+5/2}}, \end{aligned}$$

uniformly with respect to $\varepsilon \in [0, 1]$, $\mu \in (0, 1)$ and $\delta < \frac{1}{1-H_1}$ such that $\varepsilon_2 = \varepsilon\delta \in [0, \varepsilon_2^{\max}]$ and $\mu_2 = \frac{\mu}{\delta^2} \in (0, \mu_2^{\max})$.

Remark 18. When $\varepsilon \sim \varepsilon_2 \sim \mu \sim \mu_2 \ll 1$ (and thus $\delta \sim 1$), one deduces from the above corollary that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = -\delta\nabla\psi_1 + O(\varepsilon)$, which is consistent with the asymptotic expansion provided by Corollary 3. A similar matching would have been observed for the next order terms if we had computed them in Corollary 4.

Remark 19. When $d = 1$, one has $\delta\mathfrak{Q}[\varepsilon_2\zeta](h_1\nabla\psi_1) = \delta\frac{h_1}{h_2}\partial_x\psi_1$.

Proof. Since (10), (12) and the condition $\delta(1 - H_1) < 1$ imply that $|\varepsilon_2\zeta|_\infty < 1$, one can use Lemma 3 and the computations above indicate that

$$\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}\underline{\Phi}_{app} = \mu_2^2 \nabla_{X,z} \cdot Q_1 \nabla_{X,z}\Phi_1 = \mu_2^{3/2}\nabla^{\mu_2}_{X,z} \cdot \mathbf{h},$$

with

$$\mathbf{h} = \begin{pmatrix} I_{d\times d} & 0 \\ 0 & \sqrt{\mu_2} \end{pmatrix} Q_1 \nabla_{X,z}\Phi_1.$$

It is also easy to check that

$$\begin{aligned} \partial_n \underline{\Phi}_{app}|_{z=0} &= \delta\mu_2\nabla\cdot(h_1\nabla\psi_1) + \mu_2^{3/2}\mathbf{e}_z \cdot \mathbf{h}|_{z=0}, \\ \partial_n \underline{\Phi}_{app}|_{z=-1} &= \mu_2^{3/2}\mathbf{e}_z \cdot \mathbf{h}|_{z=-1}. \end{aligned}$$

Thus, the difference $u = \underline{\Phi}_{app} - \underline{\Phi}_2$ satisfies the boundary value problem

$$\begin{cases} \nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z} u = \mu_2^{3/2}\nabla^{\mu_2}_{X,z} \cdot \mathbf{h}, \\ \partial_n u|_{z=0} = \sqrt{\mu_2}\,\nabla\cdot V + \mu_2^{3/2}\mathbf{e}_z \cdot \mathbf{h}|_{z=0}, \qquad \partial_n u|_{z=-1} = \mu_2^{3/2}\mathbf{e}_z \cdot \mathbf{h}|_{z=-1}, \end{cases}$$

with $V = h_1\delta\sqrt{\mu_2}\,\nabla\psi_1 - V^\mu[\varepsilon\zeta]\psi_1$, where $V^\mu[\varepsilon\zeta]\psi_1$ is given by Lemma 1. One concludes from Proposition 3 that $|\nabla u_{|_{z=0}}|_{H^s}$ is bounded from above by the quantity

$$C\bigg(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{s+3/2}}\bigg)\bigg(\mu_2\|\mathbf{h}\|_{H^{s+1/2,1}} + \delta\bigg|h_1\nabla\psi_1 - \frac{1}{\sqrt{\mu}}V^\mu[\varepsilon\zeta]\psi_1\bigg|_{H^{s+1}}\bigg).$$

The result is a direct consequence of Proposition 2, since

$$\|\mathbf{h}\|_{H^{s+1/2,1}} \leqslant \delta C\bigg(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{s+3/2}}\bigg)|\nabla\psi_1|_{H^{s+3/2}}. \qquad \square$$

2.2.3. The Shallow-water/Small-amplitude regime: $\mu \ll 1$, $\varepsilon_2 \ll 1$

It is now presumed that both μ and ε_2 are small, but no such restriction is laid upon ε nor μ_2. So, this regime is not a subcase of the regimes investigated in Sections 2.2.1 and 2.2.2. We construct an approximate solution $\underline{\Phi}_{app}$ to (19) exactly as in Section 2.2.1, but only a first-order approximation of the form,

$$\underline{\Phi}_{app} = \Phi^{(0)},$$

will be required. Since $\mu \ll 1$ here, Proposition 1 may be utilized to write

$$\frac{1}{\delta} G^{\mu}[\varepsilon\zeta]\psi_1 = \frac{\mu}{\delta}\nabla\cdot(h_1\nabla\psi_1) + O\left(\frac{\mu^2}{\delta}\right).$$

Just as in Section 2.2.1, it can be shown that $\Phi^{(0)}$ must solve the boundary-value problem

$$\begin{cases} \Delta^{\mu_2}_{X,z}\Phi^{(0)} = 0, \\ \partial_z\Phi^{(0)}|_{z=0} = \frac{\mu}{\delta}\nabla\cdot(h_1\nabla\psi_1), \qquad \partial_z\Phi^{(0)}|_{z=-1} = 0, \end{cases}$$

which is to say that

$$\Phi^{(0)}(X,z) = \sqrt{\mu}\frac{\cosh(\sqrt{\mu_2}(z+1)|D|)}{\cosh(\sqrt{\mu_2}|D|)}\frac{1}{|D|\tanh(\sqrt{\mu_2}|D|)}\nabla\cdot(h_1\nabla\psi_1).$$

The following result is proved exactly as was Corollary 1.

Corollary 5 *(Shallow water/Small amplitude regime). Let $t_0 > d/2$, $s \geqslant t_0 + 1/2$, and $\zeta \in H^{s+3/2}(\mathbb{R}^d)$ be such that* (10) *and* (12) *are satisfied. Then, for all ψ_1 such that $\nabla\psi_1 \in H^{s+5/2}(\mathbb{R}^d)$, it is the case that*

$$\begin{aligned}&\left|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 + \sqrt{\mu}|D|\coth\left(\sqrt{\mu_2}|D|\right)\Pi(h_1\nabla\psi_1)\right|_{H^s}\\ &\quad\leqslant \frac{\mu^{3/2}+\varepsilon_2\sqrt{\mu}}{\sqrt{\mu_2}}C\left(\frac{1}{H_1},\frac{1}{H_2},\delta^{\max},\mu_2^{\max},|\zeta|_{H^{s+3/2}}\right)|\nabla\psi_1|_{H^{s+5/2}},\end{aligned}$$

where $h_1 = 1-\varepsilon\zeta$ and $\Pi = -\frac{\nabla\nabla^T}{|D|^2}$ is given by (22)*. This estimate is uniform with respect to $\varepsilon \in [0,1]$, $\mu \in (0,1)$ and $\delta \in (0,\delta^{\max})$ such that $\mu_2 = \frac{\mu}{\delta^2} \in (0,\mu_2^{\max})$.*

Remark 20. Several regimes fall within the range of Corollary 5.

