Justifying asymptotics for $3D$ water-waves

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Abstract

The aim of this article is to sketch the main steps of a general method to fully justify asymptotic models for $3D$ water-waves. The strategy adopted here is the one developed in full details in [1]. The key step is to prove a large time existence result for the nondimensionalized water-waves equations written in terms of the water elevation and the velocity potential at the surface. The theorem also furnishes a bound on a special energy introduced to have uniform control on the solution (with respect to the nondimensionalization parameters). We then describe a systematic way to provide asymptotic expansions on the Dirichlet-Neumann operator involved in the water-waves equations, and deduce asymptotic models in different physical regimes. Full justification of $2DH$ Boussinesq systems, of $2DH$ shallow water equations, and of the Kadomtsev-Petviashvili approximation are sketched with some details as an illustration of these results.

1 Introduction

1.1 General setting

The motion of a perfect, incompressible and irrotational fluid under the influence of gravity is described by the free surface Euler (or water-waves) equations. These equations have a very rich structure and many famous equations of mathematical physics can be obtained as asymptotic limits: the Korteweg-de Vries (KdV) and Kadomtsev-Petviashvilii (KP) equations, the Boussinesq systems, the shallow water equations, deep water models etc. Each of these asymptotic limits corresponds to a very specific physical regime which determines its range of validity as a tool in oceanography.

While the derivations of these models goes back to the XIXth century, their mathematical justification is a much more recent concern (by mathematical justification, we mean the rigorous proof that the solution of the water-waves equations is well approximated by the solution of the asymptotic model corresponding to the physical regime under consideration). So far, the only asymptotic models fully justified are the KdV equations and the $1DH$-Boussinesq

systems (see [8, 24, 4]) and some variants in presence of surface tension [25], bottom topography [13], or higher order terms [27]; note also that Kano and Nishida [15] gave a justification of the $1DH$-shallow water equations under some restrictions (analytic and small data).
For the $2DH$-case and the other regimes mentioned above, there is no full rigorous justification; one of the reasons for this is the complexity of the water-waves equations for which local well-posedness and error estimates are nontrivial. Following the pioneer works in $1DH$ of Ovsjannikov [23] and Nalimov [22] (see also Yosihara [30, 31]), Craig [8] and Kano and Nishida [15] managed to give the first justification of the KdV and $1DH$ Boussinesq and shallow water approximations, but the comprehension of the well-posedness theory for the water-waves equations hindered the perspective of justifying the other asymptotic regimes until the breakthroughs of S. Wu ([28] and [29] respectively for the $1DH$ and $2DH$ case, in infinite depth, and without restrictive assumptions). Since when, the literature on free surface Euler equations has been very active: the case of finite depth was proved in [17], and in the related case of the study of the free surface of a liquid in vacuum with zero gravity, Lindblad [19, 20] and more recently Coutand and Shkoller [7] and Shatah and Zeng [26] managed to remove the irrotationality condition and/or took into account surface tension effects.
Though quite numerous now, the results on the well-posedness of the water-waves equations cannot be directly applied to justify rigorously asymptotic models because the estimates they give on the existence time are far too rough and only provide existence over an interval of time asymptotically shrinking to zero (relatively to the pertinent time scale). This difficulty is well illustrated by the early works of Kano and Nishida [15] and Kano [14] where the KdV and KP approximations are justified (for analytic and small data) for times $t = O(1)$ while the relevant time scale for the asymptotics is $t = O(1/\varepsilon)$ (with the notations used in the present paper). In [8] this confusion is not made, and the proof relies on a large time (i.e. $O(1/\varepsilon)$) existence theorem for the water-waves equations in the particular "long-waves" scaling. It was recently shown in [4] that the $2DH$ Boussinesq systems are justified with sharp error estimates if solutions to the water-waves equations in the long-waves regime exist over the time scale $t = O(1/\varepsilon)$ and are bounded in regular enough Sobolev spaces. Similarly, it is proved in [18] that the rigorous justification of the KP approximation follows from such a large time existence theorem, and from (unexpected enough) bounds on the solution.

Regardless of the physical regime investigated, the key steps in the process of justification of asymptotic equations is thus the following:

1. Formally derive the asymptotic equations and identify the relevant time scale of their dynamics;
2. Prove an existence result for the water-waves equations for this time scale (this is what we call here "large time" existence) and bounds on the solution;
3. Perform error estimates to control the error between the exact solution of

the water-waves equations and the solution furnished by the asymptotic model.

The first step of this procedure can be done at the formal level, while the third one can be done assuming the second one (as in [4, 6, 18]). Therefore, it turns out that the proof of a large time existence theorem is the key step of the process. Before explaining the approach developped here, let us review quickly existing results for the main physical regimes: in order to do this, let us denote by a the typical amplitude of the waves, by h the mean depth, and by λ the wavelength of the waves.

- Shallow-water equations (i.e. $h^2/\lambda^2 \ll 1$). In $1DH$, steps 1 to 3 of the above procedure are done in [15], with some restrictions in Step 2 (analytic and small data). In $2DH$, Steps 2 and 3 remain open.

- Long-wave regime (i.e. $h^2/\lambda^2 \sim a/h \ll 1$). The justification process is complete in $1DH$ [8, 24, 4, 13]; in $2DH$, Steps 1 and 3 are done in [4] (flat bottom) and [6] (uneven bottom) but Step 2 is open.

- KP or weakly transverse regime (this regime is the same as the $2DH$ long-wave regime, but with a wavelength in the transverse direction much larger than in the longitudinal direction). As said above, [18] shows that only Step 2 remains to be done.

- Serre approximation (i.e. $h/\lambda \sim a/h \ll 1$). These equations are commonly used in oceanography (see for instance Chapter 5.7 of [12]), but no mathematical justification exists.

- Deep water models. In deep water ($h^2/\lambda^2 \gg 1$) the asymptotic expansions are commonly made in terms of the slope of the waves ($a/\lambda \ll 1$). For instance, Matsuno [21] proposed (without justification) a model with full dispersion valid for deep water in $1D$.

Instead of developing an existence theory for each physical scaling, we develop here a global method which allows one to justify all the asymptotics mentioned above at once. In order to do that, we nondimensionalize the water-waves equations, and keep track of the four physical quantities which characterize the dynamics of the water-waves: amplitude, depth, wavelength in the longitudinal direction and wavelength in the transverse direction (for the sake of simplicity, we only consider in this note flat bottoms; in the case of uneven bottoms, a fifth parameter must be introduced, the amplitude of the bottom variations).
Our main theorem gives an estimate of the existence time of the solution of the water-waves equations in terms of these four parameters. It is worth remarking that this estimate is *uniform* with respect to these parameters (though they may grow to infinity or decay to zero, depending on the physical regime investigated). In order to prove this theorem we introduce an energy which involves the aforementioned parameters and use it to construct our solution by an iterative scheme. This energy provides moreover bounds on the solution which

appear to be exactly those needed for the error estimates of Step 3.
Having proved such a large time existence result, we derive the asymptotic models for the regimes mentioned above in a systematic way, and use the bounds on the solution provided by the energy to proceed with Step 3.