- The SW/FD regime: when $\mu \ll 1$, $\varepsilon_2 \ll 1$ and $\varepsilon \sim \mu_2 \sim 1$ (and thus $\delta^2 \sim \mu \sim \varepsilon_2^2$); the precision of the approximation is $O(\mu)$.
- The ILW regime: if $\mu \sim \varepsilon^2 \ll 1$ and $\mu_2 \sim 1$ (and thus $\delta^2 \sim \mu \sim \varepsilon_2$); in this case, the estimate in the corollary can be simplified without adverse effects on the precision of the approximation to simply,
$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = -\sqrt{\mu}|D|\coth\left(\sqrt{\mu_2}|D|\right)\nabla\psi_1 + O(\mu). \tag{24}$$
- The BO regime: if $\mu \ll 1$ and $\delta = 0$ (and thus $\mu_2 = \infty$, $\varepsilon_2 = 0$), one gets formally from (24) that
$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 \sim -\sqrt{\mu}|D|\nabla\psi_1. \tag{25}$$

3. Asymptotic models for internal waves

The preliminary analysis in Section 2 allows us to derive the various asymptotic models referred to in the Introduction.

3.1. The Small amplitude/Small amplitude regime: $\varepsilon \ll 1$, $\varepsilon_2 \ll 1$

Derived first are various models corresponding to the case wherein the interface deformation are small for both fluids. Different systems of equations obtain, depending on the sizes of the parameters ε, μ and δ (and thus ε_2 and μ_2).

3.1.1. The Full dispersion/Full dispersion regime: $\varepsilon \sim \varepsilon_2 \ll 1$ and $\mu \sim \mu_2 = O(1)$

An asymptotic model can be derived from (14) by replacing the operators $G^\mu[\varepsilon\zeta]$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ by their asymptotic expansions, provided by Proposition 1 and Corollary 1 in the present regime. The following theorem shows that in the present regime, the full internal wave equations (14) are consistent with following FD/FD system,

$$\begin{cases} \partial_t\zeta + \frac{1}{\sqrt{\mu}}\frac{\nabla}{|D|}\cdot\Big(\frac{\mathbf{T}_\mu\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v}\Big) + \frac{\varepsilon_2}{\sqrt{\mu}}\frac{\nabla}{|D|}\cdot\Big(\frac{\mathbf{T}_\mu\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}B\Big(\zeta,\frac{\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v}\Big)\Big) \\ \quad -\varepsilon\nabla\cdot\Big(\zeta\frac{\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v}\Big) + \varepsilon|D|\mathbf{T}_\mu\Big(\zeta\frac{\nabla}{|D|}\cdot(\frac{\mathbf{T}_\mu\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v})\Big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \frac{\varepsilon}{2}\nabla\Big(\Big|\frac{\mathbf{T}_\mu}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v}\Big|^2 - \gamma\Big|\frac{\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v}\Big|^2\Big) + \varepsilon\frac{\gamma-1}{2}\nabla\Big(\frac{\nabla}{|D|}\cdot\Big(\frac{\mathbf{T}_\mu\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v}\Big)\Big)^2 = 0, \end{cases} \tag{26}$$

where as before, $\mathbf{T}_\mu = \tanh(\sqrt{\mu}|D|)$, $\mathbf{T}_{\mu_2} = \tanh(\sqrt{\mu_2}|D|)$ and the bilinear mapping $B(\cdot,\cdot)$ is given in (22).

Theorem 1. *Let $0 < \delta^{\min} < \delta^{\max}$. The internal waves equations* (14) *are consistent with the FD/FD equations* (26) *in the sense of Definition* 3*, with a precision $O(\varepsilon^2)$, and uniformly with respect to $\varepsilon \in [0,1]$, $\mu \in (0,\mu^{\max})$ and $\delta \in [\delta^{\min},\delta^{\max}]$.*

Remark 21. One can give a more precise version of the estimate, as in Corollary 1 for instance. It simplifies the exposition to use the notation $O(\varepsilon^2)$ and the associated rough estimate of the precision. We follow this policy throughout the discussion.

Remark 22. It is straightforward to check that the dispersion relation of (26) is exactly the same as (15), which is the reason we refer to (26) as a "full dispersion" model. In particular, (26) is linearly well-posed provided that $\gamma < 1$.

Proof. First, notice that with the range of parameters considered in the theorem, one has $\varepsilon \sim \varepsilon_2$ when $\varepsilon \to 0$, while $\mu \sim \mu_2 = O(1)$. By the definition (16) of $\mathbf{v}$ and using Proposition 1 and Corollary 1, one deduces from (14) that

$$\begin{cases} \partial_t\zeta - \frac{1}{\sqrt{\mu}}\frac{\nabla}{|D|}\cdot(\mathbf{T}_\mu\nabla\psi_1) + \varepsilon\nabla\cdot(\zeta\nabla\psi_1) - \varepsilon|D|\mathbf{T}_\mu\Big(\zeta\frac{\nabla}{|D|}\cdot(\mathbf{T}_\mu\nabla\psi_1)\Big) = O(\varepsilon^2), \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \frac{\varepsilon}{2}\nabla\big(\big|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1\big|^2 - \gamma|\nabla\psi_1|^2\big) + \varepsilon\frac{\gamma-1}{2}\nabla\Big(\frac{\nabla}{|D|}\cdot(\mathbf{T}_\mu\nabla\psi_1)\Big)^2 = O(\varepsilon^2). \end{cases}$$

It follows from Corollary 1 and the relation $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \mathbf{v} + \gamma\nabla\psi_1$ that

$$\nabla\psi_1 = -\frac{\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\Big(\mathbf{v} + \varepsilon_2 B\Big(\zeta,\frac{\mathbf{T}_{\mu_2}}{\gamma\mathbf{T}_{\mu_2}+\mathbf{T}_\mu}\mathbf{v}\Big)\Big) + O(\varepsilon^2).$$

The result is view is now apparent. □

3.1.2. The Boussinesq/Full dispersion regime $\mu \sim \varepsilon \ll 1$, $\mu_2 \sim 1$

We show here that in this regime (for which one also has $\delta^2 \sim \varepsilon$ and thus $\varepsilon_2 \sim \varepsilon^{3/2} \ll 1$), the internal waves equations (14) are consistent with the *three-parameter family* of Boussinesq/FD systems

$$\begin{cases} (1-\mu b\Delta)\partial_t\zeta + \frac{1}{\gamma}\nabla\cdot\big((1-\varepsilon\zeta)\mathbf{v}_\beta\big) - \frac{\sqrt{\mu}}{\gamma^2}|D|\coth\big(\sqrt{\mu_2}|D|\big)\nabla\cdot\mathbf{v}_\beta \\ \quad + \frac{\mu}{\gamma}\Big(a - \frac{1}{\gamma^2}\coth^2\big(\sqrt{\mu_2}|D|\big)\Big)\Delta\nabla\cdot\mathbf{v}_\beta = 0, \\ (1-\mu d\Delta)\partial_t\mathbf{v}_\beta + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla|\mathbf{v}_\beta|^2 + \mu c(1-\gamma)\Delta\nabla\zeta = 0, \end{cases} \tag{27}$$

where $\mathbf{v}_\beta = (1-\mu\beta\Delta)^{-1}\mathbf{v}$ and the constants a, b, c and d are defined now.

Theorem 2. *Let* $0 < c^{\min} < c^{\max}$, $0 < \mu_2^{\min} < \mu_2^{\max}$, *and set*:

$$a = \frac{1}{3}(1 - \alpha_1 - 3\beta), \qquad b = \frac{1}{3}\alpha_1, \qquad c = \beta\alpha_2, \qquad d = \beta(1 - \alpha_2),$$

with $\alpha_1 \geqslant 0$, $\beta \geqslant 0$ *and* $\alpha_2 \leqslant 1$. *With these choices of parameters, the internal wave equations* (14) *are consistent with the Boussinesq/FD equations* (27) *in the sense of Definition* 3, *with a precision* $O(\varepsilon^{3/2})$, *and uniformly with respect to* $\varepsilon \in [0,1]$, $\mu \in (0,1)$ *and* $\delta \in (0,1)$ *satisfying the conditions*

$$c^{\min} \leqslant \frac{\varepsilon}{\mu} \leqslant c^{\max} \quad \textit{and} \quad \mu_2^{\min} \leqslant \frac{\mu}{\delta^2} \leqslant \mu_2^{\max}.$$

Remark 23. The dispersion relation associated to (27) is

$$\omega^2 = \frac{1-\gamma}{\gamma}|\mathbf{k}|^2\big(1 - \mu c|\mathbf{k}|^2\big)\frac{1 - \frac{\sqrt{\mu}}{\gamma}|\mathbf{k}|\coth(\sqrt{\mu_2}|\mathbf{k}|) - \mu|\mathbf{k}|^2(a - \frac{1}{\gamma^2}\coth^2(\sqrt{\mu_2}|\mathbf{k}|))}{(1 + \mu b|\mathbf{k}|^2)(1 + \mu d|\mathbf{k}|^2)},$$

and (27) is therefore linearly well-posed when $b, d \geqslant 0$ and $a, c \leqslant 0$. Notice that in the case $\alpha_1 = \alpha_2 = \beta = 0$, one has $a = \frac{1}{3}$ and $b = c = d = 0$ and the corresponding system is thus linearly ill-posed. The freedom to choose a well-posed model is just one of the advantages of a three-parameter family of formally equivalent systems. The same remark has already been made about the Boussinesq systems for wave propagation in the case of surface gravity waves [8,10].

Proof. The proof is made in several steps, corresponding to particular assumptions about the parameters α_1, α_2 and β. Throughout, use will be freely made of the relations $\mu \sim \varepsilon$ and $\mu_2 \sim 1$.

Step 1. The case $\alpha_1 = 0$, $\beta = 0$, $\alpha_2 = 0$. From the expansion of the Dirichlet–Neumann operator, see Remark 11, it follows as in the previous section that

$$\begin{cases} \partial_t\zeta - \nabla\cdot\big((1 - \varepsilon\zeta)\nabla\psi_1\big) - \frac{\mu}{3}\nabla\cdot\Delta\nabla\psi_1 = O\big(\varepsilon^2\big), \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \frac{\varepsilon}{2}\nabla\big(\big|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1\big|^2 - \gamma|\nabla\psi_1|^2\big) = O\big(\varepsilon^2\big), \end{cases}$$

where the fact that $O(\mu) = O(\varepsilon)$ has been used. From the relation $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \mathbf{v} + \gamma\nabla\psi_1$, and Corollary 2, it is seen that

$$\nabla\psi_1 = -\frac{1}{\gamma}\mathbf{v} - \frac{\sqrt{\mu}}{\gamma}\frac{|D|}{\mathbf{T}_{\mu_2}}\bigg[1 + \frac{\mu}{3}\Delta - \varepsilon\Pi(\zeta\cdot)\bigg]\nabla\psi_1 + O\big(\varepsilon^2\big).$$

Again using the fact that $O(\mu) = O(\varepsilon)$, one concludes that

$$\nabla\psi_1 = -\frac{1}{\gamma}\mathbf{v} + \frac{\sqrt{\mu}}{\gamma^2}\frac{|D|}{\mathbf{T}_{\mu_2}}\mathbf{v} + \frac{\mu}{\gamma^3}\frac{\Delta}{\mathbf{T}_{\mu_2}^2}\mathbf{v} + O\big(\varepsilon^{\frac{3}{2}}\big),$$

and the result follows.

Step 2. The case $\alpha_1 \geqslant 0$, $\beta = 0$, $\alpha_2 = 0$. We use here the classical BBM trick [7]. It is clear from the first equation that

$$\partial_t\zeta = -\frac{1}{\gamma}\nabla\cdot\mathbf{v} + O\big(\varepsilon^{1/2}\big),$$

from which it is inferred that

$$\nabla\cdot\mathbf{v} = (1 - \alpha_1)\nabla\cdot\mathbf{v} - \alpha_1\gamma\partial_t\zeta + O\big(\varepsilon^{1/2}\big).$$

Replacing $\nabla\cdot\mathbf{v}$ by this expression in the component $\frac{\mu}{3\gamma}\Delta\nabla\cdot\mathbf{v}$ of the first equation of the system derived in Step 1, leads to the desired result.