1.2 Presentation of the results

Parameterizing the free surface by $z = \zeta(t, X)$ (with $X = (x, y) \in \mathbb{R}^2$) and the bottom by $z = -h$ (with $h > 0$ constant − uneven bottoms are considered in [1]), one can use the incompressibility and irrotationality conditions to write the water-waves equations under Bernouilli's formulation, in terms of a velocity potential Φ (i.e., the velocity field is given by $\mathbf{v} = \nabla_{X,z}\Phi$):

$$\begin{cases} \partial_x^2\Phi + \partial_y^2\Phi + \partial_z^2\Phi = 0, & -h \le z \le \zeta, \\ \partial_n\Phi = 0, & z = -h, \\ \partial_t\zeta + \nabla\zeta \cdot \nabla\Phi = \partial_z\Phi, & z = \zeta, \\ \partial_t\Phi + \frac{1}{2}\big(|\nabla\Phi|^2 + (\partial_z\Phi)^2\big) + \zeta = 0, & z = \zeta, \end{cases} \qquad (1)$$

where $\nabla = (\partial_x, \partial_y)^T$ and $\partial_n\Phi$ is the normal derivative.

The qualitative study of the water-waves equations is made easier by the introduction of dimensionless variables and unknowns. This requires the introduction of various orders of magnitude linked to the physical regime under consideration. As said in the introduction, these quantities are:

- a is the order of amplitude of the waves;
- h is the mean depth;
- λ is the wavelength of the waves in the x direction;
- λ/γ is the wavelength of the waves in the y direction.

We also introduce the following dimensionless parameters

$$\varepsilon = \frac{a}{h}, \quad \mu = \frac{\lambda^2}{h^2}, \quad \nu = \frac{1}{1+\sqrt{\mu}}; \qquad (2)$$

the parameter ε is often called *nonlinearity* parameter, while μ is the *shallowness* parameter. The parameter ν is a *transition* parameter which takes into account the fact that different nondimensionalizations are used in shallow and deep water.

Zakharov [32] remarked that the system (1) could be written in Hamiltonian form in terms of the free surface elevation ζ and of the trace of the velocity potential at the surface $\psi = \Phi_{|z=\zeta}$ and Craig, Sulem and Sulem [11] and Craig, Schanz and Sulem [10] used the fact that (1) could be reduced to a system of two evolution equations on ζ and ψ to prove the consistency of the Schrödinger and Davey-Stewartson approximation; this formulation has commonly been used since when. In Section 2, we derive the following dimensionless form of this formulation, which involves the parameters introduced in (2):

$$\left\{\begin{array}{l} \partial_t\zeta - \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu}\frac{(\frac{1}{\mu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi + \varepsilon\nabla^\gamma\zeta\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0, \end{array}\right. \quad (3)$$

where $\nabla^\gamma = (\partial_x, \gamma\partial_y)^T$ and $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = (\partial_z\Phi - \mu\varepsilon\nabla^\gamma\zeta\cdot\nabla^\gamma\Phi)_{|z=\varepsilon\zeta}$, with Φ solving the boundary value problem

$$\left\{\begin{array}{ll} \partial_z^2\Phi + \mu\partial_x^2\Phi + \gamma\mu\partial_y^2\Phi = 0, & -1 < z < \varepsilon\zeta \\ \Phi_{|z=\varepsilon\zeta} = \psi, \qquad \partial_z\Phi_{|z=-1} = 0. & \end{array}\right. \quad (4)$$

Section 3 is devoted to the asymptotic expansion of the Dirichlet-Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in terms of the parameters ε, γ and μ; we show how to get explicit expansions in the cases mentioned above.
Section 4 is devoted to the study of the well-posedness of the water-waves equations for large times. With the notations above, we show that solutions to (3) exist and are unique over times $t = O(\frac{1}{\varepsilon/\nu})$; we also prove that the energy

$$|\zeta|_{H^s} + \Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^s} \quad (\text{with } |D^\gamma| = \sqrt{D_x^2 + \gamma^2 D_y^2});$$

remains bounded over this time scale. It turns our that this existence time and this bound on the solution are exactly those needed to justify rigorously all the models described above. This is sketched in Section 5 for three asymptotic models: the shallow water equations, the Boussinesq system, and the Kadomtsev-Petviashvili (KP) approximation.

1.3 Notations

- When we want to insist on the dependence of some constant C on various parameters $p_1, p_2, \ldots$, we write $C = C(p_1, p_2, \ldots)$, and always assume that the dependence on the parameters is *nondecreasing*.
- For all tempered distribution $u \in \mathfrak{S}'(\mathbb{R}^d)$, we denote by $\widehat{u}$ its Fourier transform.
- Fourier multipliers: For all rapidly decaying $u \in \mathfrak{S}(\mathbb{R}^d)$ and all $f \in C(\mathbb{R}^d)$ with tempered growth, $f(D)$ is the distribution defined by

$$\forall \xi \in \mathbb{R}^d, \qquad \widehat{f(D)u}(\xi) = f(\xi)\widehat{u}(\xi); \quad (5)$$

(this definition can be extended to wider spaces of functions).
- We write $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, $\Lambda = \langle D\rangle$ and $\xi^\gamma = (\xi, \gamma\xi_2)$.
- For all $1 \le p \le \infty$, $|\cdot|_p$ denotes the classical norm of $L^p(\mathbb{R}^d)$ while $\|\cdot\|_p$ stands for the canonical norm of $L^p(\mathcal{S})$, with $\mathcal{S} = \mathbb{R}^d \times (-1, 0)$.
- For all $s \in \mathbb{R}$, $H^s(\mathbb{R}^d)$ is the classical Sobolev space defined as

$$H^s(\mathbb{R}^d) = \{u \in \mathfrak{S}'(\mathbb{R}^d), |u|_{H_s} := |\Lambda^s u|_2 < \infty\}.$$

- For all $\gamma > 0$, we write $\nabla^\gamma = (\partial_x, \gamma\partial_y)^T$, so that ∇^γ coincides with the usual gradient when $\gamma = 1$. We also use the Fourier multiplier $|D^\gamma|$ defined as

$$|D^\gamma| = \sqrt{D_x^2 + \gamma^2 D_y^2},$$

as well as the anisotropic divergence operator

$$\mathrm{div}_\gamma = (\nabla^\gamma)^T.$$

- We write $X = (x, y)$ and $\nabla_{X,z} = (\partial_x, \partial_y, \partial_z)^T$.
- The notation $a \lesssim b$ means that $a \leq Cb$, for some nonnegative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*).
- We use the condensed notation

$$A_s = B_s + \langle C_s \rangle_{s>\underline{s}} \tag{6}$$

to say that $A_s = B_s$ if $s \leq \underline{s}$ and $A_s = B_s + C_s$ if $s > \underline{s}$.
- When the notation $\partial_n u_{|\partial\Omega}$ is used for boundary conditions of an elliptic equation of the form $\nabla_{X,z} \cdot P\nabla_{X,z} u = h$ in some open set Ω, it stands for the *conormal derivative* associated to this operator, namely,

$$\partial_n u_{|\partial\Omega} = \mathbf{n} \cdot P\nabla_{X,z} u_{|\partial\Omega}, \tag{7}$$

$\mathbf{n}$ standing for the *outward* unit normal vector to $\partial\Omega$.