Step 3. The case $\alpha_1 \geqslant 0$, $\beta \geqslant 0$, $\alpha_2 = 0$. Replacing $\mathbf{v}$ by $(1 - \mu\beta\Delta)\mathbf{v}_\beta$ in the system of equations derived in Step 2, and neglecting the $O(\varepsilon^{3/2})$ terms is all that is required in this case.

Step 4. The case $\alpha_1 \geqslant 0$, $\beta \geqslant 0$, $\alpha_2 \leqslant 1$. We use once again the BBM trick. From the second equation in the system derived in Step 3, one obtains that for all $\alpha_2 \leqslant 1$,

$$\partial_t \mathbf{v}_\beta = (1-\alpha_2)\partial_t \mathbf{v}_\beta - \alpha_2(1-\gamma)\nabla\zeta + O(\varepsilon).$$

If this relationship is substituted into the system derived in Step 3, the result follows. □

3.1.3. The Boussinesq/Boussinesq regime $\varepsilon \sim \mu \sim \varepsilon_2 \sim \mu_2 \ll 1$

In this regime, the nonlinear and dispersive effects are of the same size for both fluids; the systems of equations that are derived from the internal waves equations (14) in this situation are the following *three-parameter family* of Boussinesq/Boussinesq systems, viz.

$$\begin{cases} (1-\mu b\Delta)\partial_t\zeta + \dfrac{1}{\gamma+\delta}\nabla\cdot\mathbf{v}_\beta + \varepsilon\dfrac{\delta^2-\gamma}{(\gamma+\delta)^2}\nabla\cdot(\zeta\mathbf{v}_\beta) + \mu a\nabla\cdot\Delta\mathbf{v}_\beta = 0, \\ (1-\mu d\Delta)\partial_t\mathbf{v}_\beta + (1-\gamma)\nabla\zeta + \dfrac{\varepsilon}{2}\dfrac{\delta^2-\gamma}{(\delta+\gamma)^2}\nabla|\mathbf{v}_\beta|^2 + \mu(1-\gamma)c\Delta\nabla\zeta = 0, \end{cases} \tag{28}$$

where $\mathbf{v}_\beta = (1-\mu\beta\Delta)^{-1}\mathbf{v}$, and where the coefficients a, b, c, d are provided in the statement of the next theorem.

Theorem 3. *Let $0 < c^{\min} < c^{\max}$, $0 < \delta^{\min} < \delta^{\max}$, and set*

$$a = \frac{(1-\alpha_1)(1+\gamma\delta) - 3\delta\beta(\gamma+\delta)}{3\delta(\gamma+\delta)^2}, \qquad b = \alpha_1\frac{1+\gamma\delta}{3\delta(\gamma+\delta)},$$
$$c = \beta\alpha_2, \qquad d = \beta(1-\alpha_2),$$

with $\alpha_1 \geqslant 0$, $\beta \geqslant 0$ and $\alpha_2 \leqslant 1$. With this specification of the parameters, The internal wave equations (14) *are consistent with the Boussinesq/Boussinesq equations* (28) *in the sense of Definition* 3, *with a precision $O(\varepsilon^{3/2})$, and uniformly with respect to $\varepsilon \in [0,1]$, $\mu \in (0,1)$ and $\delta \in [\delta^{\min}, \delta^{\max}]$ such that $c^{\min} < \frac{\varepsilon}{\mu} < c^{\max}$.*

Remark 24. Taking $\gamma = 0$ and $\delta = 1$ in the Boussinesq/Boussinesq equations (28), reduces them to the system

$$\begin{cases} \left(1-\mu\dfrac{\alpha_1}{3}\Delta\right)\partial_t\zeta + \nabla\cdot\big((1+\varepsilon\zeta)\mathbf{v}\big) + \mu\dfrac{1-\alpha_1-3\beta}{3}\nabla\cdot\Delta\mathbf{v} = 0, \\ \big(1-\mu\beta(1-\alpha_2)\Delta\big)\partial_t\mathbf{v} + \nabla\zeta + \dfrac{\varepsilon}{2}\nabla|\mathbf{v}|^2 + \mu\beta\alpha_2\Delta\nabla\zeta = 0, \end{cases}$$

which is exactly the family of formally equivalent Boussinesq systems derived in [8,10].

Remark 25. The dispersion relation associated to (28) is

$$\omega^2 = |\mathbf{k}|^2\frac{(\frac{1}{\gamma+\delta} - \mu a|\mathbf{k}|^2)(1-\gamma-\mu c|\mathbf{k}|^2)}{(1+\mu b|\mathbf{k}|^2)(1+\mu d|\mathbf{k}|^2)}.$$

It follows that (28) is linearly well-posed when $a, c \leqslant 0$ and $b, d \geqslant 0$. The system corresponding to $\alpha_1 = \alpha_2 = \beta = 0$ is ill-posed (one can check that $a = \frac{1+\gamma\delta}{3\delta(\gamma+\delta)^2} > 0$). This system corresponds to a Hamiltonian system derived in [17] (see their formula (5.10)). As mentioned before, the present, three-parameter family of systems allows one to circumvent the problem of ill-posedness without the need of taking into account higher-order terms in the expansion, as in [17].

Proof. The proof is again made based on various possibilities for the parameters in the problem. For this regime, we have that $\varepsilon \sim \mu \sim \varepsilon_2 \sim \mu_2$ as $\varepsilon \to 0$. The overall idea of the argument is the same as evinced in the proof of Theorem 1.

Step 1. The case $\alpha_1 = 0$, $\beta = 0$, $\alpha_2 = 0$. Using Remark 11 and Corollary 3 (instead of Proposition 1 and Corollary 1 as in the last theorem) one checks immediately that

$$\nabla\psi_1 = -\frac{1}{\gamma+\delta}\left[1 + \mu\frac{1}{3\delta}\frac{1-\delta^2}{\gamma+\delta}\Delta + \varepsilon_2\frac{1+\delta}{\gamma+\delta}\Pi(\zeta\,\cdot)\right]\mathbf{v} + O\big(\varepsilon^2\big);$$

(the nonlocal operator Π does not appear in the final equations because of the identity $\nabla \cdot \Pi V = \nabla \cdot V$ for all $V \in H^1(\mathbb{R}^d)^d$).

Step 2. The case $\alpha_1 \geqslant 0$, $\beta = 0$, $\alpha_2 = 0$. To use the BBM-trick, remark that for all $\alpha_1 \geqslant 0$,

$$\nabla \cdot \mathbf{v} = (1-\alpha_1)\nabla \cdot \mathbf{v} - \alpha_1(\gamma+\delta)\partial_t \zeta + O(\varepsilon).$$

Substitute this relation into the third-derivative term of the first equation of the system derived in Step 1.

Step 3. The case $\alpha_1 \geqslant 0$, $\beta \geqslant 0$, $\alpha_2 = 0$. It suffices to replace $\mathbf{v}$ by $(1-\mu\beta\Delta)\mathbf{v}_\beta$ in the system of equations derived in Step 2.

Step 4. The case $\alpha_1 \geqslant 0$, $\beta \geqslant 0$, $\alpha_2 \leqslant 1$. This is exactly as in Step 4 of Theorem 2. □

3.2. The Shallow water/Shallow water regime: $\mu \sim \mu_2 \ll 1$

Contrary to the regimes investigated above, large amplitude interfacial deformations are allowed for both fluids, as $\varepsilon \sim \varepsilon_2 = O(1)$. As in the previous section, an asymptotic model can be derived from (14) by replacing the operators $G^\mu[\varepsilon\zeta]$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]$ by their asymptotic expansions, provided by Proposition 2 and Corollary 4 in the present regime. The following theorem shows that the internal wave equations are consistent in this regime with the shallow water/shallow water system,

$$\begin{cases} \partial_t \zeta + \dfrac{1}{\gamma+\delta}\nabla \cdot \left(h_1 \mathfrak{Q}\left[\dfrac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta\right](h_2\mathbf{v})\right) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta + \dfrac{\varepsilon}{2}\nabla\left(\left|\mathbf{v} - \dfrac{\gamma}{\gamma+\delta}\mathfrak{Q}\left[\dfrac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta\right](h_2\mathbf{v})\right|^2 - \dfrac{\gamma}{(\gamma+\delta)^2}\left|\mathfrak{Q}\left[\dfrac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta\right](h_2\mathbf{v})\right|^2\right) = 0, \end{cases} \tag{29}$$

where $h_1 = 1 - \varepsilon\zeta$, $h_2 = 1 + \varepsilon\delta\zeta$, and the operator $\mathfrak{Q}$ is defined in Lemma 3.

Theorem 4. *Let* $0 < \delta^{\min} < \delta^{\max} \leqslant \frac{1}{1-H_1}$. *The internal waves equations* (14) *are consistent with the SW/SW equations* (29) *in the sense of Definition* 3, *with a precision $O(\mu)$, and uniformly with respect to $\varepsilon \in [0,1]$, $\mu \in (0,1)$ and* $\delta \in [\delta^{\min}, \delta^{\max}]$.

Remark 26. Taking $\gamma = 0$ and $\delta = 1$ in the SW/SW equations (29) yields the usual shallow water equations for surface water waves (recall that it follows from Lemma 3 that $\nabla \cdot [(1-\varepsilon\zeta)\mathfrak{Q}[-\varepsilon\zeta]((1+\varepsilon\zeta)\mathbf{v})] = \nabla \cdot ((1+\varepsilon\zeta)\mathbf{v})$).

Remark 27. In the one-dimensional case $d = 1$, one has

$$\frac{1}{\gamma+\delta}\mathfrak{Q}\left[\frac{\gamma-1}{\gamma+\delta}\varepsilon\delta\zeta\right](h_2\mathbf{v}) = \frac{h_2}{\delta h_1 + \gamma h_2}\mathbf{v},$$

and Eqs. (29) take the simpler form

$$\begin{cases} \partial_t \zeta + \partial_x\left(\dfrac{h_1 h_2}{\delta h_1 + \gamma h_2}\mathbf{v}\right) = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\partial_x\zeta + \dfrac{\varepsilon}{2}\partial_x\left(\dfrac{(\delta h_1)^2 - \gamma h_2^2}{(\delta h_1 + \gamma h_2)^2}|\mathbf{v}|^2\right) = 0, \end{cases}$$

which coincides of course with the system (5.26) of [17]. The presence of the nonlocal operator $\mathfrak{Q}$, which does not seem to have been noticed before, appears to be a purely two dimensional effect.

Proof. First remark that with the range of parameters considered in the theorem, one has $\mu \sim \mu_2$ as $\mu \to 0$ while $\varepsilon \sim \varepsilon_2 = O(1)$.

By the definition (16) of $\mathbf{v}$ and using Proposition 2 and Corollary 4, one deduces from (14) that

$$\begin{cases} \partial_t\zeta - \nabla\cdot\big((1-\varepsilon\zeta)\nabla\psi_1\big) = O(\mu), \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta + \frac{\varepsilon}{2}\nabla\big(\big|\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1\big|^2 - \gamma|\nabla\psi_1|^2\big) = O(\mu). \end{cases} \tag{30}$$

Recall now that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \mathbf{v} + \gamma\nabla\psi_1$; since moreover one also gets from Corollary 4 that $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = -\delta\mathfrak{Q}[\varepsilon_2\zeta](h_1\nabla\psi_1) + O(\mu)$, it is straightforward to deduce that

$$\mathbf{v} + \gamma\nabla\psi_1 = -\delta\mathfrak{Q}[\varepsilon_2\zeta](h_1\nabla\psi_1) + O(\mu).$$