2 Nondimensionalization(s) of the equations

Depending on the value of μ, two distinct nondimensionalizations are commonly used in oceanography (see for instance [12]). Namely, with dimensionless quantities denoted with a prime:

- Shallow-water, i.e. $\mu \ll 1$, one writes

$$x = \lambda x', \quad y = \frac{\lambda}{\gamma} y', \qquad z = hz', \quad t = \frac{\lambda}{\sqrt{gh}} t',$$
$$\zeta = a\zeta', \quad \Phi = \frac{a}{h}\lambda\sqrt{gh}\Phi'.$$

- Deep-water, i.e. $\mu \gg 1$, one writes

$$x = \lambda x', \quad y = \frac{\lambda}{\gamma} y', \qquad z = \lambda z', \quad t = \frac{\lambda}{\sqrt{g\lambda}} t',$$
$$\zeta = a\zeta', \quad \Phi = a\sqrt{g\lambda}\Phi'.$$

Remarking that when $\mu \sim 1$, that is when $\lambda \sim h$, both nondimensionalizations are equivalent, we introduce the following general nondimensionalization, which is valid for all $\mu > 0$:

$$x = \lambda x', \quad y = \frac{\lambda}{\gamma} y', \qquad z = h\nu z', \quad t = \frac{\lambda}{\sqrt{gh\nu}} t',$$
$$\zeta = a\zeta', \quad \Phi = \frac{a}{h}\lambda\sqrt{\frac{gh}{\nu}}\Phi', \quad b = Bb',$$

where $\nu = \frac{1}{1+\sqrt{\nu}}$ is a smooth function of μ such that $\nu \sim 1$ when $\mu \ll 1$ and $\nu \sim \mu^{-1/2} (= \lambda/h)$ when $\mu \gg 1$ (in [21], the parameter κ plays a similar role).

The equations of motion (1) then become (after dropping the primes for the sake of clarity):

$$\begin{cases} \nu^2\mu\partial_x^2\Phi + \nu^2\gamma^2\mu\partial_y^2\Phi + \partial_z^2\Phi = 0, & -\frac{1}{\nu} \leq z \leq \frac{\varepsilon}{\nu}\zeta, \\ -\nu^2\mu\nabla^\gamma(\frac{\beta}{\nu}b)\cdot\nabla^\gamma\Phi + \partial_z\Phi = 0, & z = -\frac{1}{\nu}, \\ \partial_t\zeta - \frac{1}{\mu\nu^2}\big(-\nu^2\mu\nabla^\gamma(\frac{\varepsilon}{\nu}\zeta)\cdot\nabla^\gamma\Phi + \partial_z\Phi\big) = 0, & z = \frac{\varepsilon}{\nu}\zeta, \\ \partial_t\Phi + \frac{1}{2}\big(\frac{\varepsilon}{\nu}|\nabla^\gamma\Phi|^2 + \frac{\varepsilon}{\mu\nu^3}(\partial_z\Phi)^2\big) + \zeta = 0, & z = \frac{\varepsilon}{\nu}\zeta, \end{cases} \tag{8}$$

with $\nabla^\gamma = (\partial_x, \gamma\partial_y)^T$.
In order to reduce this set of equations into a system of two evolution equations, define the Dirichlet-Neumann operator $\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\cdot$ as

$$\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\psi = \sqrt{1+|\nabla(\frac{\varepsilon}{\nu}\zeta)|^2}\partial_n\Phi_{|_{z=\frac{\varepsilon}{\nu}\zeta}},$$

with Φ solving the boundary value problem

$$\begin{cases} \nu^2\mu\partial_x^2\Phi + \nu^2\gamma^2\mu\partial_y^2\Phi + \partial_z^2\Phi = 0, & -\frac{1}{\nu} \leq z \leq \frac{\varepsilon}{\nu}\zeta, \\ \Phi_{|_{z=\frac{\varepsilon}{\nu}\zeta}} = \psi, \qquad \partial_n\Phi_{|_{z=\frac{1}{\nu}(-1+\beta b)}} = 0, \end{cases}$$

(as always in this paper, $\partial_n\Phi$ stands for the outward conormal derivative associated to the elliptic equation). As remarked by in [32, 11, 10], the equations (8) are equivalent to a set of two equations on the free surface parameterization ζ and the trace of the velocity potential at the surface $\psi = \Phi_{|_{z=\varepsilon/\nu\zeta}}$ involving the Dirichlet-Neumann operator $\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]$. Namely,

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu\nu^2}\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu^3}\frac{(\frac{1}{\mu}\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\psi + \nu\nabla^\gamma(\varepsilon\zeta)\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0. \end{cases} \tag{9}$$

In order to derive the system (3), let $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\cdot$ be the Dirichlet-Neumann operator $\mathcal{G}^\nu_{\mu,\gamma}[\varepsilon\zeta]\cdot$ corresponding to the case $\nu = 1$. One will easily check that

$$\forall\nu > 0, \qquad \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\cdot = \frac{1}{\nu}\mathcal{G}^\nu_{\mu,\gamma}[\frac{\varepsilon}{\nu}\zeta]\cdot,$$

so that plugging this relation into (9) yields

$$\begin{cases} \partial_t\zeta - \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = 0, \\ \partial_t\psi + \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu}\frac{(\frac{1}{\mu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi + \nabla^\gamma(\varepsilon\zeta)\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)} = 0. \end{cases}$$

3 Asymptotic expansion of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$

Throughout this section, we assume that the water height is always positive, that is

$$\exists h_0 > 0, \qquad \inf_{\mathbb{R}^d}(1+\varepsilon\zeta) \geq h_0. \tag{10}$$

3.1 The case of small amplitude waves ($\varepsilon \ll 1$)

Expansions of the Dirichlet-Neumann operator for small amplitude waves has been developed in [11, 10]. This method is very efficient to compute the formal expansion, but instead of adapting it in the present case to give uniform estimates on the truncation error, we rather propose a very simple method based on the following explicit formula for the derivative of the mapping $\zeta \mapsto \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\underline{\psi}$, which is a particular case of Theorem 3.20 of [17].

Theorem 1 *Let $t_0 > 1$, $s \geq t_0$ and $\underline{\zeta} \in H^{s+3/2}(\mathbb{R}^2)$ be such that (10) is satisfied for some $h_0 > 0$. For all $\underline{\psi} \in H^{s+3/2}(\mathbb{R}^2)$, the mapping*

$$\zeta \mapsto \mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\underline{\psi} \in H^{s+1/2}(\mathbb{R}^2)$$

is well defined and differentiable in a neighborhood of $\underline{\zeta}$ in $H^{s+3/2}(\mathbb{R}^2)$, and

$$\forall h \in H^{s+3/2}(\mathbb{R}^2), \qquad d_{\underline{\zeta}}\mathcal{G}_{\mu,\gamma}[\varepsilon\cdot]\underline{\psi}\cdot h = -\varepsilon\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}](h\underline{Z}) - \varepsilon\mu\nabla^\gamma\cdot(h\underline{\mathbf{v}}),$$

with

$$\begin{aligned} \underline{Z} &= \frac{1}{1+\varepsilon^2\mu|\nabla^\gamma\underline{\zeta}|^2}(\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\underline{\psi} + \varepsilon\mu\nabla^\gamma\underline{\zeta}\cdot\nabla^\gamma\underline{\psi}), \\ \underline{\mathbf{v}} &= \nabla^\gamma\underline{\psi} - \varepsilon\underline{Z}\nabla^\gamma\underline{\zeta}. \end{aligned}$$

We can now state the following proposition, which gives an expansion of the Dirichlet-Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in terms of ε, and uniform with respect to $\gamma \in (0,1]$ and $\mu > 0$.

Proposition 1 *Let $s \geq t_0 > 1$, $\psi \in H^{s+4}(\mathbb{R}^2)$ and $\zeta \in H^{s+9/2}(\mathbb{R}^2)$ be such that (10) is satisfied for some $h_0 > 0$. Then one has*

$$\begin{aligned} &\big|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi - \big[\mathcal{G}_{\mu,\gamma}[0] - \varepsilon\mathcal{G}_{\mu,\gamma}[0]\big(\zeta(\mathcal{G}_{\mu,\gamma}[0]\psi)\big) - \varepsilon\mu\nabla^\gamma\cdot(\zeta\nabla^\gamma\psi)\big]\big|_{H^s} \\ &\leq \varepsilon^2\mu^{3/2}C\Big(\frac{1}{h_0}, \varepsilon\sqrt{\mu}, |\zeta|_{H^{s+9/2}}, \big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\big|_{H^{s+7/2}}\Big). \end{aligned}$$

Proof.
An order two expansion of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ gives

$$\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = \mathcal{G}_{\mu,\gamma}[0]\psi + d_0\mathcal{G}_{\mu,\gamma}[\varepsilon\cdot]\psi\cdot\zeta + \int_0^1(1-z)d^2_{z\zeta}\mathcal{G}[\varepsilon\cdot]\psi\cdot(\zeta,\zeta)dz.$$

Using Theorem 1, one computes

$$d_0\mathcal{G}_{\mu,\gamma}[\varepsilon\cdot]\psi\cdot\zeta = -\varepsilon\mathcal{G}_{\mu,\gamma}[0]\big(\zeta(\mathcal{G}_{\mu,\gamma}[0]\psi)\big) - \varepsilon\mu\nabla^\gamma\cdot(\zeta\nabla^\gamma\psi)$$

and, with some more work, one also gets an explicit expression for the second derivative $d^2_{z\zeta}\mathcal{G}[\varepsilon\cdot]\psi\cdot(\zeta,\zeta)$. It appears that one can write

$$d^2_{z\zeta}\mathcal{G}[\varepsilon\cdot]\psi\cdot(\zeta,\zeta) = \varepsilon^2\mu^{3/2}F(z,\varepsilon,\mu,\gamma,\zeta,\psi),$$

and that

$$|F(z,\varepsilon,\mu,\gamma,\zeta,\psi)|_{H^s} \leq C\Big(\varepsilon\sqrt{\mu},\frac{1}{h_0},|\zeta|_{H^{s+9/2}},\big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\big|_{H^{s+7/2}}\Big),$$

uniformly with respect to all the parameters; an important step in the above estimate is the following estimate on the operator norm of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]$:

$$\forall s\geq t_0>1,\qquad |\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi|_{H^{s-1/2}} \leq C(\frac{1}{h_0},|\zeta|_{H^{s+1}})\big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\big|_{H^s}.$$

■

We can now give asymptotic expansions of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in the different regimes mentioned in the introduction. The first one is the long-waves regime (see also [4] for a different proof based on a BKW expansion of the velocity potential).

Corollary 1 (Long-Waves regime) *Let $\varepsilon_0>0$, $s\geq t_0>1$, $\psi\in H^{s+6}(\mathbb{R}^2)$ and $\zeta\in H^{s+9/2}(\mathbb{R}^2)$ be such that (10) is satisfied for some $h_0>0$.*
If $\gamma=1$ then for all $0<\varepsilon=\mu<\varepsilon_0$, one has

$$\begin{aligned}\big|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi \quad &- \quad \big[-\varepsilon\Delta\psi-\varepsilon^2(\frac{1}{3}\Delta^2\psi+\nabla\cdot(\zeta\nabla\psi)\big]\big|_{H^s}\\ &\leq \quad \varepsilon^3 C\Big(\frac{1}{h_0},\varepsilon_0,|\zeta|_{H^{s+9/2}},\big|\nabla\psi\big|_{H^{s+5}}\Big).\end{aligned}$$

Proof.
Under the long-waves regime, one can compute explicitly $\mathcal{G}_{\mu,\gamma}[0]\psi=\mathcal{G}_{\varepsilon,1}[0]\psi$ (see Proposition 3 below for the computation):

$$\mathcal{G}_{\varepsilon,1}[0]\psi = \sqrt{\varepsilon}|D|\tanh(\sqrt{\varepsilon}|D|)\psi,$$

and a second order Taylor expansion of the function $\varepsilon\mapsto\sqrt{\varepsilon}z\tanh(\sqrt{\varepsilon}z)$ at the origin gives therefore

$$\big|\mathcal{G}_{\varepsilon,1}[0]\psi-\big[-\varepsilon\Delta\Psi-\varepsilon^2\frac{1}{3}\Delta^2\Psi\big]\big|_{H^s}\lesssim\varepsilon^3|\nabla\psi|_{H^{s+5}}.$$

Since $\mu=\varepsilon$, $\gamma=1$ and $\nu\sim1$ in the present scaling, one also deduces that

$$\big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\big|_{H^s}\leq\big|\nabla\psi|_{H^s},$$

uniformly with respect to ε, and the corollary follows therefore from Proposition 1.

■

In the case of the KP regime (or weakly transverse long-waves), which is the same as the long-wave regime described above, but with $\gamma = \sqrt{\varepsilon}$, one has (see also [18]):

Corollary 2 (KP regime) *Let $\varepsilon_0 > 0$, $s \geq t_0 > 1$, $\psi \in H^{s+6}(\mathbb{R}^2)$ and $\zeta \in H^{s+9/2}(\mathbb{R}^2)$ be such that (10) is satisfied for some $h_0 > 0$.*
Then for all $0 < \varepsilon = \mu = \gamma^2 < \varepsilon_0$, one has

$$\begin{aligned}\big|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi \quad &- \quad \big[-\varepsilon\partial_x^2\psi - \varepsilon^2\big(\frac{1}{3}\partial_x^4\psi + \partial_x(\zeta\partial_x\psi) + \partial_y^2\psi\big) - \varepsilon^3\partial_y(\zeta\partial_y\psi)\big]\big|_{H^s} \\ &\leq \quad \varepsilon^3 C\Big(\frac{1}{h_0}, \varepsilon_0, |\zeta|_{H^{s+9/2}}, |\nabla\psi|_{H^{s+5}}\Big).\end{aligned}$$