Multiplying this relation by h_2 and taking the divergence, one gets

$$\begin{aligned} \nabla\cdot(h_2\mathbf{v}) + \gamma\nabla\cdot(h_2\nabla\psi_1) &= -\delta\nabla\cdot\big(h_2\mathfrak{Q}[\varepsilon_2\zeta](h_1\nabla\psi_1)\big) + \nabla\cdot O(\mu) \\ &= -\delta\nabla\cdot(h_1\nabla\psi_1) + \nabla\cdot O(\mu), \end{aligned}$$

where the second equality comes from the definition of the operator $\mathfrak{Q}[\varepsilon_2\zeta]$. We thus have

$$\nabla\cdot\left(\left(1 + \frac{\gamma-1}{\gamma+\delta}\varepsilon_2\zeta\right)\nabla\psi_1\right) = -\frac{1}{\gamma+\delta}\nabla\cdot(h_2\mathbf{v}) + \nabla\cdot O(\mu),$$

and we can therefore use Lemma 3 to conclude that

$$\nabla\psi_1 = -\frac{1}{\gamma+\delta}\mathfrak{Q}\left[\frac{\gamma-1}{\gamma+\delta}\varepsilon_2\zeta\right](h_2\mathbf{v}) + O(\mu),$$

and consequently,

$$\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1 = \mathbf{v} + \gamma\nabla\psi_1 = \mathbf{v} - \frac{\gamma}{\gamma+\delta}\mathfrak{Q}\left[\frac{\gamma-1}{\gamma+\delta}\varepsilon_2\zeta\right](h_2\mathbf{v}) + O(\mu).$$

Replacing $\nabla\psi_1$ and $\mathbf{H}^{\mu,\delta}[\varepsilon\zeta]\psi_1$ by these two expressions in (30) yields the result. □

3.3. The Shallow water/Small amplitude regime: $\mu \ll 1$, $\varepsilon_2 \ll 1$

Derived here are various models corresponding to the case when the upper fluid layer is shallow, but this restriction is not required of the lower layer. The interfacial deviations are thus not necessarily small relative to the upper fluid depth, but they are small relative to the undisturbed depth of the lower layer. Different systems of equations obtain, depending on the sizes of the parameters ε, μ and δ (and thus ε_2 and μ_2).

3.3.1. The Shallow water/Full dispersion regime: $\mu \sim \varepsilon_2^2 \ll 1$, $\varepsilon \sim \mu_2 \sim 1$

In this regime, the internal waves equations are consistent with the shallow water/full dispersion system

$$\begin{cases} \partial_t\zeta + \frac{1}{\gamma}\nabla\cdot(h_1\mathbf{v}) - \frac{\sqrt{\mu}}{\gamma^2}\nabla\cdot\big(h_1|D|\coth\big(\sqrt{\mu_2}|D|\big)\Pi(h_1\mathbf{v})\big) = 0, \\ \partial_t\mathbf{v} + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla\left[|\mathbf{v}|^2 - 2\frac{\sqrt{\mu}}{\gamma}\mathbf{v}\cdot\big(|D|\coth\big(\sqrt{\mu_2}|D|\big)\Pi(h_1\mathbf{v})\big)\right] = 0, \end{cases} \tag{31}$$

where $h_1 = 1 - \varepsilon\zeta$ and $\Pi = -\frac{\nabla\nabla^T}{\Delta}$.

Theorem 5. *Let $0 < c^{\min} < c^{\max}$ and $\mu_2^{\min} < \mu_2 < \mu_2^{\max}$. The internal waves equations* (14) *are consistent with the SW/FD equations* (31) *in the sense of Definition* 3*, with a precision $O(\mu)$, and uniformly with respect to $\varepsilon \in [0,1]$, $\mu \in (0,1)$ and $\delta \in (0,1)$ satisfying the conditions*

$$c^{\min} < \frac{\mu}{\varepsilon^2\delta^2} < c^{\max} \quad \textit{and} \quad \mu_2^{\min} < \frac{\mu}{\delta^2} < \mu_2^{\max}.$$

Remark 28. The SW/FD system (31), which as far as we know is new, is a generalization of the results of Section 5.4 of [17] to the two-dimensional case $d = 2$ and to the case of a lower layer of finite depth (the case of an infinite lower layer is formally recovered here by taking $\mathbf{T}_{\mu_2} = 1$ in (31)).

Proof. First remark that with the range of parameters considered in the theorem, one has $\varepsilon_2^2 \sim \mu$ and $\varepsilon \sim \mu_2 \sim 1$ as $\mu \to 0$.

Proposition 2 implies that $\frac{1}{\mu} G^\mu[\varepsilon\zeta]\psi_1 = \nabla \cdot (h_1 \nabla \psi_1) + O(\mu)$ while it follows from the definition of $\mathbf{v}$ and Corollary 5 that

$$\nabla \psi_1 = -\frac{1}{\gamma}\mathbf{v} + \frac{\sqrt{\mu}}{\gamma^2}\frac{|D|}{\mathbf{T}_{\mu_2}} \Pi(h_1 \mathbf{v}) + O(\mu).$$

One then concludes the proof exactly as in the previous sections. □

3.3.2. The Intermediate long wave regime: $\mu \sim \varepsilon^2 \sim \varepsilon_2 \ll 1$, $\mu_2 \sim 1$

In this regime, a *one-parameter family* of intermediate long wave systems may be derived from the internal waves equations. These depend upon the parameter α and have the form

$$\begin{cases} \left[1 + \sqrt{\mu}\frac{\alpha}{\gamma}|D|\coth\left(\sqrt{\mu_2}|D|\right)\right]\partial_t \zeta + \frac{1}{\gamma}\nabla \cdot \left((1-\varepsilon\zeta)\mathbf{v}\right) - (1-\alpha)\frac{\sqrt{\mu}}{\gamma^2}|D|\coth\left(\sqrt{\mu_2}|D|\right)\nabla \cdot \mathbf{v} = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla|\mathbf{v}|^2 = 0. \end{cases} \tag{32}$$

Theorem 6. *Let $0 < c^{\min} < c^{\max}$, $\mu_2^{\min} < \mu_2 < \mu_2^{\max}$. The internal wave equations* (14) *are consistent with the ILW system* (32) *in the sense of Definition* 3, *with a precision $O(\mu)$, and uniformly with respect to $\varepsilon \in [0,1]$, $\mu \in (0,1)$ and $\delta \in (0,1)$ satisfying the conditions*

$$c^{\min} < \frac{\mu}{\varepsilon^2} < c^{\max} \quad \text{and} \quad \mu_2^{\min} < \frac{\mu}{\delta^2} < \mu_2^{\max}.$$

Remark 29. In dimension $d = 1$ and with $\alpha = 0$, (32) corresponds to (5.47) of [17]. However this system is not linearly well-posed. It is straightforward to ascertain that the condition $\alpha \geqslant 1$ insures that (32) is linearly well-posed for either $d = 1$ or $d = 2$.