Proof.
In this regime, $\mu = \varepsilon = \gamma^2$, so that $\mathcal{G}_{\mu,\gamma}[0]\psi = \mathcal{G}_{\varepsilon,\sqrt{\varepsilon}}[0]\psi$, which can be explicitly computed (see Proposition 1):

$$\mathcal{G}_{\varepsilon,\sqrt{\varepsilon}}[0]\psi = \sqrt{\varepsilon}|D^{\sqrt{\varepsilon}}|\tanh\big(\sqrt{\varepsilon}|D^{\sqrt{\varepsilon}}|\big)\psi,$$

where we recall that $|D^{\sqrt{\varepsilon}}| = \sqrt{D_x^2 + \varepsilon D_y^2}$. An order 3 Taylor expansion at the origin of this expression gives

$$\big|\mathcal{G}_{\varepsilon,\sqrt{\varepsilon}}[0]\psi - \big[-\varepsilon(\partial_x^2 + \varepsilon\partial_y^2)\Psi - \varepsilon^2\frac{1}{3}(\partial_x^2 + \varepsilon\partial_y^2)^2\Psi\big]\big|_{H^s} \lesssim \varepsilon^3\big||D^{\sqrt{\varepsilon}}|\psi|_{H^{s+5}}. \quad (11)$$

Remarking that under the present scaling, one has

$$\big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\big|_{H^s} \leq \big||D^{\sqrt{\varepsilon}}|\psi|_{H^s} \leq |\partial_x\psi|_{H^s} + |\sqrt{\varepsilon}\partial_y\psi|_{H^s},$$

the corollary follows from Proposition 1 and (11).

■

Remark 1 i. *The method used above to give an expansion of the Dirichlet-Neumann operator $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ is general and can be used for other scalings, and in particular for the Serre approximation mentioned in the introduction, and for which $\gamma = 1$, $\mu = \varepsilon^2 \ll 1$.*
ii. *The two corollaries given above concern shallow-water models ($\mu \ll 1$), but Proposition 1 is also valid in deep water. In this case, one has $\nu \sim \mu^{-1/2}$ and the quantity one has to expand in the first equation of (3) is therefore $\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$. Remarking also that $\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[0]$ is uniformly bounded (as an operator of order 1), Proposition 1 furnishes an expansion of $\frac{1}{\sqrt{\mu}}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi$ in terms of $\varepsilon\sqrt{\mu}$. Going back to the definition of ε and μ, one can check that $\varepsilon\sqrt{\mu} = a/\lambda$. This is the* slope *of the wave, used in oceanography as small parameter in deep water.*

3.2 The case of large amplitude waves ($\varepsilon = 1$)

The shallow-water regime (for instance) assumes that $\mu \ll 1$ but deals with waves of large amplitude for which $\varepsilon = 1$. In this kind of situation, we cannot use Proposition 1 to obtain an expansion of $\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi = \mathcal{G}_{\mu,1}[\zeta]\psi$. However, one can quite easily construct by a standard BKW procedure an (explicit) approximation Φ_{app} of the velocity potential Φ (which solves (4)). We then write

$$\begin{aligned}\mathcal{G}_{\mu,1}[\zeta]\psi &= \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi|_{z=0}\\ &= \sqrt{1+|\nabla\zeta|^2}\partial_n\Phi_{app}|_{z=0} + \sqrt{1+|\nabla\zeta|^2}\partial_n\big(\Phi-\Phi_{app}\big)|_{z=0}.\end{aligned}$$

The first component of the last equality gives the asymptotic expansion on $\mathcal{G}_{\mu,1}[\zeta]\psi$ since the formula giving Φ_{app} is explicit. The second term of the equality is the truncation error and can be controlled by elliptic estimates (see [6] for such estimates in a quite general framework). With this method, one obtains the following proposition:

Proposition 2 (Shallow water regime) *Let $\mu_0 > 0$, $\varepsilon = \gamma = 1$, $s \geq t_0 > 1$, $\psi \in H^{s+4}(\mathbb{R}^2)$ and $\zeta \in H^{s+1}(\mathbb{R}^2)$ be such that (10) is satisfied for some $h_0 > 0$. Then for all $0 < \mu < \mu_0$, one has*

$$\big|\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi - \mu\big(-(1+\zeta)\Delta\psi - \nabla\psi\cdot\nabla\zeta\big)\big|_{H^s} \leq \mu^2 C\Big(\frac{1}{h_0}, \mu_0, |\zeta|_{H^{s+1}}, |\nabla\psi\big|_{H^{s+3}}\Big).$$

4 A large time existence result for the water-waves equations

This section is devoted to the proof of the main theorem of this article. We state it below and refer to [1] for full details on the proof. We just hint here at the main steps of the proof. Section 4.1 explains the structure of the linearized equations (we do it in Section 4.1.1 with elementary tools when the reference state is zero, and explain how to treat the general case in Section 4.2). The main steps of the proof of the theorem are given in Section 4.3.
Before stating the theorem, let us introduce the energy $\mathcal{E}^s$:

$$\forall s \geq 0, \qquad \mathcal{E}^s\big((\zeta,\psi)\big) := |\zeta|_{H^s} + \Big|\frac{\nu^{-1/2}|D^\gamma|}{(1+\sqrt{\mu}|D^\gamma|)^{1/2}}\psi\Big|_{H^s}. \tag{12}$$

Theorem 2 *Let $s \geq 0$ and $U^0 = (\zeta^0, \psi^0)$ be such that $\mathcal{E}^{\underline{s}}(U^0) < \infty$ for some $\underline{s} = \underline{s}(s)$ large enough. If moreover $\inf_{\mathbb{R}^d}(1+\varepsilon\zeta^0) = h_0 > 0$, then there exists $T = T(\mathcal{E}^{\underline{s}}(U^0), \frac{1}{h_0}, \varepsilon\sqrt{\mu}) > 0$ and a unique solution $U = (\zeta,\psi)$ to (3) with $(\zeta, \psi - \psi^0) \in C([0, \frac{T}{\varepsilon/\nu}]; H^s \times H^{s+1/2}(\mathbb{R}^d))$; moreover, one has*

$$\sup_{0\leq t\leq \frac{T}{\varepsilon/\nu}} \mathcal{E}^s(U(t)) \leq C(T, \mathcal{E}^{\underline{s}}(U^0), \frac{1}{h_0}, \varepsilon\sqrt{\mu}).$$

Remark 2 **i.** *The "large time" evoked in the title of this section is thus* $O(\frac{1}{\varepsilon/\nu})$. *In the shallow-water regime one has* $\varepsilon/\nu = 1$, *so that the existence time furnished by the theorem is* $O(1)$; *it is however "large" in the sense that it is uniform with respect to* $\mu \ll 1$ *(and in particular, it does not shrink to zero when* $\mu \to 0$*).*
ii. *The scale* $O(\frac{1}{\varepsilon/\nu})$ *appears to be the pertinent scale of the dynamics of the asymptotics in all the regimes mentioned in the introduction.*
iii. *The theorem requires that* $\varepsilon\sqrt{\mu}$ *remains bounded in order to have a useful control of the energy. As remarked previously,* $\varepsilon\sqrt{\mu} = a/\lambda$ *is the slope of the waves and it is not restrictive at all to assume that it remains bounded (in all the regimes considered here,* $\varepsilon\sqrt{\mu} \ll 1$*).*

4.1 The linearized equations

Let us rewrite the water-waves equations (3) in condensed form as

$$\partial_t U + \mathcal{F}_{\varepsilon,\mu,\gamma}[U] = 0,$$

with $U = (\zeta, \psi)^T$ and $\mathcal{F}_{\varepsilon,\mu,\gamma}[U]$ given by

$$\mathcal{F}_{\varepsilon,\mu,\gamma}[U] = \Big(-\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi, \zeta + \frac{\varepsilon}{2\nu}|\nabla^\gamma\psi|^2 - \frac{\varepsilon\mu}{\nu}\frac{(\frac{1}{\mu}\mathcal{G}_{\mu,\gamma}[\varepsilon\zeta]\psi + \varepsilon\nabla^\gamma\zeta\cdot\nabla^\gamma\psi)^2}{2(1+\varepsilon^2\mu|\nabla^\gamma\zeta|^2)}\Big)^T.$$

By definition, the linearized operator $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ around some reference state $(\underline{\zeta},\underline{\psi})^T$ is given by

$$\mathcal{L}_{(\underline{\zeta},\underline{\psi})} = \partial_t + d_{\underline{U}}\mathcal{F}_{\varepsilon,\mu,\gamma};$$

the goal of this section is to give energy estimates on the initial value problem

$$\begin{cases} \mathcal{L}_{(\underline{\zeta},\underline{\psi})}U = \frac{\varepsilon}{\nu}G \\ U_{|_{t=0}} = U^0; \end{cases} \tag{13}$$

4.1.1 The linearized equations around the rest state

We assume here that $\underline{U}$ is the rest state: $\underline{U} = (0,0)^T$. In this particular case, the computation of $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ can be directly computed:

$$\mathcal{L}_{(0,0)} = \partial_t + \begin{pmatrix} 0 & -\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[0]\cdot \\ 1 & 0 \end{pmatrix},$$

and moreover, one has an explicit expression for $\mathcal{G}_{\mu,\gamma}[0]\cdot$:

Proposition 3 *The operator* $\mathcal{G}_{\mu,\gamma}[0]\cdot$ *is given by the Fourier multiplier*

$$\mathcal{G}_{\mu,\gamma}[0]\cdot = \sqrt{\mu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\cdot .$$

Proof.
By definition of the operator $\mathcal{G}_{\mu,\gamma}[0]\cdot$, one has $\mathcal{G}_{\mu,\gamma}[0]\psi = \partial_z\Phi_{|_{z=0}}$, where Φ solves the Laplace equation

$$\begin{cases} \partial_z^2\Phi^2 + \mu\partial_x^2\Phi + \gamma^2\mu\partial_y^2\Phi = 0 \\ \Phi_{|_{z=0}} = \psi, \qquad \partial_z\Phi_{|_{z=-1}} = 0. \end{cases}$$

One can take the Fourier transform in the horizontal variables, which yields the second order ODE on $\widehat{\Phi}$:

$$\partial_z^2\widehat{\Phi} - \mu|\xi^\gamma|^2\widehat{\Phi} = 0,$$

which can be explicitly solved thanks to the boundary conditions. Taking the inverse Fourier transform of the solution then yields

$$\Phi(\cdot, z) = \frac{\cosh(\sqrt{\mu}(z+1)|D^\gamma|)}{\cosh(\sqrt{\mu}|D^\gamma|)}\psi,$$

so that one gets by direct computation $\partial_z\Phi_{|_{z=0}} = \sqrt{\mu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\psi$, which proves the proposition.

■

It follows from the proposition that $\mathcal{L}_{(0,0)}$ takes the explicit form

$$\mathcal{L}_{(0,0)} = \partial_t + \begin{pmatrix} 0 & -\frac{1}{\sqrt{\mu}\nu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\cdot \\ 1 & 0 \end{pmatrix}.$$

Quite obviously, this operator is non strictly hyperbolic (0 is a double eigenvalue of the principal symbol, and there is a Jordan block), and any symmetrizer, if it exists will be non-homogeneous (thus inducing a shift of derivatives in the energy – see for instance [9] for a discussion on this point). Here, a symmetrizer is obviously given by

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{\mu}\nu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\cdot \end{pmatrix},$$

which motivates the following choice of the energy

$$\begin{aligned} \mathfrak{E}^s(U)^2 &= (\Lambda^s U, S\Lambda^s U) \\ &= |\zeta|_{H^s}^2 + \big(\Lambda^s\psi, \frac{1}{\sqrt{\mu}\nu}|D^\gamma|\tanh(\sqrt{\mu}|D^\gamma|)\Lambda^s\Psi\big). \end{aligned}$$

It worth noticing that $\mathfrak{E}^s(U) \sim \mathcal{E}^s(U)$ (and the equivalence is *uniform* with respect to the parameters μ and γ). This shows in particular that this energy controls the truncation error in Proposition 1.

Remark 3 *It is true that one has the following equivalence*

$$\mathfrak{E}^s(U) \sim |U|_{H^s\times H^{s+1/2}};$$

however, this equivalence is completely useless because the equivalence is not uniform with respect to γ and μ. The fact that one cannot use such an equivalence complicates considerably the proof and compels us to use more structural properties of the water-waves equations than in [17] for instance.

By very standard techniques, one then obtains the following proposition:

Proposition 4 *Assume that* $\underline{U} = (0,0)$ *and let* $s \geq 0$*. Assume also that* $G \in C([0, \frac{T}{\varepsilon/\nu}]; H^s \times H^{s+1/2}(\mathbb{R}^2))$ *and* $U^0 \in H^s \times H^{s+1/2}(\mathbb{R}^2)$*. Then there exists a unique solution* $U \in C([0, \frac{T}{\varepsilon/\nu}]; H^s \times H^{s+1/2}(\mathbb{R}^2))$ *to (13) and one has*

$$\forall t \in [0, \frac{\nu}{\varepsilon}T], \qquad \mathfrak{E}^s(U(t)) \leq \mathfrak{E}^s(U^0) + T \sup_{t\in[0,\nu T/\varepsilon]} \mathfrak{E}^s(G(t)).$$

Remark 4 *One could of course have solved (13) explicitly and deduce the estimates, but our purpose here is to introduce the methods used in the study of (13) when* $\underline{U}$ *is not necessarily zero.*

4.2 The linearized equations in the general case

In the general case, i.e. when $\underline{U}$ is not zero, the computation of $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ is not straightforward. However, assuming that $\underline{U}$ is such that the assumptions of Theorem 1 are satisfied, one obtains as in [17] an explicit expression for $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$,

$$\mathcal{L}_{(\underline{\zeta},\underline{\psi})} =$$
$$\partial_t + \begin{pmatrix} \frac{\varepsilon}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}](\underline{Z}\cdot) + \frac{\varepsilon}{\nu}\nabla^\gamma\cdot(\cdot\underline{\mathbf{v}}) & -\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \\ \frac{\varepsilon^2}{\mu\nu}\underline{Z}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}](\underline{Z}\cdot) + (1 + \frac{\varepsilon^2}{\nu}\underline{Z}\nabla^\gamma\cdot\underline{\mathbf{v}}) & \frac{\varepsilon}{\nu}\underline{\mathbf{v}}\cdot\nabla^\gamma\cdot - \frac{\varepsilon}{\mu\nu}\underline{Z}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \end{pmatrix},$$

where $\underline{\mathbf{v}}$ and $\underline{Z}$ are as in the statement of Theorem 1.