Remark 30. The ILW equation derived in [23,25] is obtained as the unidirectional limit of the one dimensional ($d = 1$) version of (32)—see, for instance, Section 5.5 of [17].

Proof. Step 1. The case $\alpha = 0$. We are working with the regime $\mu \sim \varepsilon^2 \sim \varepsilon_2 \ll 1$ and $\mu_2 \sim 1$ as $\mu \to 0$. In this situation, Proposition 2 allows us to write $\frac{1}{\mu}G^\mu[\varepsilon\zeta]\psi_1 = \nabla \cdot ((1-\varepsilon\zeta)\nabla\psi_1) + O(\mu)$ while it follows from the definition of $\mathbf{v}$ and (24) that

$$\nabla \psi_1 = -\frac{1}{\gamma}\mathbf{v} + \frac{\sqrt{\mu}}{\gamma^2}\frac{|D|}{\mathbf{T}_{\mu_2}}\mathbf{v} + O(\mu).$$

Substituting these two relations into the internal wave equations (14) leads to the advertised result with $\alpha = 0$.

Step 2. The case $\alpha \geqslant 0$. This result follows from Step 1 and the observation that

$$\nabla \cdot \mathbf{v} = (1-\alpha)\nabla \cdot \mathbf{v} - \alpha\gamma\partial_t\zeta + O(\varepsilon, \sqrt{\mu}).$$

As mentioned already, the restriction on α is not to obtain consistency, but rather to ensure linear well-posedness. □

3.3.3. The Benjamin–Ono regime: $\mu \sim \varepsilon^2 \ll 1$, $\mu_2 = \infty$

For completeness, we investigate the Benjamin–Ono regime, characterized by the assumption $\delta = 0$ (the lower layer is of infinite depth). Taking $\mu_2 = \infty$ in (32) leads one to replace $\coth(\sqrt{\mu_2}|D|)$ by 1. The following two-dimensional generalization of the system (5.31) in [17] emerges in this situation.

$$\begin{cases} \left[1 + \sqrt{\mu}\frac{\alpha}{\gamma}|D|\right]\partial_t \zeta + \frac{1}{\gamma}\nabla \cdot \left((1-\varepsilon\zeta)\mathbf{v}\right) - (1-\alpha)\frac{\sqrt{\mu}}{\gamma^2}|D|\nabla \cdot \mathbf{v} = 0, \\ \partial_t \mathbf{v} + (1-\gamma)\nabla\zeta - \frac{\varepsilon}{2\gamma}\nabla|\mathbf{v}|^2 = 0. \end{cases} \tag{33}$$

Neglecting the $O(\sqrt{\mu}) = O(\varepsilon)$ terms, one finds that ζ must solve a wave equation (with speed $\sqrt{\frac{1-\gamma}{\gamma}}$). Thus, in the case of horizontal dimension $d = 1$, any interfacial perturbation splits up at first approximation into two counter-propagating waves. If one includes the $O(\sqrt{\mu}, \varepsilon)$ terms, one obtains the one-parameter family

$$\left(1 + \sqrt{\mu}\frac{\alpha}{\gamma}|\partial_x|\right)\partial_t\zeta + c\partial_x\zeta - \varepsilon\frac{3}{4}c\partial_x\zeta^2 - \frac{\sqrt{\mu}}{2\gamma}c(1-2\alpha)|\partial_x|\partial_x\zeta = 0, \tag{34}$$

of *regularized Benjamin–Ono equations* (see [11]). Here, $c = \sqrt{\frac{1-\gamma}{\gamma}}$. The usual Benjamin–Ono equation is recovered by taking $\alpha = 0$.

Appendix A. Proof of Proposition 3

The proof is made in five steps.

Step 1. *Coercivity of the operator* $\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}$. Exactly as in Proposition 2.3 of [2], one may check that

$$\forall \Theta \in \mathbb{R}^{d+1}, \quad \Theta \cdot Q^{\mu_2}[\varepsilon_2\zeta]\Theta \geqslant \frac{1}{k}|\Theta|^2,$$

with $k = k(\frac{1}{H_2}, \varepsilon\sqrt{\mu}, \varepsilon_2|\zeta|_{W^{1,\infty}}) > 0$.

Step 2. *Existence of a unique solution to (21).* Owing to Step 1, existence of a solution and uniqueness up to a constant is provided by classical theorems (e.g., Section V.7 of [34]), provided that the source terms and Neumann conditions satisfy the compatibility condition

$$\int_{\mathcal{S}} \nabla^{\mu_2}_{X,z} \cdot \mathbf{h} = \int_{\{z=0\}} \left(\sqrt{\mu_2}\,\nabla \cdot V + \mathbf{e}_z \cdot \mathbf{h}\right) - \int_{\{z=-1\}} \mathbf{e}_z \cdot \mathbf{h}.$$

This latter restriction is valid in the present circumstances on account of the divergence theorem.

Step 3. *L^2-estimate on* $\nabla^{\mu_2}_{X,z}u$. Multiplying (21) by u, integrating by parts on both sides, and using the Neumann conditions leads to

$$\int_{\mathcal{S}} \nabla^{\mu_2}_{X,z}u \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}u = -\int_{\{z=0\}} V \cdot \sqrt{\mu_2}\,\nabla u + \int_{\mathcal{S}} \mathbf{h} \cdot \nabla^{\mu_2}_{X,z}u.$$

A direct consequence of Step 1 and the Cauchy–Schwarz inequality is the inequality,

$$\left\|\nabla^{\mu_2}_{X,z}u\right\|^2 \leqslant k\left(\|\mathbf{h}\|\left\|\nabla^{\mu_2}_{X,z}u\right\| + |V|_{H^{1/2}}\left|\sqrt{\mu_2}\,\nabla u\right|_{H^{-1/2}}\right).$$

It follows from the trace theorem that

$$\begin{aligned}\left|\sqrt{\mu_2}\,\nabla u\right|_{H^{-1/2}} &\leqslant \mathrm{Cst}\left(\left\|\sqrt{\mu_2}\,\nabla u\right\| + \left\|\Lambda^{-1}\sqrt{\mu_2}\partial_z\nabla u\right\|\right)\\ &\leqslant \mathrm{Cst}\left(\left\|\nabla^{\mu_2}_{X,z}u\right\| + \sqrt{\mu_2}\left\|\nabla^{\mu_2}_{X,z}u\right\|\right).\end{aligned}$$