A study of the principal symbol of this operator shows that, as for the linearization around zero, $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ is not strictly hyperbolic (the double eigenvalue is now $\frac{\varepsilon}{\nu}\underline{\mathbf{v}}\cdot\xi^\gamma$). It was shown in Prop. 4.2 of [17] that a simple change of basis can be used to put the principal symbol of $\mathcal{L}_{(\underline{\zeta},\underline{\psi})}$ under a canonical trigonal form. This result is generalized to the present case. More precisely, with

$$\underline{\mathfrak{a}} = 1 + \frac{\varepsilon}{\nu}\underline{\mathfrak{b}}, \quad \text{ and } \quad \underline{\mathfrak{b}} = \varepsilon\underline{\mathbf{v}}\cdot\nabla^\gamma\underline{Z} + \nu\partial_t\underline{Z}, \tag{14}$$

and defining the operator $\mathcal{M}_{(\underline{\zeta},\underline{\psi})} = \partial_t + M_{(\underline{\zeta},\underline{\psi})}$ with

$$M_{(\underline{\zeta},\underline{\psi})} = \begin{pmatrix} \frac{\varepsilon}{\nu}\nabla^\gamma\cdot(\cdot\underline{\mathbf{v}}) & -\frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \\ \underline{\mathfrak{a}} & \frac{\varepsilon}{\nu}\underline{\mathbf{v}}\cdot\nabla^\gamma. \end{pmatrix}, \tag{15}$$

one reduces the study of (13) to the study of the initial value problem

$$\begin{cases} \mathcal{M}_{(\underline{\zeta},\underline{\psi})}V = \frac{\varepsilon}{\nu}H \\ V_{|_{t=0}} = V^0, \end{cases} \tag{16}$$

as shown in the following proposition (whose proof relies on simple computations and is omitted).

Proposition 5 *The following two assertions are equivalent:*

- *The pair* $U = (\zeta,\psi)^T$ *solves (13);*

- *The pair* $V = (\zeta, \psi - \varepsilon \underline{Z}\zeta)^T$ *solves (16), with* $H = (G_1, G_2 - \varepsilon \underline{Z} G_1)^T$ *and* $V^0 = (\zeta^0, \psi^0 - \varepsilon \underline{Z}_{|t=0}\zeta^0)^T$.

In view of this proposition, it is a key step to understand (16), and the rest of this subsection shows the way to prove energy estimates for this initial value problem.

First remark that a symmetrizer for $\mathcal{M}_{(\underline{\zeta},\underline{\psi})}$ is given by

$$S = \begin{pmatrix} \underline{\mathfrak{a}} & 0 \\ 0 & \frac{\varepsilon^2}{\nu^2} + \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\cdot \end{pmatrix}, \tag{17}$$

so that (provided that $\underline{\mathfrak{a}}$ is nonnegative), a natural energy for the ivp (16) is given by

$$\begin{aligned} E^s(V)^2 &= (\Lambda^s V, S\Lambda^s V) \\ &= |\sqrt{\underline{\mathfrak{a}}}\Lambda^s V_1|_2^2 + \frac{\varepsilon^2}{\nu^2}|V_2|_{H^s}^2 + (\Lambda^s V_2, \frac{1}{\mu\nu}\mathcal{G}_{\mu,\gamma}[\varepsilon\underline{\zeta}]\Lambda^s V_2). \end{aligned} \tag{18}$$

Remark 5 *The term* $\frac{\varepsilon^2}{\nu^2}|V_2|_{H^s}^2$ *in (18) is due to the* $\frac{\varepsilon^2}{\nu^2}$ *in the second coefficient of the diagonal of (17). Removing this term would not affect the energy estimate given below; however, thanks to it, the energy controls the low frequencies of* V_2*, which is very important in the iterative scheme used to solve to full water-waves equations.*

The energy (18) is the right one to obtain energy estimates on (16), but the reference state $\underline{U}$ must be admissible in the following sense:

Definition 1 *Let* $t_0 > 1$ *and* $T > 0$*. We say that* $\underline{U} = (\underline{\zeta}, \underline{\psi})$ *is* admissible on $[0, \frac{T}{\varepsilon/\nu}]$ *if*

- $(\underline{\zeta}, \nabla\underline{\psi}) \in C^2([0, \frac{T}{\varepsilon/\nu}]; H^\infty(\mathbb{R}^d)^{1+2})$*;*
- *The surface parameterization* $\underline{\zeta}$ *satisfies (10) for some* $h_0 > 0$*, uniformly on* $[0, \frac{T}{\varepsilon/\nu}]$*;*
- *There exists* $c_0 > 0$ *such that* $\underline{\mathfrak{a}} \geq c_0$*, uniformly on* $[0, \frac{T}{\varepsilon/\nu}]$*.*

We can now give the energy estimate associated to (16); it can be seen as a generalization of Proposition 4. We refer to [1] for the proof (in the statement of the proposition, $E^s_T(H)$ stands for $E^s_T(H) = \sup_{0\leq \nu T/\varepsilon} E^s(H(t))$).

Proposition 6 *Let* $t_0 > 1$*,* $T > 0$*, and assume that* $\underline{U} = (\underline{\zeta}, \underline{\psi})$ *is admissible on* $[0, \frac{T}{\varepsilon/\nu}]$ *for some* $h_0 > 0$ *and* $c_0 > 0$*.*
Then, for all $H \in C([0, \frac{T}{\varepsilon/\nu}]; H^\infty(\mathbb{R}^2)^2)$*, there exists a unique solution* $V \in C([0, \frac{T}{\varepsilon/\nu}]; H^\infty(\mathbb{R}^2)^2)$ *to (16) and, for all* $s \geq 0$*, and* $0 \leq t \leq \frac{T}{\varepsilon/\nu}$*,*

$$E^s(V(t)) \leq \underline{C} \times \Big[E^s(V^0) + TE^s_T(H) + \Big\langle \big(E^{t_0+1}(V^0) + TE^{t_0+1}_T(H)\big)\underline{D}_s\Big\rangle_{s>t_0+1}\Big],$$

where
$$\underline{D}_s = \mathcal{E}_T^{s+7/2}(\underline{U}) + \mathcal{E}_T^{s+2}(\partial_t \underline{U})$$
and
$$\underline{C} = C\Big(T, \frac{1}{h_0}, \frac{1}{c_0}, \frac{\varepsilon}{\nu}, \mathcal{E}_T^{t_0+9/2}(\underline{U}), \mathcal{E}_T^{t_0+2}(\partial_t \underline{U}), \mathcal{E}_T^{t_0+3/2}(\partial_t^2 \underline{U})\Big).$$

4.3 Main steps of the proof of Theorem 2

The energy estimate given in Proposition 6 concerns the initial value problem (16). Using Proposition 5 once again, but with the other side of the equivalence, we deduce an energy estimate for the initial value problem (13).
This energy estimates does not allow a standard Picard iterative scheme because it exhibits losses of derivatives (in Proposition 6 for instance, one needs an energy of order $s+7/2$ on $\underline{U}$ to control an energy of order s on V). However, this energy is *tame* in the sense that the s-dependent terms on the right hand side are all linear. This allows, as in [17], the use of a Nash-Moser type iterative scheme. The order $O(\frac{1}{\varepsilon/\nu})$ of the existence time furnished by the Nash-Moser fixed point theorem follows from the fact that the energy estimate of Proposition 6 depends only on T for times $\frac{T}{\varepsilon/\nu}$ (note that we use here a special Nash-Moser theorem with parameters for evolution equations developed in [2]).
The last points to comment on are the two conditions required in the definition of an admissible reference state. Quite obviously, the condition on the water depth will remain true for T small enough (but uniformly with respect to the parameters) if it is initially true. The second condition, on the sign of $\underline{\mathfrak{a}}$, is not that clear. In fact, it follows from the works of S. Wu [28, 29], generalized in [17] for the case of finite depth, that one has necessarily $\underline{\mathfrak{a}} > 0$ for *exact solutions* of the water-waves equations. Choosing the first term of the iterative scheme in such a way that it solves the water-waves equations at $t = 0$, there exists c_0 such that $\underline{\mathfrak{a}}(t = 0) > 2c_0$; it is then possibly to maintain the condition $\underline{\mathfrak{a}}(t) > c_0$ on $[0, \frac{T}{\varepsilon/\nu}]$ (taking a smaller T if necessary).

5 Asymptotics for $3D$ water-waves

As an illustration of the methods developed in this note, we sketch here how to give a full justification of asymptotic models for $3D$ water-waves in three different regimes: shallow water, long waves, and KP regime.

5.1 Shallow-water equations

We recall that the so-called "shallow-water" regime corresponds to the condition $\mu \ll 1$ (so that $\nu \sim 1$) and that $\varepsilon = \gamma = 1$. It follows therefore from Theorem 2 that there exists $T > 0$ independent of μ such that solutions to (3) exist on $[0, T]$. Moreover, the energy bound provided by the theorem ensures that ζ and $V := \nabla\psi$ are uniformly bounded on $[0, T]$ in Sobolev spaces. Plugging the expansion furnished by Proposition 2 into (3) and taking the gradient of the second equations in order to obtain a system of equations on ζ and $V=\nabla\psi$, one gets

$$\left\{\begin{array}{l} \partial_t V+\nabla\zeta+\frac{1}{2}\nabla|V|^2=\mu R_1^\mu,\\ \partial_t\zeta+\nabla\cdot(1+\zeta V)=\mu R_2^\mu, \end{array}\right. \tag{19}$$

with R_1^μ and R_2^μ uniformly bounded in Sobolev spaces on the time interval $[0,T]$. An energy estimate on (19) thus shows that the error made by using exact solutions of the shallow water equations (namely, (19) with zero on the right hand side) instead of (3) is $O(\mu)$ on $[0,T]$. In other words, the $2DH$ shallow water model is fully justified.

5.2 Long-waves regime

The long-wave regime is characterized by the scaling $\gamma=1$, $\mu=\varepsilon\ll 1$, so that one has $\nu\sim 1$. It follows from Theorem 2 that there exists $T>0$ independent of ε and a unique solution $U=(\zeta,\psi)$ to (3) on the time interval $[0,\frac{T}{\varepsilon}]$ such that the energy

$$|\zeta(t)|_{H^s}+\big|\frac{|D|}{(1+\sqrt{\varepsilon}|D|)^{1/2}}\psi(t)\big|_{H^s}$$

remains bounded on $[0,\frac{T}{\varepsilon}]$. Defining $V=\nabla\psi$, this implies that ζ and V remain bounded on $[0,\frac{T}{\varepsilon}]$ in Sobolev spaces. This is was exactly the condition that was needed in [4] to fully justify the $2DH$ Boussinesq systems. For the sake of completeness, we recall briefly the strategy of [4].

Plugging the asymptotic expansion of the Dirichlet-Neumann operator given by Corollary 1 into (3) and taking the gradient of the equation on ψ, one gets

$$\left\{\begin{array}{l} \partial_t V+\nabla\zeta+\varepsilon\frac{1}{2}\nabla|V|^2=\varepsilon^2 R_1^\varepsilon,\\ \partial_t\zeta+\nabla\cdot V+\varepsilon\big(\frac{1}{3}\Delta\nabla\cdot V+\nabla\cdot(\zeta V)\big)=\varepsilon^2 R_2^\varepsilon, \end{array}\right. \tag{20}$$

where, as a consequence of Corollary 1 and Theorem 2, $R_j^\varepsilon=R_j^\varepsilon(\zeta,\psi)$ $(j=1,2)$ are uniformly bounded on the time interval $[0,T/\varepsilon]$ in Sobolev spaces. The Boussinesq system (20), however, is not well-posed, and one cannot directly conclude as in the shallow-water regime. Using linear manipulations (set forth in a systematic way in [5]) and a nonlinear change of variables introduced in [4], one can construct an infinity of Boussinesq systems, formally equivalent to (20), some of which being well-posed. Making the energy estimates on such a well-posed system, one can show that the approximations furnished by the Boussinesq systems have a precision of order $O(\varepsilon^2 t)$ on $[0,T/\varepsilon]$.

5.3 The Kadomtsev-Petviashvilii approximation

We recall that the KP regime is the same as the long-waves regime, but with $\gamma=\sqrt{\varepsilon}$. Theorem 2 then furnishes a solution of (3) on a time interval $[0,T/\varepsilon]$; moreover, the energy bound shows that ζ, $\partial_x\psi$ and $\sqrt{\varepsilon}\partial_y\psi$ are bounded on $[0,T/\varepsilon]$ in Sobolev spaces. This was exactly the assumption made in [18] to justify the Katomtsev-Petviashvili equations which states that the water elevation

ζ is approximated on $[0, T/\varepsilon]$ by

$$\zeta(t,x) \sim \zeta_+(\varepsilon t, x-t, \sqrt{\varepsilon}y) + \zeta_-(\varepsilon t, x+t, \sqrt{\varepsilon}y),$$

where $\zeta_\pm(\tau, \widetilde{x})$ solves

$$\partial_\tau \zeta_\pm \pm \frac{1}{2}\partial_{\widetilde{x}}^{-1}\partial_y^2\zeta_\pm \pm \frac{1}{6}\partial_{\widetilde{x}}^3\zeta_\pm + \frac{3}{2}\zeta_\pm\partial_{\widetilde{x}}\zeta_\pm = 0.$$

The strategy of [18] to justify the KP approximation from the large time existence theorem and the bounds on ζ, $\partial_x\psi$ and $\sqrt{\varepsilon}\partial_y\psi$ consists in justifying first a class of *weakly transverse* Boussinesq systems along the lines described in Section 5.2; the KP approximation is then justified from these systems with nonlinear optics methods, as in [3].

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