It is concluded that

$$\left\|\nabla^{\mu_2}_{X,z}u\right\| \leqslant C\left(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{W^{1,\infty}}\right)\left(\|\mathbf{h}\| + |V|_{H^{1/2}}\right).$$

Step 4. *H^s-estimate ($s \geqslant 0$) on* $\nabla^{\mu_2}_{X,z}u$. Let $v = \Lambda^s u$. Multiplying (21) by Λ^s on both sides, it results that v solves the system

$$\begin{cases}\nabla^{\mu_2}_{X,z} \cdot Q^{\mu_2}[\varepsilon_2\zeta]\nabla^{\mu_2}_{X,z}v = \nabla^{\mu_2}_{X,z} \cdot \widetilde{\mathbf{h}}, & \text{in } \mathcal{S},\\ \partial_n v|_{z=0} = \sqrt{\mu_2}\,\nabla \cdot \Lambda^s V + \mathbf{e}_z \cdot \widetilde{\mathbf{h}}|_{z=0}, & \partial_n v|_{z=-1} = \mathbf{e}_z \cdot \widetilde{\mathbf{h}}|_{z=-1},\end{cases}$$

with $\widetilde{\mathbf{h}} = \Lambda^s \mathbf{h} + [Q^{\mu_2}[\varepsilon_2\zeta], \Lambda^s]\nabla^{\mu_2}_{X,z} u$. From Step 3 and the definition of v, it is thus deduced that $\|\Lambda^s \nabla^{\mu_2}_{X,z} u\|$ is bounded from above by

$$C\left(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{W^{1,\infty}}\right)\left(\|\Lambda^s \mathbf{h}\| + |V|_{H^{s+1/2}} + \left\|\left[Q^{\mu_2}[\varepsilon_2\zeta], \Lambda^s\right]\nabla^{\mu_2}_{X,z} u\right\|\right).$$

Using the expression for $Q^{\mu_2}[\varepsilon_2\zeta]$ and the commutator estimate,

$$\left|\left[\Lambda^s, f\right]g\right|_2 \leqslant C|\nabla f|_{H^{\max\{t_0, s-1\}}}|g|_{H^{s-1}},$$

which holds for some constant C which depends upon $s > -\frac{d}{2}$ and $t_0 > \frac{d}{2}$ (see Theorem 6 of [27]), we obtain

$$\left\|\left[Q^{\mu_2}[\varepsilon_2\zeta], \Lambda^s\right]\nabla^{\mu_2}_{X,z} u\right\| \leqslant C\left(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{\max\{t_0+2, s+1\}}}\right)\left(\left\|\Lambda^{s-1}\nabla^{\mu_2}_{X,z} u\right\|\right).$$

We thus get an estimate on $\|\Lambda^s \nabla^{\mu_2}_{X,z} u\|$ in terms of $\|\Lambda^{s-1}\nabla^{\mu_2}_{X,z} u\|$ which, together with Step 3 (i.e. $s = 0$) allows us to derive the following relation by induction (and interpolation when $s \in (0,1)$)

$$\forall s \geqslant 0, \quad \left\|\Lambda^s \nabla^{\mu_2}_{X,z} u\right\| \leqslant C\left(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{\max\{t_0+2, s+1\}}}\right)\left(\left\|\Lambda^s \mathbf{h}\right\| + |V|_{H^{s+1/2}}\right).$$

Step 5. *H^s-estimate ($s \geqslant 0$) on $\partial_z \nabla^{\mu_2}_{X,z} u$.* First remark that using the equation yields the formula

$$\begin{aligned}\frac{1 + \mu\varepsilon^2(z+1)^2|\nabla\zeta|^2}{1+\varepsilon_2\zeta}\partial_z^2 u &= \nabla^{\mu_2}_{X,z}\cdot \mathbf{h} - \sqrt{\mu_2}\,\nabla\cdot\left((1+\varepsilon_2\zeta)\sqrt{\mu_2}\,\nabla u - \sqrt{\mu}\varepsilon(z+1)\nabla\zeta\,\partial_z u\right) \\ &\quad + \sqrt{\varepsilon}\mu\nabla\zeta\cdot\left(\sqrt{\mu_2}\,\nabla u\right) - 2\mu\varepsilon^2(z+1)\frac{|\nabla\zeta|^2}{1+\varepsilon_2\zeta}\partial_z u,\end{aligned}$$

from which one obtains the estimate

$$\left\|\Lambda^s \partial_z^2 u\right\| \leqslant C\left(\varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{\max\{t_0+2, s+1\}}}\right)\left(\left\|\Lambda^s \nabla^{\mu_2}_{X,z}\mathbf{h}\right\| + \sqrt{\mu_2}\left\|\Lambda^{s+1}\nabla^{\mu_2}_{X,z} u\right\|\right).$$

Use this to write

$$\begin{aligned}\left\|\Lambda^s \partial_z \nabla^{\mu_2}_{X,z} u\right\| &\leqslant \sqrt{\mu_2}\left\|\Lambda^s\partial_z\nabla u\right\| + \left\|\Lambda^s \partial_z^2 u\right\| \\ &\leqslant C\left(\varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{\max\{t_0+2, s+1\}}}\right)\left(\left\|\Lambda^s \nabla^{\mu_2}_{X,z}\mathbf{h}\right\| + \sqrt{\mu_2}\left\|\Lambda^{s+1}\nabla^{\mu_2}_{X,z} u\right\|\right).\end{aligned}$$

With the help of Step 4, one obtains the inequality

$$\left\|\Lambda^s \partial_z \nabla^{\mu_2}_{X,z} u\right\| \leqslant C\left(\frac{1}{H_2}, \varepsilon_2^{\max}, \mu_2^{\max}, |\zeta|_{H^{\max\{t_0+2, s+1\}}}\right)\left(\|\mathbf{h}\|_{H^{s+1,1}} + |V|_{H^{s+3/2}}\right).$$

Step 6. *Conclusion.* By the trace theorem we may assert that for all $s \geqslant 0$,

$$|\nabla u_{|z=0}|_{H^s} \leqslant \mathrm{Cst}\,\|\nabla u\|_{H^{s+1/2,1}} \leqslant \frac{\mathrm{Cst}}{\sqrt{\mu_2}}\left\|\nabla^{\mu_2}_{X,z} u\right\|_{H^{s+1/2,1}}.$$

The desired result now follows from Steps 4 and 5.

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